Optimal Portfolio Analysis over the Life Cycle

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Abstract

This paper solves the calibrated partial equilibrium model over the life cycle and explores the optimal consumption and portfolio allocation. We find that after plugging in the minimum equity investment requirement, the allocation on the stock market has hump shaped pattern. Wealthier households also have smoother equity holding over the whole life cycle. These replicate the empirical evidence.

1 Introduction

For many individuals, the investment decision and the quantity of retirement saving are among the most important choices over the life cycle. Samuelson (1969) and Merton (1969) first addressed this problem in discrete and continuous time, respectively, under the assumption of complete market, and without labor income, etc. Merton (1971) then introduced risk insured labor income into complete market. Recently, several literatures discuss uninsurable labor income and incomplete market, and the effect of labor income risk on portfolio allocation over the life cycle. These are explored by Bertaut and Haliassos (1997), Benzoni, Collin-Dufresne and Goldstein (2004), Gakidis (1997), Storesletten, Telmer, and Yaron (2000), and Viceira (2001). Other papers, such as Balduzzi and Lynch (1997), Brennan and Xia (1998), Campbell and Viceira (1999), Kim and Omberg (1996), study the effects caused by the changes of riskfree interest rate or equity premium over the time. Carroll (2002), Campbell and Cochrane (1999) propose models in which risk aversion varies with wealth.

This paper solves for the optimal decision on consumption and portfolio choices in the calibrated life cycle model. In addition to the uninsurable labor incomes that other literatures have already explored, our model also captures the minimum equity investment requirement on the stock account and the average wage-indexed earnings (AWIE) scheme for each individual's retirement income. This retirement benefit that individual receives will replicate the redistributive property of Social Security, which is heavily biased in favor of those low income workers. We also find empirical evidence on household asset allocation from Survey of Consumer Finances 2004. This paper will examine the asset allocation from our model, focusing on how the minimum equity investment requirement and borrowing constraints influence individual's decision making process on optimal portfolio choices, and replicate the empirical facts.

The organization of the paper is as follows. Section 2 describes the empirical evidence. Section 3 sets up the model and discusses the assumption and numerical methods. Section 4 lays out the calibration of the parameters. Section 5 presents the simulation results. Section 6 gives the conclusion.

2 Empirical Evidence

Survey of Consumer Finances (SCF) is generally considered as the best data for information on financial wealth situations of U.S. families. It provides good coverage of characteristics, such as age, education, etc., which are related to many aspects of financial behavior. Besides total wealth, the data set also measures many categories of the wealth. This detailed breakdown makes our investigation on the asset allocations become possible. The SCF uses a dual-frame sample design. One part is selected to obtain a sufficiently large and unbiased sample. The other part is designed to disproportionately select wealthier families. We use the weights to adjust the unequal probabilities of selection in the survey and for nonresponse. SCF is conducted every three years with different families. It's not panel data. Thus we choose 2004 data set, which is the most recent one.

Figure 1, Figure 2 and Figure 3 illustrate the mean stock allocation in different age groups for no high school education families, high school education families and college education families, separately. These graphs show a hump-shaped stock proportion over the life-cycle. Through different education groups, although their equity holdings in the middle-age are all around 70 percent, their allocations in the early age and after retirement do vary. Households with more education experience tend to hold more equity when they are young or after they retire. In another word, the equity allocation through the whole life is relatively smoother for those households that receive more education. Appendix A describes the definitions of variables from SCF 2004.

We construct a theoretical model and calibrate it to replicate some facts about stock allocation in the next several sections.

3 Specification of the Model

3.1 The Life Cycle

Each individual i starts working from age t_0 to retirement age M. Individual can live up to age T with surviving probability p_t, which is the probability that the individual is still alive at age $t+1$, conditional on being alive at age t.

During individual's working age, their earnings change in a predictable pattern, as well as random shocks, whose distributions are known. Individuals tax their income, participate into the Social Security system during their working ages and receive retirement benefits after that. They can satisfy their consumption needs in retirement by accumulating savings in their working ages.

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Individuals make their consumption decision in the beginning of each period. For the rest of the wealth that individuals do not consume, they hold them in two forms: bonds and stocks. The returns of the bonds are assumed to be fixed and riskless. In contrast, the returns of the stocks are composed by a constant excess return and random innovations to returns, with a known distribution. Individuals then make the decision on their asset allocation, restricted to a borrowing constraint.

At each age of their life, individuals don't know their income and stock returns in the next period. But they are aware of the corresponding distributions. Agents are fully rational, so they make the decision on their consumption and investment to maximize their expected utility.

3.2 Preferences

It's common to assume that Individuals have constant relative risk aversion preferences for consumption. The preferences are additively time-separable, which is a general assumption in life cycle models. $\beta < 1$ is the time preference rate. Preferences are identical across individuals as

$$\max E_{t_0} \sum_{t=t_0}^{T} \beta^{t-t_0} \left(\prod_{j=t_0-1}^{t-2} p_j \right) \left(p_{t-1} \frac{C_{i,t}^{1-\gamma}}{1-\gamma} + b(1-p_{t-1}) \frac{W_{i,t}^{1-\gamma}}{1-\gamma} \right) \tag{1}$$

$C_{i,t} > 0$ represents individual i's consumption at age t. $\gamma > 0$ is the coefficient of relative risk aversion. $W_{i,t}$ is the amount of bequest to the individual's heir at the time of the death. We assume that the utility function applied to the bequest is identical to the one applied to the general consumption. Parameter b identifies the intensity of bequests.

