Actuarially Consistent Valuation of Catastrophe Derivatives

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Abstract

In this article, we investigate the valuation of insurance derivatives which facilitate the trading of insurance risks on capital markets, such as catastrophe derivatives that were traded at the Chicago Board of Trade. These instruments have to be priced relative to observed insurance premiums that are written on the same underlying risks to exclude any arbitrage opportunities. We derive a representation of those catastrophe derivative price processes that are *actuarially consistent* and determine several situations in which *actuarial consistency* leads to a complete market for catastrophe derivatives.

JEL Classification: G12, G13

Key Words: Insurance derivatives; securitization; actuarially consistent pricing; Fourier transform; incomplete markets; compound Poisson process; Lévy process.

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1 INTRODUCTION

The insurance and reinsurance industry has become increasingly concerned about the concentration of exposures linked to a single event such as a natural catastrophe or a terrorist attack. Those industries traditionally provided the only vehicle in the private sector for spreading risks of such magnitudes across a society. In 1992, Hurricane Andrew caused $16 billion in insured losses and more than sixty insurance companies became insolvent. In need for alternative means of risk spreading that would add capacity to the market dealing with such extreme events, the private sector has been exploring the potential inherent in risk securitization. Consequentially, financial products emerged over the last decade that capture insurance related risks, such as catastrophe derivatives that were introduced at the Chicago Board of Trade (CBoT) in 1993. These exchange-traded financial derivatives were based on underlying indexes that encompass insured property losses due to natural catastrophes. Insurance futures and options on insurance futures were the first contracts of catastrophe derivatives traded at the CBoT. Due to lack of trading activity, they were replaced in 1995 by catastrophe spread options based on underlying loss indexes which are not traded themselves and provided by Property Claim Services (PCS). We refer to D'Arcy and France (1992) for a detailed description of insurance futures and to O'Brien (1997) for catastrophe spread options.

The idea behind trading insurance related risk additionally on capital markets is twofold. It attempts at attracting new capital linked to natural catastrophic risk from investors for whom those derivatives provide an excellent opportunity for portfolio diversification. Additionally, it allows insurance and reinsurance companies to dynamically adjust their exposure to natural catastrophic risk through hedging with those standardized financial contracts at low transaction costs.

The introduction of insurance derivatives raises the question of how those financial contracts are valued in relation to existing insurance contracts. In an established insurance market, information about the underlying natural catastrophic risk is reflected in insurance premiums. As the same risk underlies insurance derivatives, their prices must be linked to premiums of existing insurance contracts to exclude any arbitrage opportunity that may arise from trading in both insurance and financial contracts. The idea of this article is to determine price processes of catastrophe derivatives that are *actuarially consistent* with existing insurance premiums.

Cummins and Geman (1995) examined the valuation of insurance futures and options on insurance futures—the first generation of catastrophe derivatives traded at the CBoT. The authors model the increments of the underlying index as a geometric Brownian motion plus a jump process that is assumed to be a Poisson process with constant jump size. Since the futures' price, the underlying asset for those insurance derivatives, is traded on the market and the jump size is assumed to be constant, the market is complete and unique pricing is possible solely based on assuming absence of arbitrage opportunities. While the completeness of the market is convenient, the assumption of constant loss sizes is questionable in the context of natural catastrophic risk.

Aase (1999) takes a different, more realistic modeling approach. He models the dynamics of the underlying loss index as a compound Poisson process with random jump sizes and investigates the valuation of insurance futures and options on insurance futures. The market in this framework is incomplete due to the randomness in jump sizes. The author specifies the preferences of market participants by a utility function and determines unique price processes within the framework of partial equilibrium theory under uncertainty. Closed pricing formulae are derived under the assumption of negative exponential utility function and Gamma distributed loss sizes. Embrechts and Meister (1997) generalize Aase's analysis by allowing for mixed compound Poisson processes with stochastic frequency rate.

Geman and Yor (1997) investigate the valuation of the second generation of catastrophe derivatives, catastrophe options, that are based on a non-traded underlying PCS loss index, which as in Cummins and Geman (1995) is modeled as a geometric Brownian motion plus a Poisson process with constant jump sizes. Market incompleteness in this setup arises out of the fact that the underlying index is not traded on the market. The authors base their arbitrage arguments on the existence of a vast class of layers of reinsurance with different attachment points to guarantee completeness of the catastrophe derivative market. An Asian options approach is used to obtain semi-analytical solutions for call option prices in form of their Laplace transform.