3.3 Financial Assets

Investors have access to two investment instruments: riskfree bonds and risky stocks. The riskless bonds, often Treasury bills, pay a constant gross real return, the logarithm of which is R_f, while the risky asset pays a logarithm of gross real return R_t^S represented by

$$R_t^S = R^f + \mu + \eta_t \tag{2}$$

where η_t is the innovation to excess returns of time t. We assume η_t to be independently and identically distributed (i.i.d.) over time and the logarithm of η_t is distributed as normal distribution $N(0, \sigma_\eta)$.

3.4 Labor Income

Because of income uncertainty, individuals save for future contingencies and retirement. Literatures have demonstrated that these income shocks lead to hump-shaped consumption profiles due to precautionary savings (Hubband, Skinner and Zeldes (1994)). Zeldes(1989), Carroll (1992) and Deaton (1991) also conclude that consumers are impatient in the sense of if future incomes were known with certainty, individuals would borrow in order to maintain a high level of current consumption more than their current incomes.

With prudent and impatient consumers, recent theoretical work has shown that individual's savings play a role of a "buffer stock" against income shocks

(Deaton (1991), Zeldes(1989), and Carroll (1997)). Buffer stock savers have a target wealth level so that, if assets are below the level, prudence dominates and the individuals save for precautionary motive, else if assets are above the target level, impatience dominates and assets are declining.

Following Carroll (1997) and Gourinchas and Parker (2002), which proof that buffer stock model is superior to other models in terms of a better explanation for "aggregated consumption/income parallel" and "consumption/income divergence" facts, the income process is modeled as follow. $Y_{i,t}$ is exogenous labor income.

$$Y_{i,t} = \exp(y_{i,t}) \tag{3}$$

Before retirement, it's composed by an idiosyncratic temporary shock $\omega_{i,t}$ and a permanent shock $z_{i,t}$.

$$y_{i,t} = g(t, F_{i,t}) + z_{i,t} + \omega_{i,t} \tag{4}$$

where $g(t, F_{i,t})$ is a deterministic function of age t and other individual's characteristics $F_{i,t}$. $\omega_{i,t}$ is distributed i.i.d. and as $N(0, \sigma_\omega)$. $z_{i,t}$ is a first order autoregressive process given by

$$z_{i,t} = \theta z_{i,t-1} + \xi_t + \phi_{i,t} \tag{5}$$

θ is the autocorrelation coefficient. Permanent shock u_t is composed by aggregate component ξ_t and idiosyncratic component $\phi_{i,t}$, which are distributed i.i.d. and as $N(0, \sigma_\xi)$ and $N(0, \sigma_\phi)$, separately. Finally, we assume the correlation between innovation to excess stock returns η_t and the aggregate labor income shock ξ_t as $\rho_{\xi\eta}$.

3.5 Wage-Indexed Social Security

Investors participate in social insurance program that taxes their income at a rate of τ_{SS} through their working years. The disposable income for the investors after taxes is as follows.

$$Y_{i,t}^d = (1 - \tau_{SS})Y_{i,t} \text{ for } t < M \tag{6}$$

The social security benefits after retirement is based upon the investor's Average Wage-Indexed Earnings (AWIE) $\overline{Y}_{j,M}$ at retirement and the progressive Social Security benefit formula.

3.5.1 Average Wage-Indexed Earnings

In each period from working age t_0 to retirement age M, individual earning is indexed by the aggregate wage level of the base year (chosen to be retirement age) to that of the period. Defining Y_t^A as the aggregate wage level at time t, the investor's AWIE at retirement will be,

$$\overline{Y}_{j,M} = \frac{Y_{j,t_0}\frac{Y_M^A}{Y_{t_0}^A} + Y_{j,m+1}\frac{Y_M^A}{Y_{t_0+1}^A} + \cdots + Y_{j,M}\frac{Y_M^A}{Y_{M-1}^A}}{M-t_0} \tag{7}$$

$$= \frac{\left(\left(Y_{j,t_0}\frac{Y_{t_0+1}^A}{Y_{t_0}^A} + Y_{j,t_0+1}\right)\frac{Y_{t_0+2}^A}{Y_{t_0+1}^A} + Y_{j,t_0+2}\right)\cdots}{M-t_0}$$

The average wage level is modeled as an AR(1) process with a unit-root assumption and the risky component.

$$Y_{t+1}^A = Y_t^A \cdot \exp \xi_{t+1} \tag{8}$$

Since

$$\frac{Y_{t+1}^A}{Y_t^A} = \exp \xi_{t+1} \tag{9}$$

We can model the investor's AWIE at each period based on the AWIE one period before.

$$\overline{Y}_{j,t+1} = \frac{(t-t_0)\overline{Y}_{j,t} \cdot \exp \xi_{t+1} + Y_{j,t+1}}{t-t_0+1} \tag{10}$$

The wage-indexed benefits in our model mimic the Old-Age portion of the US Social Security system.

3.6 Optimization Problem

In each period t, individual i has cash on hand $X_{i,t}$, which is composed by the wealth $W_{i,t}$ at the beginning of each period, which is also the bequest in the case that the individual dies, and the realized income $Y_{i,t}$. h_t denotes the proportion of income in housing expenditures. Individual then decides $C_{i,t}$, how much to consume in this period. Individual then determines how much of the remaining to be invested into bonds and stocks, separately.

$$W_{i,t+1} = R_{i,t}(W_{i,t} + (1-h_t)Y_{i,t}^d - C_{i,t}) \tag{11}$$

$$X_{i,t} = W_{i,t} + (1-h_t)Y_{i,t}^d \tag{12}$$

$\alpha_{i,t}$ is the proportion that the individual invests into risky asset. We assume $\alpha_{i,t} \in [0,1]$, so that the investor's allocation to bonds and stocks has to be positive at any period. $R_{i,t}$ is the total gross real return on the portfolio at time t.

$$R_{i,t} = \alpha_{i,t} \cdot \exp R_{i,t}^S + (1-\alpha_{i,t}) \cdot \exp R_f \tag{13}$$

Each period, there is a minimum investment requirement for risky assets. In order to reduce the numbers of state variables, we compute two situations. One is that the individual can invest into both bonds and stocks, where the amount that is put into risky account has to be equal or greater than m. The other is that the individual can only invest into bond market. Then we compare the value of these two situations for each state variables and pick the larger one.