In this article, we contribute to the literature mentioned above by deriving the link between insurance premiums and insurance derivative prices. We follow the modeling approach of Aase (1999) and examine the pricing of catastrophe derivatives based on a non-traded loss index as in Geman and Yor (1997). Similar to options that are priced relative to their underlying stock to exclude arbitrage opportunities, insurance derivatives must be valued relative to insurance contracts that indirectly allow trading in the underlying loss index. Without specifying investors' preferences, we give a representation of the set of no-arbitrage catastrophe derivative prices that are *actuarially consistent* with observed insurance premiums. We then show that actuarial consistency leads to unique price processes in case of constant jump sizes–as in Cummins and Geman (1995) and Geman and Yor (1997)–risk neutrality with respect to jump sizes–as in Aase (1992,1993)–or negative exponential utility functions representing investors' preferences–as in Aase (1999). In these situations, actuarial consistency thus "completes" the market for catastrophe derivatives whose underlying assets are not traded directly on the capital market. Fourier analysis proves to be crucial for the aim of this article as it allows the separation of prices into two components, one accounting for the underlying stochastic risk structure which underlies both the insurance and financial contract and the other representing the specific contractual structure. It furthermore allows deducing the inverse Fourier transform of those prices in closed form and extending the analysis to general Lévy processes.

As insurance premiums are the reference point relative to which we price catastrophe derivatives, this article relates to and is based upon the literature on actuarial pricing in a dynamic economy. Delbaen and Haezendonck (1989) and Sondermann (1991) assume that the liabilities of an insurance company are tradable and derive insurance premiums based solely on the exclusion of arbitrage opportunities. Aase (1992, 1993) determines equilibrium insurance premiums in the framework of a dynamic pure risk exchange economy. We follow the preference-free valuation approach of Delbaen and Haezendonck (1989) and Sondermann (1991) to derive the representation of actuarially consistent catastrophe derivative prices and, thereafter, apply our result to the equilibrium framework of Aase (1992, 1993).

Embrechts (2000) concludes in his excellent overview on actuarial and financial pricing of insurance related risk:

"As always, a bridge can be walked in two directions. I very much hope that, besides the existence of a financial bridge to actuarial pricing my summary will also have indicated that there is something like an actuarial bridge to financial pricing."

By determining no-arbitrage catastrophe derivative prices that are anchored in insurance premiums we intend to contribute to building the latter part of the bridge.

The remainder of the article is organized as follows. In Sections 2 we introduce the model for the underlying risk process and the contracts that are available in the insurance and capital market. Section 3 is the core of the article and derives a representation of insurance derivative prices that are *actuarially consistent* with existing insurance premiums. In Section 4 we derive unique catastrophe derivative price processes in case of constant loss sizes, risk neutrality with respect to loss sizes, and in the framework of partial equilibrium analysis under uncertainty with negative exponential utility functions of investors. We conclude in Section 5.

2 THE MARKET

In this section, we introduce the model for the market dealing with natural catastrophic risk. This includes the stochastic structure of the underlying risk, the specifications of the insurance and financial contracts, and their respective price processes.

2.1 Risk Process

Uncertainty in the market is described by a probability space $(\Omega, \mathcal{F}, P)$ on which random variables will be defined. We assume that the economy is of finite horizon $T < \infty$ and the flow of information is modeled by a non-decreasing family of σ-algebras $(\mathcal{F}_t)_{0\leq t\leq T}$, a filtration. We assume that $\mathcal{F}_T = \mathcal{F}$, each $\mathcal{F}_t$ contains the events in $\mathcal{F}$ that are of P-measure zero, and the filtration is right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+}$ where $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$.

The risk faced by the insurance and reinsurance industry is inherent in their exposure to accumulated insured property losses. As natural catastrophes cause claims of unpredictable magnitude, we follow the approach of Aase (1999) and assume that the process of accumulated insured property losses follows a compound Poisson process $X = (X_t)_{0 \leq t \leq T}$. The random variable X_t thus represents the sum of insured property losses to the industry incurred in $(0, t]$, i.e.

$$X_t = \sum_{\{k | T_k \leq t\}} Y_k \;=\; \sum_{k=1}^{N_t} Y_k, \tag{1}$$

where T_k is the random time point of occurrence of the k^{th} catastrophe that causes a corresponding insured property loss of size Y_k, and N_t is a random variable counting catastrophes up to and including time t. We assume that the counting process $N = (N_t)_{0 \leq t \leq T}$ is a Poisson process with intensity λ, and $Y_1, Y_2, ...$ are nonnegative, independent and identically distributed random variables, all independent of the counting process N. Let G be the distribution function of Y_k with support $[0, \infty)$. The parameters $(\lambda, dG(y))$ are called the characteristics of the process X.

Under these assumptions, X is thus a time-homogeneous process with independent increments. Actuarial studies (see Levi and Partrat, 1991) have shown that these assumptions are reasonable in the context of losses arising from windstorm, hail and flood. Earthquakes are described as events arising from a superposition of events caused by several independent sources. Accumulated insured losses due to earthquakes can therefore be approximated by a compound Poisson process. The assumption on time-homogeneity is questionable for the case of hurricanes which occur seasonally. In the context of catastrophe derivatives, however, contracts linked to regions, that are exposed to hurricane risk, all track quarterly loss periods to account for seasonal effects.

We assume that the past evolution and current state of the risk process X is observable by every agent in the economy, i.e. X is assumed to be adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$. We thus exclude any effects that asymmetric information may have on the market through moral hazard or adverse selection. For simplicity, it is assumed that X generates the flow of information, i.e. $\mathcal{F}_t = \sigma(\sigma(X_s,\, s \leq t) \cup \mathcal{N})$ where $\mathcal{N}$ denotes the events of P-measure zero.

2.2 Equivalent Probability Measures

Changing the probability measure plays a central role in the context of no-arbitrage valuation of contracts as their discounted price processes follow martingales under the appropriate probability measure. For compound Poisson processes, Delbaen and Haezendonck (1989) characterized the set of probability measures Q on $(\Omega, \mathcal{F})$ that are equivalent to the "reference" measure P and that preserve the structure of the underlying risk process X, i.e. such that X is a compound Poisson process under the new probability measure Q. This set can be parameterized by a pair $(\kappa, v(\cdot))$ where κ is a non-negative constant κ and $v : \mathbb{R}_+ \to \mathbb{R}$ is a non-negative, measurable function with $\mathbf{E}^P[v(Y_1)] = 1$. The density process $\xi_t = \mathbf{E}^P[\xi_T \mid \mathcal{F}_t]$ of the Radon-Nikodym derivative $\xi_T = \frac{dQ}{dP}$ is then given by

$$\xi_t = \exp\left(\sum_{k=1}^{N_t} \ln(\kappa v(Y_k)) + \lambda(1-\kappa)t\right), \tag{2}$$

for any $0 \leq t \leq T$. Let $P^{\kappa,v}$ denote the equivalent probability measure that corresponds to the constant κ and the function $v(\cdot)$. Under the new measure $P^{\kappa,v}$ the process X is then a compound Poisson process with characteristics $(\lambda\kappa, v(y)\,dG(y))$.

Aase (1992, 1993) interprets the Radon-Nikodym derivative (2) as the marginal disutility of the market which endogenously arises in equilibrium of a dynamic pure risk exchange economy. He therefore calls $v(\cdot)$ the market's marginal disutiltiy of loss size risk and κ the market's attitude towards frequency risk.

2.3 The Contracts

We assume the existence of an insurance market which is in equilibrium and specifies a premium process $p = (p_t)_{0 \leq t \leq T}$ for the industry's overall insured losses $X = (X_t)_{0 \leq t \leq T}$. A European-style catastrophe derivative with maturity T is then introduced to the market which is written on the same underlying risk process X. The payoff of the catastrophe derivative thus depends on the realization of X_T only and specifies a price process $\pi = (\pi_t)_{0 \leq t \leq T}$. We assume that the equilibrium of the insurance market and therewith the premium process is not altered by the introduction of the catastrophe derivative. By assuming the same risk process underlying both contracts, we do not allow for any other stochastic factor that would influence one but not the other contract, i.e. we do not consider basis risk.

2.3.1 The Insurance Contract

We consider the setup of Delbaen and Haezendonck (1989) in which a reinsurance contract allows the insurance company to sell off the risk $X_T - X_t$ of the remaining period $(t, T]$ for a premium p_t. The premium process $p = (p_t)_{0 \leq t \leq T}$ is a stochastic process that is assumed to be predictable, i.e. it is adapted to $(\mathcal{F}_{t-})_{0 \leq t \leq T}$, where $\mathcal{F}_{t-} = \bigvee_{s<t} \mathcal{F}_s$.

Remark 2.1 *Sondermann (1991) considers dynamic reinsurance policies, i.e. the insurance company can decide to sell off a certain fraction of their risk and adjust their decision continuously. If the insurance company is allowed to only adjust at finitely many times this approach can be embedded in the framework of Delbaen and Haezendonck (1989) by defining the maturities of subsequent contracts accordingly.*

A trading strategy in this setup means the possibility of 'take-over' and the company's liabilities $(X_t + p_t)_{0 \leq t \leq T}$ thus represent the underlying price process. According to the fundamental theorem of asset pricing, the absence of arbitrage strategies implies the existence of a probability measure $P^{\kappa,v}$ that is equivalent to the "reference" measure P and under which the price process $(X_t + p_t)_{0 \leq t \leq T}$ follows a martingale. If one further assumes that the predictable process $p = (p_t)_{0 \leq t \leq T}$ under $P^{\kappa,v}$ is linear, i.e. of the form

$$p_t = p\left(P^{\kappa,v}\right) \cdot (T - t) \tag{3}$$

for some premium lump sum $p\left(P^{\kappa,v}\right)$. Delbaen and Haezendonck (1989) conclude that the existence of sufficiently many reinsurance markets implies that the risk process X under $P^{\kappa,v}$ is still a compound Poisson process. As the risk process X has stationary and independent increments, the martingale property implies that the premium process is of the form

$$\begin{aligned} p_t &= \mathbf{E}^{P^{\kappa,v}}\left[X_1\right](T-t) \\ &= \mathbf{E}^{P^{\kappa,v}}\left[N_1\right]\mathbf{E}^{P^{\kappa,v}}\left[Y_1\right](T-t) \\ &= \lambda\kappa \cdot \mathbf{E}^{P}\left[Y_1 \cdot v\left(Y_1\right)\right](T-t). \end{aligned} \tag{4}$$

From this representation, we conclude that there are infinitely many market prices of risk $(\kappa, v(\cdot))$ and therefore equivalent probability measures that lead to the same premium process $p = (p_t)_{0 \le t \le T}$.