Thus, in case one, an individual characterized by age, wealth and permanent income shock solves the Bellman equation at time t, with the state variables $X_{i,t}$, $z_{i,t}$, and other parameters given

$$V^1_{i,t}(X_{i,t}, z_{i,t}) = \max_{C^1_{i,t}, \alpha^1_{i,t}} \left\{ \frac{\left(C^1_{i,t}\right)^{1-\gamma}}{1-\gamma} + \beta \left[p_t \cdot E_t V^1_{i,t+1}\left(X^1_{i,t+1}, z_{i,t+1}\right) + b(1-p_t) \cdot E_t \left(\frac{\left(W^1_{i,t+1}\right)^{1-\gamma}}{1-\gamma} \right) \right] \right\} \tag{14}$$

Subject to

$$\alpha_{i,t} \in [0, 1] \tag{15}$$

Where

$$X^1_{i,t+1} = Y_{i,t+1} + (\alpha^1_{i,t} R^S_{i,t} + (1-\alpha^1_{i,t}) R_f)(X_{i,t} - C^1_{i,t}) \tag{16}$$

$$\alpha^1_{i,t}(X_{i,t} - C^1_{i,t}) \geqslant m \tag{17}$$

In case two,

$$V^2_{i,t}(X_{i,t}, z_{i,t}) = \max_{C^2_{i,t}} \left\{ \frac{\left(C^2_{i,t}\right)^{1-\gamma}}{1-\gamma} + \beta \left[p_t \cdot E_t V^2_{i,t+1}\left(X^2_{i,t+1}, z_{i,t+1}\right) + b(1-p_t) \cdot E_t \left(\frac{\left(W^2_{i,t+1}\right)^{1-\gamma}}{1-\gamma} \right) \right] \right\} \tag{18}$$

Where

$$X^2_{i,t+1} = Y_{i,t+1} + R_f(X_{i,t} - C^2_{i,t}) \tag{19}$$

Finally, we pick the larger value

$$V_{i,t}(X_{i,t}, z_{i,t}) = \max\left[V^1_{i,t}(X_{i,t}, z_{i,t}), V^2_{i,t}(X_{i,t}, z_{i,t})\right]$$

$\{t, X_{i,t}, z_{i,t}\}^T_{t=t_0}$ are the state variables. The control variables are $\{C_{i,t}, \alpha_{i,t}\}^T_{t=t_0}$.

3.7 Numerical Solution

The result of the optimization problem is a set of policy function for each state variable node. This problem can not be solved analytically. The backward iteration derives the policy functions numerically and solves for $C_{i,t}(X_{i,t}, z_{i,t})$ and $\alpha_{i,t}(X_{i,t}, z_{i,t})$ with the following constraint

$$C_{i,t} > 0$$

$$0 \leq \alpha_{i,t} \leq 1$$

Since state spaces are continuous, they are discretized in even space grid in computation. To solve this three-dimensional problem, Schumaker shape-preserving and Bilinear interpolation constructs an approximation that interpolates the data linearly in both coordinate directions.

There exist three labor income shocks $\omega_{i,t}$, ξ_t and $\phi_{i,t}$, and the uncertainty of the innovation to excess stock returns η_t. They are continuous and their distributions are known to the consumers. Gauss-Hermite quadrature discretizes all these shocks into several nodes and evaluates these normal distributional functions using these points. Since the innovation to excess stock returns is assumed to be correlated with aggregate permanent income shock, the transformation of Gauss-Hermite quadrature for bivariate normal distribution is shown in Appendix B.

With bequest motive $b < 1$, individual consumes all the cash on hand in the last period. If $b \geqslant 1$, we assume that individual consumes one tenth of the last period cash-on-hand, and leave the rest to the heir. Hence the value function at T is equivalent to the utility function with the corresponding consumption and bequests. For each other period before T, we compute the maximum value function and optimal control variables associated with each pair of discretized values for cash on hand and labor income. The value function is equal to the utility function plus the discounted expectation of future value function. The expectation is computed by Gauss quadrature weights and interpolation nodes. For the points in the state space other than the grip points, we use Schumaker shape-preserving quadratic spline interpolation which can best preserve the curvature and monotonicity properties of a function. Because it's difficult to extend Schumaker shape-preserving interpolation into multidimensional space, we explore the method of combination of Schumaker and Bilinear interpolation to approximate the three dimensional problems. Details about the integration process are illustrated in the Appendix B.

We use Euler Equation Errors to verify the accuracy of the numerical solution. Appendix C explains the details.

4 Calibration

4.1 Preferences

Individual starts working from age 20 for households without a college degree, and from age 22 for households with a college degree. The retirement age is set to be 65 for all households. Individual could live up to age 100. We set the discount factor β to 0.90 and the coefficient of relative risk aversion γ to 5. The mortality data for the whole population follow the National Vital Statistics

Reports from National Center for Health Statistics (Elizabeth Arias 2004). The benchmark bequest intensity parameter is set to be 0.

4.2 Financial Assets

We consider μ a conservative value of 4.00%. The riskfree return R_f is set to be 2.00%. The standard deviation of the innovation to excess returns σ_η is set to be the historical value 0.157. According to the regular mutual fund, we set $3000 as the amount of minimum investment requirement.