Remark 2.2 *Delbaen and Haezendonck (1989) show that representation* (4) *includes commonly used premium calculation principles as certain loading factors can be generated by choosing the equivalent probability measure accordingly.*

2.3.2 The Catastrophe Derivative

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function that specifies the payoff at maturity to the buyer of the catastrophe derivative, i.e. at T the buyer receives $\phi(X_T)$. The price process $\pi = (\pi_t)_{0 \le t \le T}$ defines at any time t the price π_t the buyer has to pay to enter into the financial contract. In the absence of arbitrage strategies, the fundamental theorem of asset pricing implies that the price process $\pi = (\pi_t)_{0 \le t \le T}$ is a martingale under an equivalent probability measure $P^{\kappa,v}$. It can therefore be represented as

$$\pi_t^{\kappa,v} = \mathbf{E}^{P^{\kappa,v}} \left[\phi(X_T) \, | \mathcal{F}_t \right] \tag{5}$$

for all $0 \le t \le T$. As the underlying risk process X is a Markov process and generates the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ π_t is of the form

$$\pi_t^{\kappa,v} = f^{\kappa,v}(X_t, t) = \mathbf{E}^{P^{\kappa,v}} \left[\phi(X_T) \, | X_t \right] \tag{6}$$

for some measurable function $f^{\kappa,v}$ with $f^{\kappa,v}(X_T, T) = \phi(X_T)$.

In the following section—the core of the article—we derive a representation of those price processes $(f^{\kappa,v}(X_t, t))_{0 \le t \le T}$ that are consistent with the observed premium process $(p_t)_{0 \le t \le T}$.

3 ACTUARIALLY CONSISTENT PRICING

The market for catastrophe derivatives is incomplete for two reasons: the underlying asset X is not traded itself in the market and it exhibits unpredictable movements through jumps of random size. It is thus not possible to determine a unique price process for catastrophe derivatives purely based on the exclusion of arbitrage opportunities. The contribution of this article is to consider the fact that the underlying asset X is indirectly traded through insurance contracts. Catastrophe derivatives must therefore be priced relative to insurance premiums to exclude arbitrage opportunities. In different words, the market prices of frequency and severity risk $(\kappa, v(\cdot))$ reflected by the price process of a catastrophe derivative must be consistent with observed premium process. We call this requirement on financial prices *actuarial consistency* based on which we will derive a link between the insurance premium

$$p_t = \mathbf{E}^{P^{\kappa,v}}[X_1](T-t) \tag{7}$$

and the financial price

$$f^{\kappa,v}(X_t,t) = \mathbf{E}^{P^{\kappa,v}}[\phi(X_T)|X_t]. \tag{8}$$

The same argument applies for example to a European call option that is written on a stock and hence priced relative to the observed stock price. The difference to our setup is that the underlying asset X is not traded directly on the capital market. It is therefore inevitable to extract the common underlying risk process X from the specifications of both contracts. In the following Theorem, we apply Fourier analysis to the representation (8) which allows us to carry out this extraction and derive a representation of those catastrophe derivative price processes that are actuarially consistent.

Theorem 3.1 *Let $X = (X_t)_{0\le t\le T}$ be a compound Poisson process with characteristics $(\lambda, dG(y))$ and let $(p_t)_{0\le t\le T}$ be the linear premium process specified in the insurance market and defined by (7). Suppose the function $\phi : \mathbb{R} \to \mathbb{R}$ specifies the payoff of the financial contract at time T and satisfies the integrability condition $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$ for some $k \in \mathbb{R}$. Then, for a given market disutility of severity risk $v : \mathbb{R}_+ \to \mathbb{R}$ with $\mathbf{E}^P[v(Y_1)] = 1$, the function $f^v : \mathbb{R}_+ \times \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ defining the financial price process $(f^v(p_t, X_t, t))_{0\le t\le T}$ that excludes arbitrage strategies and is actuarially consistent with the premium process can be represented as*

$$f^v(p_t, X_t, t) = \int_{-\infty}^{\infty} \exp\left(iuX_t + \frac{\mathbf{E}^P\left[\left(e^{iuY_1}-1\right)\cdot v(Y_1)\right]}{\mathbf{E}^P[Y_1 \cdot v(Y_1)]} \cdot p_t\right) \check{\varphi}(u)\, du + k, \tag{9}$$

where $\check{\varphi}(\cdot)$ *is the inverse Fourier transform of* $\phi(\cdot) - k$*.*

Proof. There is no generally agreed definition of the Fourier transform. We follow Davies (2002) and define the Fourier transform of a function $f \in \mathbf{L}^2(\mathbb{R})$ as

$$F(f)(u) = \int_{-\infty}^{\infty} e^{iuz} f(z)\, dz. \tag{10}$$

The Fourier transform is a one-to-one mapping of $\mathbf{L}^2(\mathbb{R})$ onto itself and the Inversion Formula states that the identity

$$f \equiv F\left(F^{-1}(f)\right) \tag{11}$$

holds for the inverse Fourier transform

$$F^{-1}(f)(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz} f(z)\, dz. \tag{12}$$

Applying the identity (11) to the function $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$ yields

$$\phi(x) - k = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iux} e^{-iuz} (\phi(z) - k)\, dz du. \tag{13}$$

For the financial price (8) we thus deduce

$$\begin{aligned}
f^{\kappa,v}(X_t, t) &= \mathbf{E}^{P^{\kappa,v}}\left[\phi(X_T) \,|X_t\right] \\
&= \mathbf{E}^{P^{\kappa,v}}\left[\phi(X_T) - k \,|X_t\right] + k \\
&= \frac{1}{2\pi}\mathbf{E}^{P^{\kappa,v}}\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{iuX_T} e^{-iuz} (\phi(z) - k)\, dz du \,|X_t\right] + k \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{E}^{P^{\kappa,v}}\left[e^{iuX_T} \,|X_t\right] e^{-iuz} (\phi(z) - k)\, dz du + k \\
&= \int_{-\infty}^{\infty} \mathbf{E}^{P^{\kappa,v}}\left[e^{iuX_T} \,|X_t\right] \check{\varphi}(u)\, du + k,
\end{aligned}$$

where we applied Fubini's theorem and $\check{\varphi}(\cdot)$ denotes the inverse Fourier transform of $\phi(\cdot) - k$, i.e.

$$\check{\varphi}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz} (\phi(z) - k) \, dz. \tag{14}$$

Since a compound Poisson process is a stochastic process with stationary and independent increments, we have

$$\begin{aligned} \mathbf{E}^{P^{\kappa,v}} \left[e^{iuX_T} | X_t \right] &= e^{iuX_t} \mathbf{E}^{P^{\kappa,v}} \left[e^{iu(X_T - X_t)} | X_t \right] \\ &= e^{iuX_t} \mathbf{E}^{P^{\kappa,v}} \left[e^{iuX_{T-t}} \right]. \end{aligned} \tag{15}$$

$\mathbf{E}^{P^{\kappa,v}} \left[e^{iuX_{T-t}} \right]$ is the characteristic function of the random variable X_{T-t} under the probability measure $P^{\kappa,v}$ and given by

$$\chi_{T-t}^{\kappa,v}(u) = \exp\left(\lambda\kappa \cdot \mathbf{E}^{P} \left[\left(e^{iuY_1} - 1 \right) v(Y_1) \right] \cdot (T - t) \right) \tag{16}$$

(see Karlin and Taylor (1981) p.428). The price at time t of the catastrophe derivative can therefore be represented as

$$f^{\kappa,v}(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t} \chi_{T-t}^{\kappa,v}(u) \, \check{\varphi}(u) \, du + k. \tag{17}$$

Actuarial consistency requires that the market prices of risk $(\kappa, v(\cdot))$ characterizing the price process $(f^{\kappa,v}(X_t, t))_{0 \le t \le T}$ is consistent with the observed premium process $(p_t)_{0 \le t \le T}$, i.e. satisfy

$$p_t = \lambda\kappa \cdot \mathbf{E}^{P} [Y_1 \cdot v(Y_1)] \cdot (T - t).$$

Substituting this expression into the characteristic function (16) yields

$$\chi_{T-t}^{v}(u) = \exp\left(\frac{\mathbf{E}^{P} \left[\left(e^{iuY_1} - 1 \right) \cdot v(Y_1) \right]}{\mathbf{E}^{P} [Y_1 \cdot v(Y_1)]} \cdot p_t \right) \tag{18}$$

and (17) implies

$$f^v(p_t, X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t} \exp\left(\frac{\mathbf{E}^P\left[\left(e^{iuY_1} - 1\right) \cdot v(Y_1)\right]}{\mathbf{E}^P\left[Y_1 \cdot v(Y_1)\right]} \cdot p_t \right) \check{\varphi}(u)\, du + k.$$

■

Representation (9) of actuarially consistent price processes enables us to derive the inverse Fourier transform of the price process in closed form. For a given value of the underlying risk process $X_t = x$ and premium process $p_t = p$, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} \left(f^v(p, x, t) - k\right) dx = \chi_{T-t}^v(u) \cdot \check{\varphi}(u) \tag{19}$$

where $\chi_{T-t}^v(\cdot)$ is given by (18).