4.3 Labor Income and Housing Expenditures

The estimation of labor income process follows Cocco, Gomes and Maenhout (CGM) (2005). Here we briefly describe the data and estimation method they use.

The data they use to estimate the deterministic function of the income process is the family questionnaire of the Panel Study of Income Dynamics (PSID). They take a household as a unit.

When estimating the deterministic income function, they include a broad definition, such as labor income, unemployment compensation, workers compensation, social security, supplemental social security, other welfare, child support, and total transfers from relatives, in order to implicitly allow for self insurance so that household could partially protect themselves against labor income risk.

Distinguishing among the education is according to the finding that age profiles differ across different education groups (Attanasio 1995, Hubbard, Skinner and Zeldes 1994). $g(t, F_{i,t})$ is assumed to be additively separable in t and $F_{i,t}$, where the characteristics of the household $F_{i,t}$ include marital status and family size. Family size is defined to be the additional number of family members in the household besides the head and his spouse, if any. For the age t, a third order polynomial to the age dummies is estimated to fit the age profiles. The logarithm of labor income is regressed based on the above factors.

The variances of transitory shocks and permanent shocks on labor income are taken from Carroll (1997). The correlation coefficient between stock returns shocks and labor income shocks is taken from Campbell et al. (2001). The housing expenditures are taken exogenously from Gomes and Michaelides (2005). They use the data from PSID 1976 until 1993 to measure the ratio of annual housing expenses to annual labor income as a function of age. Table 1 reports all the parameter values and coefficients. Table 2 illustrates the income process coefficients for different education groups.

The paper by CGM uses 1992 dollar to compute all the parameters. They have used inflation to calculate real variables. All regressions and correlations were calculated using these real variables. In order to be consistent with 2004 social security bend points, we use the change in the Consumer Price Index 34.11% as the inflation rate from 1992 to 2004.

4.3.1 Progressive Social Security Benefit Formula

The social security tax rate τ_{SS} is 12.4%. After calculating the AWIE, we apply three separate rates to generate the final benefits according to the following formula. The "bend points", which are points that joint different rates, are 2004 data.

90 percent of the first \$7,344 of AWIE, plus

32 percent of the AWIE over \$7,344 through 44,268, plus

15 percent of the AWIE above 44,268

5 Results for the benchmark case

After obtaining the optimal policy function, we generate a series of random variables for labor income shocks and portfolio return shocks, based on their lognormal distribution, then simulate over 10,000 households. We consider those individuals with high school degree as baseline. Figure 4 to Figure 6 illustrates the means of the simulated profiles.

In Figure 4, over the whole life cycle, consumption increases with income in the early years, then decreases due to mortality risk. The hump shape of consumption is consistent with the literatures on life cycle consumption profiles. In Figure 5, The cash-on-hand (COH) rises with age while the individual still works, then gradually decreases after retirement due to much less retirement income.

Figure 6 illustrates the equity holding over the life cycle. Our portfolios on the stock accounts generated from model are generally higher than empirical counterparts by an average of about 20 percent. In terms of life cycle behavior, however, the model predictions have similar patterns with empirical data. In the early ages, the households expect the future increasing labor income to be riskless, which reduces households' holding of riskless assets. However, households are subject to minimum equity investment requirement and borrowing constraints, which are more important for younger households. These make their equity holdings at around 60 percent. As households work for a few years and accumulate wealth, the "crowd-out" part of labor income takes effect. Households have almost full portfolio allocation in the stocks. When households approaches age 45, their cash-on-hand increases relative to labor income. There is also no much increasing space for future labor income. The "crowd-out" effect of labor income becomes less important. Meanwhile, individuals save for after retirement life due to precautionary reason. These all induce a shift in the portfolio allocation towards riskless assets. After retirement, individuals have lower labor income compared to working ages. Thus, households start to consume not only their income but their accumulated wealth. Their cash-on-hand drops rapidly. They are again subject to minimum equity investment requirement and borrowing constraints. The shift towards riskless assets continues.

Under the same benchmark parameters, we draw the equity allocations from our model for no high school education group and college education group in

Figure 7 and Figure 8, separately. The hump-shaped pattern for no high school group coincides with empirical findings from SCF 2004 in Figure 1. It also replicates the fact that poorer people have relatively lower stock allocation in their early ages and after the retirement. The college education group has full allocation on the equity even when they are young. This is because college group households start working later than other groups. Generally, they also have higher income. Minimum equity investment requirement is less important for them.

6 Conclusion

This paper solves a calibrated life cycle model to obtain the optimal consumption and asset allocation policies. This model is expensive to compute. We use Message Passing Interface (MPI), one of the parallel computation techniques, to speed up the computation process. We found that "crowd-out" effect of labor income is generally less important when people are young and subject to minimum equity investment requirement and borrow constraints, especially for poorer households. Households demonstrate a hump shaped equity holdings over the life cycle. On the other hands, there still exists "equity premium puzzle" that empirically, people invest an average of 20 percent lower on equity than our model shows. There are some other aspects that aren't included in this paper. It doesn't include housing problem. As the most important consumption good over the life cycle, owner-occupied housing plays a significant role on individual's saving and investment decision. The portfolio constraint imposed by the demand for housing may also explain younger individuals' portfolio choice. Some literatures, such as Cocco (1998), incorporate the housing problem into the life cycle model.

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Appendix A

To construct the the household income, we include wages and salaries (X5702), practice/business/partnership/farm income (X5704), rent/trusts/royolties (X5714), unemployment or worker's compensation (X5716), child support or alimony (X5718), food stamps and welfare income (X5720) and other income (X5724).