The inverse Fourier transform is thus the product of two factors where the first—the characteristic function—contains the stochastic structure that is common to both contracts and the second solely depends on the specific payoff function. The characteristic function is thus the important component in linking financial prices with insurance premiums under the concept of actuarial consistency.

Catastrophe Spread Options The inverse Fourier transform $\check{\varphi}(\cdot)$ in (14) can be derived explicitly for catastrophe call and put spread options that were traded at the CBoT after 1995. We introduced the constant k to enlarge the set of financial contracts such that our representation (9) can be applied to call spread options.

- A call spread option is a capped call option and can be created by buying a call option with strike price K_1, and selling at the same time a call option with the same maturity but with strike price $K_2 > K_1$. The payoff function $\phi_{CS}(x)$ depends on the realization x of the underlying random variable X_T in the following way

$$\phi_{CS}(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq K_1 \\ x - K_1 & \text{if } K_1 < x \leq K_2 \\ K_2 - K_1 & \text{if } x > K_2. \end{cases} \tag{20}$$

As $X_T \geq 0$ it is sufficient that $(\phi_{CS}(\cdot) - k) \cdot \mathbf{1}_{[0,\infty)}(\cdot) \in \mathbf{L}^2(\mathbb{R})$ for some $k \in \mathbb{R}$ where $\mathbf{1}_A(\cdot)$ denotes the indicator function on a Borel set A. The integrability condition $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R}_+)$ is satisfied for $k = K_2 - K_1$ and the inverse Fourier transform is given by

$$\begin{aligned} \check{\varphi}_{CS}(u) &= \frac{1}{2\pi}\int_0^\infty e^{-iux}\left(\phi_{CS}(x) - (K_2 - K_1)\right)dx \\ &= \frac{1}{2\pi}\frac{1}{u^2}\left(e^{-iuK_2} - e^{-iuK_1} + iu(K_2 - K_1)\right). \end{aligned} \tag{21}$$

- A put spread option is a capped put option with payoff function ϕ_{PS} defined by

$$\phi_{PS}(x) = \begin{cases} K_2 - K_1 & \text{if } 0 \leq x \leq K_1 \\ K_2 - x & \text{if } K_1 < x \leq K_2 \\ 0 & \text{if } x > K_2. \end{cases} \tag{22}$$

We observe that $\phi_{PS}(\cdot) \cdot \mathbf{1}_{[0,\infty)}(\cdot) \in \mathbf{L}^2(\mathbb{R})$ and $\phi_{PS}(x) = -(\phi_{CS}(x) - (K_2 - K_1))$. Therefore

$$\begin{aligned} \check{\varphi}_{PS}(u) &= -\check{\varphi}_{CS}(u) \\ &= -\frac{1}{2\pi}\frac{1}{u^2}\left(e^{-iuK_2} - e^{-iuK_1} + iu(K_2 - K_1)\right). \end{aligned} \tag{23}$$

We also note that the put-call parity is satisfied as

$$\begin{aligned} f^v_{PS}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{u^2}\chi^v_{T-t}(u)\,e^{iux}\left(e^{-iuK_1} - e^{-iuK_2} - iu(K_2 - K_1)\right)du \\ &= -f^v_{CS}(x,t) + K_2 - K_1. \end{aligned} \tag{24}$$

Stationarity and independence of increments of X were the only assumptions on the stochastic structure of X that were necessary in equation (15) to derive representation (17). The following Corollary therefore extends our analysis to Lévy processes.

Corollary 3.2 *Let $X = (X_t)_{0 \le t \le T}$ be a Lévy process with characteristic function $\chi_t(u) = \mathbf{E}^P\left[e^{iuX_t}\right]$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\phi(\cdot) - k \in \mathbf{L}^2(\mathbb{R})$ for some $k \in \mathbb{R}$. Then the conditional expected value*

$$f(X_t, t) = \mathbf{E}^P[\phi(X_T)|X_t]$$

admits the representation

$$f(X_t, t) = \int_{-\infty}^{\infty} e^{iuX_t} \chi_{T-t}(u) \check{\varphi}(u)\, du + k,$$

where $\check{\varphi}(\cdot)$ is the inverse Fourier transform of $\phi(\cdot) - k$, i.e.

$$\check{\varphi}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz} (\phi(z) - k)\, dz.$$

Proof. analogously to the proof of Theorem 3.1. ■

Remark 3.3 *The classical Lévy-Khinchine theorem (see e.g. Rogers and Williams (2000) Section VI.2) provides a representation of the characteristic function $\chi_t(\cdot)$ of the form*

$$\chi_t(u) = e^{t\psi(u)}$$

with the cumulant characteristic function

$$\begin{aligned} \psi(u) &= i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\{|z| \ge 1\}} \left(e^{iuz} - 1\right) \nu(dz) \\ &\quad + \int_{\{|z|<1\}} \left(e^{iuz} - 1 - iuz\right) \nu(dz), \end{aligned}$$

where $\mu \in \mathbb{R}$, $\sigma \geq 0$, *and* ν *is a measure on* $R \setminus \{0\}$ *such that*

$$\int_{\mathbb{R}\setminus\{0\}} \min\left(z^2, 1\right) \nu\left(dz\right) < \infty.$$

4 EXAMPLES OF COMPLETE MARKETS

In general, catastrophe derivative prices under the requirement of actuarial consistency can not be determined uniquely. The indeterminacy derives from the fact that the underlying asset X exhibits jumps of random size which implies that there are many market prices of risk $(\kappa, v(\cdot))$ that lead to the same premium process p. In this section, we examine three situations in which this indeterminacy vanishes, constant loss sizes–as in Cummins and Geman (1995) and Geman and Yor (1997)–risk neutrality with respect to loss sizes–as in Aase (1992,1993)–and negative exponential utility functions representing investors' preferences–as in Aase (1999).

4.1 Constant Loss Size

If the loss size is constant, i.e. $Y_k = y$ for all k, the market price of claim size risk must equal 1, i.e. $v \equiv 1$. Then there exists a unique no-arbitrage, actuarially consistent price process $\left(f\left(X_t, t\right)\right)_{0 \leq t \leq T}$ which can be represented by

$$f\left(p_t, X_t, t\right) = \int_{-\infty}^{\infty} \exp\left(iuX_t + \frac{e^{iuy} - 1}{y} \cdot p_t\right) \check{\varphi}\left(u\right) du + k. \tag{25}$$

The market price for frequency risk κ is then given by

$$\kappa = \frac{1}{\lambda y} \cdot \frac{p_t}{T - t}. \tag{26}$$

4.2 Risk Neutrality with respect to Loss Size

Aase (1992, 1993) considers the situation in which agents are risk neutral with respect to loss sizes, i.e. $v \equiv 1$. In this case, the unique actuarially consistent price process $\left(f\left(X_t, t\right)\right)_{0 \leq t \leq T}$ can be represented by

$$f\left(p_t, X_t, t\right) = \int_{-\infty}^{\infty} \exp\left(iuX_t + \frac{\mathbf{E}^P\left[e^{iuY_1} - 1\right]}{\mathbf{E}^P\left[Y_1\right]} \cdot p_t\right) \check{\varphi}\left(u\right) du + k, \tag{27}$$

and the market price for frequency risk κ is given by

$$\kappa = \frac{1}{\lambda \mathbf{E}^P\left[Y_1\right]} \cdot \frac{p_t}{T-t}. \tag{28}$$

4.3 Representative Agent

Aase (1999) presents a partial equilibrium model under uncertainty for valuing insurance futures and options on insurance futures which were traded at the CBoT until 1995. In his model, investors's preferences are represented by expected utility maximization with negative exponential Bernoulli utility functions

$$u^{i\prime}\left(x, t\right) = e^{-\alpha^i x - \rho^i t}, \tag{29}$$

where $\alpha^i > 0$ is the coefficient of absolute risk aversion and $\rho^i > 0$ the time impatience rate of agent i. Under those Bernoulli utility functions, a Pareto efficient outcome can be achieved and is characterized by a linear risk-sharing rule where every investor holds a fraction of the aggregate risk. Furthermore, there exists a representative agent in the market with Bernoulli utility function

$$u'\left(x, t\right) = e^{-\alpha x - \rho t}, \tag{30}$$

where $\alpha > 0$ is the coefficient of absolute risk aversion and $\rho > 0$ the time impatience rate in the market.

In this framework, Aase (1999) shows that the market prices of frequency and severity risk $\left(\kappa, v\left(\cdot\right)\right)$ satisfy

$$\kappa v\left(y\right) = e^{\alpha y}, \tag{31}$$

for all $y \geq 0$ and $\mathbf{E}^P [v(Y_1)] = 1$ implies

$$\begin{aligned} \kappa &= \mathbf{E}^P \left[e^{\alpha Y_1} \right] \\ v(y) &= \frac{e^{\alpha y}}{\mathbf{E}^P \left[e^{\alpha Y_1} \right]}. \end{aligned}$$

Remark 4.1 *The equivalent martingale measure $P^{\kappa,v}$ can be interpreted as the one under which the representative agent calculates prices in the market. Hence, the characteristics κ and $v(\cdot)$ reflect the representative agent's market price of frequency risk and claim size risk respectively. Under risk aversion, i.e. $\alpha > 0$, $\kappa v(y) > 1$ for all $y > 0$. The risk-adjusted frequency $\lambda\kappa$ is thus larger than the "physical" frequency λ.*