Bonds and stocks are constructed as follows. All the acronym-variables are defined in the SAS program supplied by the SCF, which creates summary variables for SCF. Bonds consist of SAVING and MMA (savings and money market accounts), CDS (cerfiticates of deposit), TFBMUTF (tax free bond mutual funds), GBMUTF (government bond mutual funds), OBMUTF (other bond mutual funds), BOND (state, US government and corporate bonds), SAVBND (saving bonds) and COMUTF (combination and other mutual funds), for which we assume that half is invested to bonds. We also add ANNUIT (annuities) and TRUSTS (trusts), for which we count the full value if the individuals invest all in interest earning assets, while the percentage other than stock allocation if the individuals split the investment. Other bond investment includes bonds in IRA/KEOGH plans, bonds in account-type retirement plans and FUTPEN (other future pension benefits). We also subtract CCBAL (revolving credit card debt), OTHLOC (unsecured loans and loans secured by pensions) and other debt, which includes loans against pensions (X11027, X11127, X11327, X11427, X11527), loans against life insurance (X4010) and loans against margin loans (X3932).

Stocks are made up of STOCKS (directly held stocks), STMUTF (stock mutual funds), half of COMUTF (combination and other mutual funds), OMUTF (other non-bond mutual funds), PENEQ (thrift amounts invested in stock), ANNUIT (annuities) and TRUSTS (trusts) that are invested in stocks. Other stock investment includes stocks in IRA/KEOGH plans, stocks in account-type retirement plans and FUTPEN (other future pension benefits).

The total financial wealth is defined as the total of bonds, stocks, CHECKING (checking accounts) and CALL (call accounts).

Appendix B

$f(\cdot)$ denotes the p.d.f. of the variables. I, J, K, L are the numbers of interpolation nodes for labor income shocks and stock returns shocks η, ξ, ω, ϕ, respectively. The distributions of the shocks are summarized as,

$$\omega \xrightarrow{d} N(\mu_\omega, \sigma_\omega^2) \tag{20}$$

$$\phi \xrightarrow{d} N(\mu_\phi, \sigma_\phi^2) \tag{21}$$

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} \xrightarrow{d} N \left(\begin{array}{c} \mu_\eta \\ \mu_\xi \end{array} , \begin{array}{cc} \sigma_\eta^2 & \rho_{\xi\eta}\sigma_\xi\sigma_\eta \\ \rho_{\xi\eta}\sigma_\xi\sigma_\eta & \sigma_\xi^2 \end{array} \right) \tag{22}$$

For each individual i, at any time t and any state point $(X_{i,t}, z_{i,t})$, the expectation of the value function can be expressed as follows.

$$\begin{aligned} E_{\xi\eta\omega\phi}V_t(\xi, \eta, \omega, \phi, X_t, z_t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_t(\xi, \eta, \omega, \phi, X_t, z_t) f(\xi, \eta, \omega, \phi) d\xi d\eta d\omega d\phi \qquad (23) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_t(\xi, \eta, \omega, \phi, X_t, z_t) f(\eta|\xi) d\eta \cdot f(\xi) d\xi \cdot f(\omega) d\omega \cdot f(\phi) d\phi \end{aligned}$$

For the bivariate normal distribution, the conditional distribution for one of the variables, given the value for the other variable, is normally distributed. Therefore,

$$\eta|\xi \xrightarrow{d} N(\mu_{\eta|\xi}, \sigma_{\eta|\xi}^2) \tag{24}$$

where

$$\mu_{\eta|\xi} = \mu_\eta + \frac{\rho_{\xi\eta}\sigma_\eta}{\sigma_\xi}(\xi - \mu_\xi) \tag{25}$$

$$\sigma_{\eta|\xi}^2 = \sigma_\eta^2(1 - \rho_{\xi\eta}^2) \tag{26}$$

Following P.262 of Kenneth Judd, if a normal random variable X is distributed $N(\mu, \sigma^2)$, then the general Gauss-Hermite quadrature rule for expectation of X is as follows,

$$E(X) = \pi^{-1/2} \sum_{i=1}^{n} w_i \cdot (\sqrt{2}\sigma x_i + \mu)$$

Where w_i are the Gauss-Hermite quadrature weights, and x_i are the quadrature nodes, $i = 1, 2, ..., n$.

So, the functional approximation of Gauss-Hermite quadrature to η is

$$\begin{aligned} E_{\xi\eta\omega\phi} V_t(\xi, \eta, \omega, \phi, X_t, z_t) &= \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^{I} w_{i,\eta} V_t(\xi, \sqrt{2}\sigma_{\eta|\xi} + \mu_{\eta|\xi}, \omega, \phi, X_t, z_t) \\ &\quad \cdot f(\xi) d\xi \cdot f(\omega) d\omega \cdot f(\phi) d\phi \\ &= \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=1}^{I} w_{i,\eta} V_t(\xi, \sqrt{2}\sigma_\eta \sqrt{1-\rho_{\xi\eta}^2} \eta_i + \mu_\eta + \\ &\quad \frac{\rho_{\xi\eta}\sigma_\eta}{\sigma_\xi}(\xi - \mu_\xi), \omega, \phi, X_t, z_t) \cdot f(\xi) d\xi \cdot f(\omega) d\omega \cdot f(\phi) d\phi \end{aligned} \tag{27}$$

where $w_{i,\eta}$ is the weights, η_i is the nodes over $[-\infty, \infty]$

The procedure of discretizing ξ is to substitute $\sqrt{2}\sigma_\xi \xi_j + \mu_\xi$ for ξ, which yields the following,

$$\begin{aligned} & E_{\xi\eta\omega\phi} V_t(\xi, \eta, \omega, \phi, X_t, z_t) \\ = \; & \pi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=1}^{J} \sum_{i=1}^{I} w_{i,\eta} w_{j,\xi} V_t(\sqrt{2}\sigma_\eta (\sqrt{1-\rho_{\xi\eta}^2} \eta_j + \rho_{\xi\eta}\xi_j) + \mu_\eta, \omega, \phi, X_t, z_t) \cdot f(\omega) d\omega \cdot f(\phi) d\phi \end{aligned} \tag{28}$$