The coefficient of absolute risk aversion α determines uniquely the market prices of frequency risk κ and of jump size risk $v(\cdot)$ and thus the equivalent martingale measure $P^{\kappa,v}$. Both, the premium process $p = (p_t)_{0 \leq t \leq T}$ and the price process $(f(X_t, t))_{0 \leq t \leq T}$ of the catastrophe derivative are therefore uniquely determined by

$$p_t = \lambda \mathbf{E}^P \left[Y_1 e^{\alpha Y_1} \right] (T - t) \tag{32}$$

and

$$\begin{aligned} f(p_t, X_t, t) &= \int_{-\infty}^{\infty} \exp\left(iuX_t + \frac{\mathbf{E}^P \left[\left(e^{iuY_1} - 1 \right) e^{\alpha Y_1} \right]}{\mathbf{E}^P \left[Y_1 e^{\alpha Y_1} \right]} \cdot p_t \right) \check{\varphi}(u)\, du + k \\ & \int_{-\infty}^{\infty} \exp\left(iuX_t + \lambda \mathbf{E}^P \left[e^{\alpha Y_1} \left(e^{iuY_1} - 1 \right) \right] (T - t) \right) \check{\varphi}(u)\, du + k. \end{aligned} \tag{33}$$

Remark 4.2 *In this approach, both insurance premiums and catastrophe derivative prices are determined endogenously in equilibrium This is to be contrasted against the former analysis where catastrophe derivatives were valued relative to an exogenously observed premium process.*

5 CONCLUSION

Analogously to derivatives written on traded stocks, the exclusion of arbitrage strategies requires catastrophe derivatives to be valued relative to observed insurance premiums that are based on the same underlying risk. In this article, we introduced this concept of *actuarial consistency* and deduced a representation of no-arbitrage catastrophe derivative prices written on an underlying loss index that is modeled as a compound Poisson process. This representation is based on Fourier analysis which allowed us to extract the stochastic structure that underlies both contracts in form of the characteristic function and to derive the inverse Fourier transform of catastrophe derivative prices in closed form. This suggests that there is much to be gained by using Fast Fourier Transform as an efficient algorithm for the calculation of prices. We furthermore extended our analysis to Lévy processes.

In the situations of constant loss size, risk neutrality with respect to loss size, or a market with a representative agent actuarial consistency "completes" the market and leads to a unique no-arbitrage price process.

In our model, we excluded any effects that moral hazard, adverse selection, and basis risk may have on the valuation and structure of both contracts. It would be very interesting for future research to examine the trade-off between the effect of moral hazard and adverse selection on the insurance contract and the effect of basis risk on the catastrophe derivative.

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REFERENCES

Aase, K. (1992). "Dynamic Equilibrium and the Structure of Premiums in a Reinsurance Market", *Geneva Papers on Risk and Insurance Theory* **17**, 93-136.

Aase, K. (1993). "Premiums in a Dynamic Model of a Reinsurance Market", *Scandinavian Actuarial Journal* **2**, 134-160.

Aase, K. (1999). "An Equilibrium Model of Catastrophe Insurance Futures and Spreads", *Geneva Papers on Risk and Insurance Theory* **24**, 69-96.

Cummins, J. D., and H. Geman (1995). "Pricing Catastrophe Insurance Futures and Call Spreads: An Arbitrage Approach", *Journal of Fixed Income* **4**, 46-57.

D'Arcy, S. P., and V. G. France (1992). "Catastrophe Futures: A Better Hedge for Insurers", *The Journal of Risk and Insurance* **59**, 575-600.

Davies, B. (2002). *Integral Transforms and Their Applications* (Third Edition). Texts in Applied Mathematics 41. Springer-Verlag, Berlin, Heidelberg, New York.

Delbaen, F., and J. Haezendonck (1989). "A Martingale Approach to Premium Calculation Principles in an Arbitrage-free Market", *Insurance: Mathematics and Economics* **8**, 269-277.

Embrechts, P. (2000). "Actuarial versus Financial Pricing of Insurance", *Journal of Risk Finance* **1**, 17-26.

Embrechts, P., and S. Meister (1997). "Pricing Insurance Derivatives, the Case of CAT-Futures", Proceedings of the 1995 Bowles Symposium on Securitization of Risk, George State University Atlanta, Society of Actuaries, Monograph M-FI97-1, 15-26.

Geman, H., and M. Yor (1997). "Stochastic Time Changes in Catastrophe Option Pricing", *Insurance: Mathematics and Economics* **21**, 185-193.

Karlin, S., and H. M. Taylor (1981). *A Second Course in Stochastic Processes.* Academic Press, New York.

Levi, Ch., and Ch. Partrat (1991). "Statistical Analysis of Natural Events in the United States", *ASTIN Bulletin* **21**, 253-276.

O'Brien, T. (1997). "Hedging Strategies Using Catastrophe Insurance Options", *Insurance: Mathematics and Economics* **21**, 153-162.

Rogers, L. C. G., and D. Williams (2000). *Diffusions, Markov Processes and Martingales* (Volume 2). Cambridge University Press, Cambridge.

Sondermann, D. (1991). "Reinsurance in Arbitrage-free Markets", *Insurance: Mathematics and Economics* **10**, 191-202.