Again $w_{j,\xi}$ and ξ_j are the Gauss-Hermite quadrature weights and nodes, respetively. Proceeding with the integral over ω and ϕ,

$$\begin{aligned} & E_{\xi\eta\omega\phi} V_t(\xi, \eta, \omega, \phi, X_t, z_t) \\ = \; & \pi^{-2} \sum_{l=1}^{L} \sum_{k=1}^{K} \sum_{j=1}^{J} \sum_{i=1}^{I} w_{i,\eta} w_{j,\xi} w_{k,\omega} w_{l,\phi} \cdot V_t(\sqrt{2}\sigma_\eta (\sqrt{1-\rho_{\xi\eta}^2} \eta_j + \rho_{\xi\eta}\xi_j) + \mu_\eta, \sqrt{2}\sigma_\omega \omega_k \\ & + u_w, \sqrt{2}\sigma_\phi \phi_l + u_\phi; X_t, z_t) \end{aligned} \tag{29}$$

This model is expensive to compute. The number of nodes in the dynamic programming tree is $I \times J \times K \times L \times T \times S_1 \times S_2 \times S_3 \times 2$, where I, J, K, L are the nodes for different shocks as described above, T is the maximum age that individual can live up to, S_1, S_2 and S_3 are the numbers of nodes for three state variables separately. Taking 80 as the total life periods and 5 nodes for each shock, the number of nodes for cash on hand is 60, while the number of nodes for income and indexed wage is 10, separately. We have two situations in each optimization problem. One is with bond investment only, the other is with both riskless and risky allocation. Therefore, $I \times J \times K \times L \times T \times S_1^{\max} \times S_2^{\max} \times S_3^{\max} \times 2 = 600,000,000$.

Appendix C

Take the first-order condtion of Bellman equation

$$V_{i,t}(X_{i,t}, z_{i,t}) = \max_{C_{i,t},\alpha_{i,t}} \{U(C_{i,t}) + \beta [p_t \cdot E_t V_{i,t+1} (X_{i,t+1}, z_{i,t+1}) + b(1-p_t) \cdot E_t U(W_{i,t+1})]\} \tag{30}$$

Subject to

$$X_{i,t+1} = R_{i,t}(X_{i,t} - C_{i,t}) + Y^d_{i,t+1} \tag{31}$$

$$X_{i,t} = W_{i,t} + Y^d_{i,t} \tag{32}$$

We have

$$\frac{\partial V_{i,t}(X_{i,t}, z_{i,t})}{\partial C_{i,t}} \tag{33}$$

$$= U\prime(C_{i,t}) + \beta \{p_t \cdot E_t [V_{i,t+1}\prime (X_{i,t+1}, z_{i,t+1}) \cdot (-R_{i,t})] + b(1-p_t) \cdot E_t [U\prime(W_{i,t+1}) \cdot (-R_{i,t})]\} \tag{34}$$

Change the equation, we have

$$U\prime(C_{i,t}) = \beta \{p_t \cdot E_t[V_{i,t+1}\prime (X_{i,t+1}, z_{i,t+1}) \cdot R_{i,t}] + b(1-p_t) \cdot E_t [U\prime(W_{i,t+1}) \cdot R_{i,t}]\} \tag{35}$$

According to the Envelope Therorem, the partial derivative with respect to $X_{i,t}$ is

$$\frac{\partial V_{i,t}(X_{i,t}, z_{i,t})}{\partial X_{i,t}} = \beta \{p_t \cdot E_t [V_{i,t+1}\prime (X_{i,t+1}, z_{i,t+1}) \cdot R_{i,t}] + b(1-p_t) \cdot E_t[U\prime(W_{i,t+1}) \cdot R_{i,t}]\} \tag{36}$$

The right hand sides of Equations (34) and (35) are equal, which gives us

$$U\prime(C_{i,t}) = V_{i,t}\prime (X_{i,t}, z_{i,t}) \tag{37}$$

Thus, we rewrite Equation (34) as follows,

$$U\prime(C_{i,t}) = \beta p_t \cdot E_t [U\prime(C_{i,t+1}) \cdot R_{i,t}] + \beta b(1-p_t) \cdot E_t [U\prime(W_{i,t+1}) \cdot R_{i,t}] \tag{38}$$

$$C_{i,t} = U\prime^{-1} \{\beta p_t \cdot E_t [U\prime(C_{i,t+1}) \cdot R_{i,t}] + \beta b(1-p_t) \cdot E_t [U\prime(W_{i,t+1}) \cdot R_{i,t}]\} \tag{39}$$

We define the Euler error as

$$C_{i,t} \cdot (1+error) = U\prime^{-1} \{\beta p_t \cdot E_t [U\prime(C_{i,t+1}) \cdot R_{i,t}] + \beta b(1-p_t) \cdot E_t [U\prime(W_{i,t+1}) \cdot R_{i,t}]\} \tag{40}$$

$$error = \frac{U\prime^{-1}\left\{\beta p_t \cdot E_t\left[U\prime(C_{i,t+1}) \cdot R_{i,t}\right] + \beta b(1-p_t) \cdot E_t\left[U\prime(W_{i,t+1}) \cdot R_{i,t}\right]\right\} - C_{i,t}}{C_{i,t}} \tag{41}$$

We then take a $\log_{10}$ of the *error*, the acceptable range is around -3 or smaller.

Table 1: Benchmark Parameters

Description	Parameter Value
Start working age (t_0)	20
Retirement age (M)	65
Discount factor (β)	0.90
Coefficient of relative risk aversion (γ)	5
Riskless returns (R_f)	0.02
Mean risky returns ($R_f + \mu$)	0.06
Variance of transitory shocks (σ^2_ω)	0.01
Variance of permanent shocks (σ^2_u)	0.01
Standard deviation of stock returns (σ_η)	0.157
Correlation between stock returns and income shocks ($\rho_{\xi\eta}$)	0.15
Social security tax rate (τ_{SS})	0.124
Bequest intensity (b)	0
Marital status	single
Family size	1

Table 2: Labor Income Process

Coefficient of characteristic variables for labor income	No high school	High school	College
Constant	2.6275	2.7004	2.3831
Marital status	0.4008	0.4437	0.4831
Family size	-0.0176	-0.0236	-0.0228
Coefficient of age dummies for labor income			
Constant	-2.1361	-2.1700	-4.3148
Age	0.1684	0.1682	0.3194
$\text{Age}^2/10$	-0.0353	-0.0323	-0.0577
$\text{Age}^3/100$	0.0023	0.0020	0.0033

Figure 1:
No High School Group Stock Allocation from SCF2004

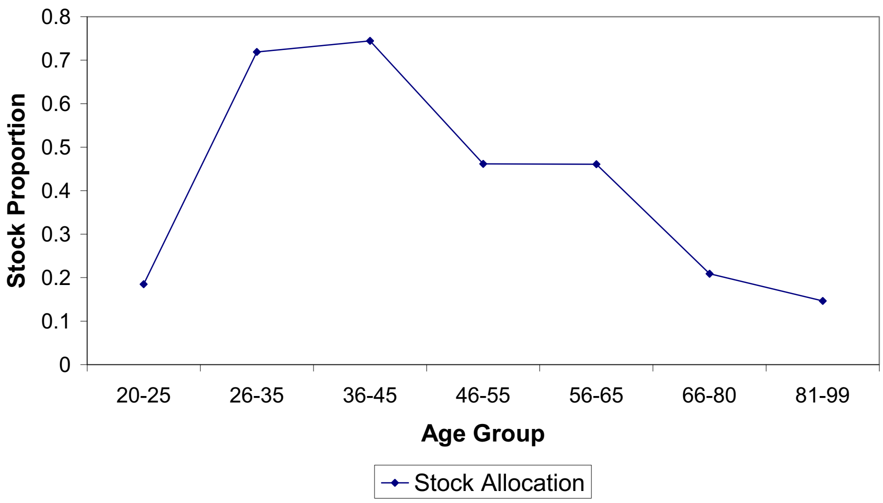

Figure 2:
High School Group Stock Allocation from SCF2004

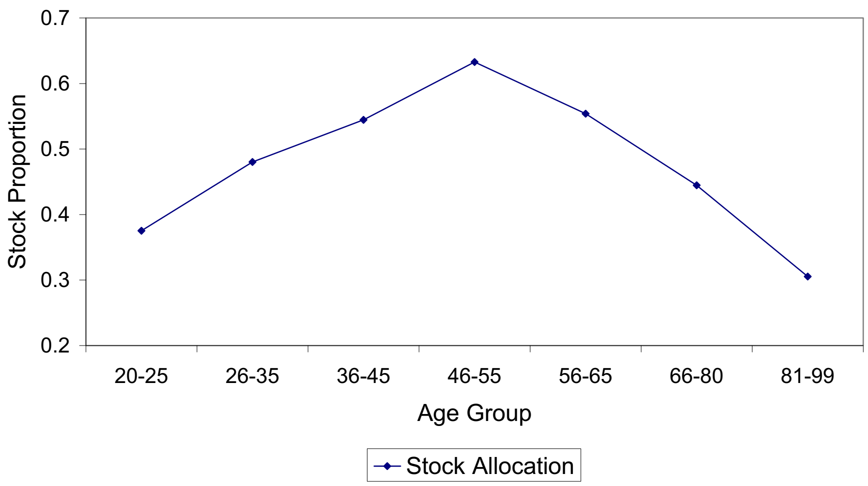

Figure 3:
College Group Stock Allocation from SCF 2004

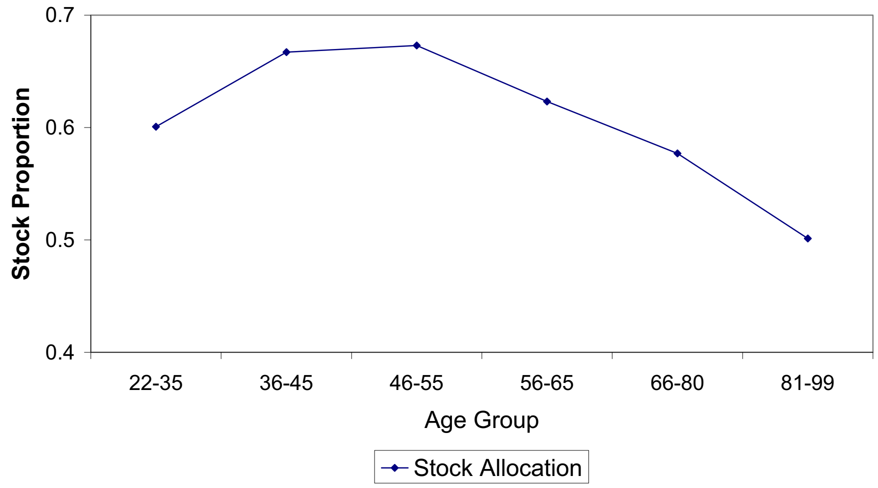

Figure 4:
High School Group Profiles of Consumption and Income

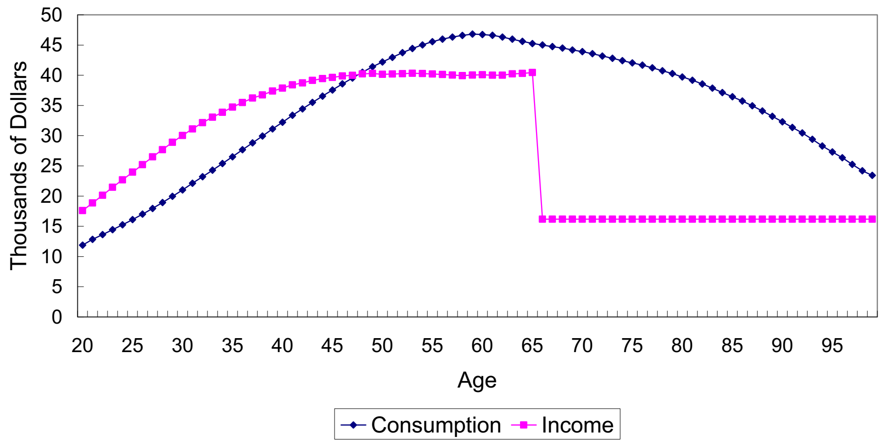

Figure 5: High School Group Wealth Profile

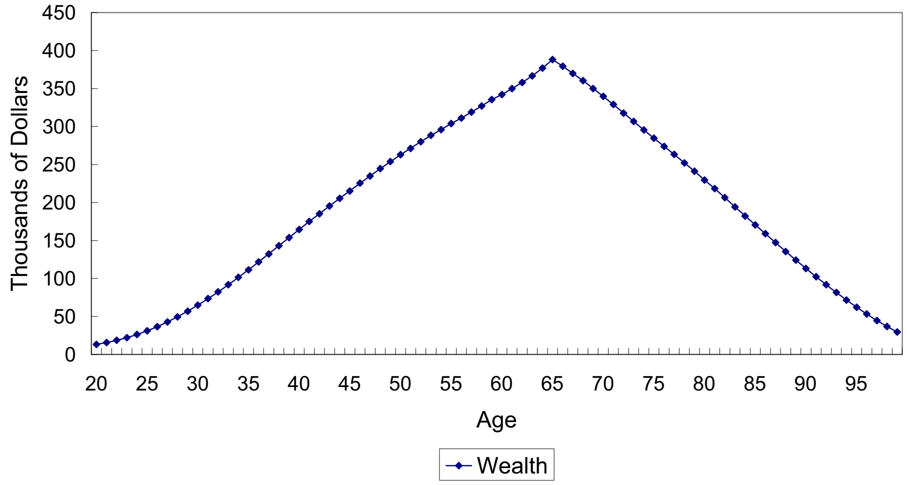

Figure 6: High School Group Stock Allocation from model

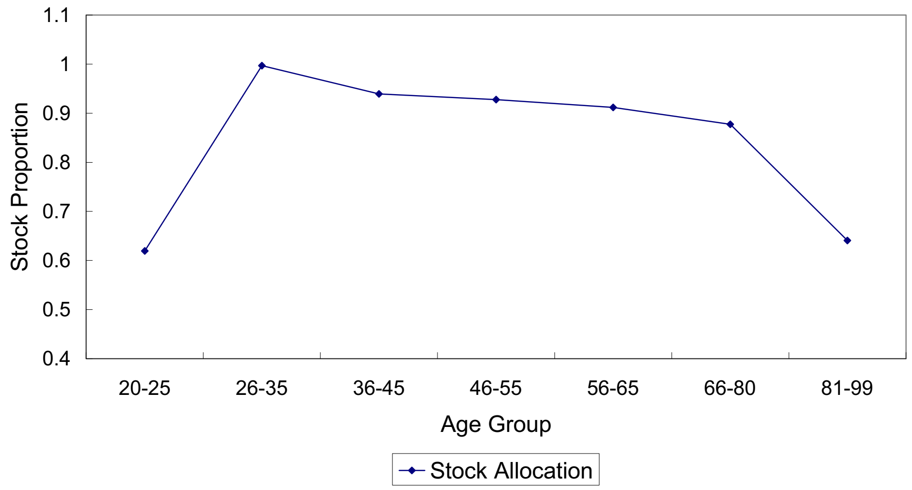

Figure 7:
No High School Group Stock Allocation from model

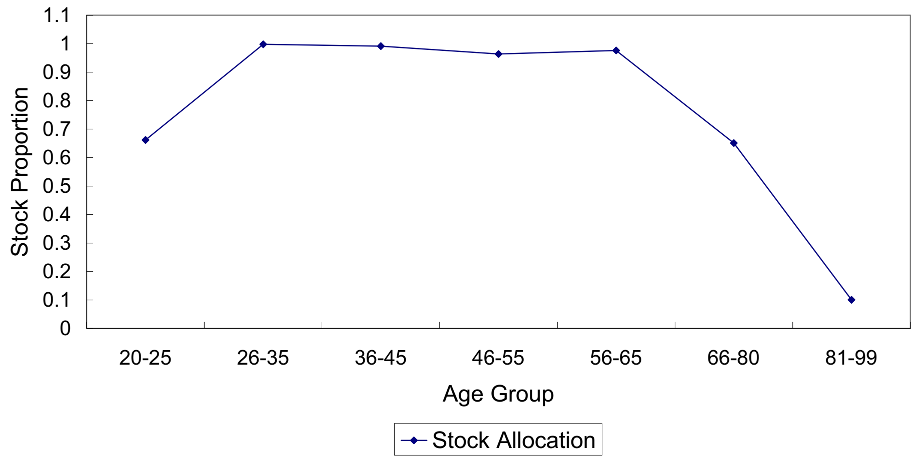

Figure 8: College Group Stock Allocation from model