Safety First and Ambiguity

Lawrence A. Berger* and Howard Kunreuther†

Abstract

There is considerable empirical evidence suggesting that ambiguity (i.e., parameter risk) impacts pricing decisions by actuaries and underwriters and their desire to provide coverage. Stone proposed a safety first model of choice that provides a possible explanation for this behavior. This paper analyzes Stone's proposed stability and survival constraints and compares the results with those predicted by expected utility theory. The analysis is motivated by insurers' increasing reluctance to provide coverage for certain specific risks such as earthquake damage insurance where the probability of loss is ambiguous. We show that such behavior is consistent with safety first but is difficult to explain using an expected utility approach.

Key words and phrases: uncertainty, catastrophic insurance, mixing distribution, stability, utility theory

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1 Introduction

Stone (1973a, 1973b) put forward a behavioral theory of insurance capacity in the spirit of a chance constrained/safety first model of choice that still stands as a possible explanation for crises of availability in insurance markets. Stone proposes that constraints of stability and survival are used by insurance companies for acceptance or rejection of risks, where stability means regularity in corporate profits over time, and survival refers to the specification of a maximum probability that aggregate losses exceed surplus.

The purpose of this paper is to conduct a more formal analysis of these constraints and to compare the results to the predictions of expected utility theory. Short-run supply functions are derived that determine the lowest price that a firm will charge to protect a certain number of risks against a particular event or, equivalently, how many risks the firm will insure at a given price. The actual price that is observed will reflect the demand for insurance. Our focus is on the first steps that firms are likely to take before entering the marketplace.

In his analysis of insurer behavior Stone suggests that “second degree uncertainty” (also termed ambiguity) influences decisions on price as well as whether a firm will want to offer coverage. Stone does not specify how ambiguity would be incorporated in his model of choice, however. In the last few years a literature on ambiguity has arisen that addresses the issue of economic behavior when there is uncertainty over the parameters of probability distributions (Kunreuther, 1989; Kunreuther, Hogarth, and Meszaros 1993).

Of interest to us is the impact of ambiguity on the premium charged by the firm and its desire to provide coverage. Our analysis is motivated by insurers’ recent difficulties in providing coverage for specific risks where the probability of a loss is ambiguous. For example, today insurers are reluctant to provide coverage to homeowners against earthquakes because of a concern that the losses from a catastrophic disaster could create capacity problems and possibly cause insolvency. Hence, the industry has argued for some type of federal earthquake insurance program just as they did for flood coverage in the 1960s; for more on this, see the Insurance Services Office (1994). As we shall demonstrate, this lack of interest by private firms in providing protection is consistent with a safety first model of firm behavior, but is difficult to explain using an expected utility approach.

2 Safet

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Under $S$ determined by $c$ to the analysis that the lost are $n$ risks, is given by

$1$ Throughout Greek letters.

$2$ Von Neumann when there is has been used many problem of a unique $u$ and (iii) the ut See, for example expected utility.
2 Safety First and Utility Theory

Suppose that an insurer is interested in offering one-period coverage for a group of $n$ mutually independent risks. At the end of the period, each risk is assumed to have a probability $\Theta$ of causing a loss of fixed amount $\ell$. (At this point $\Theta$ is assumed to be a known constant.) In Section 3, however, we will consider the case where $\Theta$ is a random variable. The insurer’s current surplus on hand, $w$, is assumed to be known with certainty. In addition, the insurer is assumed to be risk averse with a known continuous concave utility function.

Under traditional von Neumann-Morgenstern expected utility pricing, the premium is set so the insurer is indifferent between taking the risk or not. The pricing relation is given by

$$u(w) = E[u(w + n\pi - X)]$$

where:

- $u(\cdot)$ = The insurer’s utility function;
- $\pi$ = The insurance premium per risk; and
- $X$ = Aggregate losses for the $n$ risks.

where $K$ is the actual number of losses.

Under Stone’s model of safety first behavior, the premium is determined by constraints of stability and solvency. Expenses are ignored in the analysis that follows. Stability requires a probability less than $p_1$ that the loss ratio exceeds a certain target level $r^*$. Specifically, if there are $n$ risks, the premium $\pi_0$ required to satisfy the stability constraint is given by

$$Pr\left[\frac{K\ell}{n\pi_0} > r^*\right] < p_1,$$

1Throughout this paper, random variables are denoted by uppercase English or Greek letters.

2Von Neumann and Morgenstern (1947) developed expected utility theory for use when there is a process of decision making under uncertainty. Expected utility theory has been used by actuaries since the 1960s; see, for example, Borch (1968). There are many problems associated with expected utility theory, however, such as (i) the lack of a unique utility function, (ii) the utility function, even if unique, may be unknown, and (iii) the utility function may be concave in some areas and convex in other areas. See, for example, Ramsey (1993, Section 3.2) for more on the problems associated with expected utility theory.
where $K$ is the actual number of losses. Clearly $K$ is binomially distributed with parameters $n$ and $\Theta$, i.e.,

$$
\Pr[K = k] = \binom{n}{k} \Theta^k (1 - \Theta)^{n-k}.
$$

(3)

The survival constraint relates aggregate losses for the risk in question to the current surplus plus premiums written. It requires that the probability of insolvency be less than $p_2$. The premium $\pi_r$ required to satisfy the survival constraint is given by:

$$
\Pr[K \ell > w + n\pi_r] < p_2.
$$

(4)

Because the safety first constraints do not always provide a definitive premium, it is necessary to include a profit criterion as part of the pricing model. Stone indicates that insurers often specify a fixed profit margin in making their pricing decisions. Let $m$ represent the profit margin for a given risk. If one uses the expected value of losses as a reference point, then the profit criterion for any given risk would yield premium $\pi_e$ (ignoring expenses) given by:

$$
\pi_e = \Theta \ell (1 + m).
$$

Of course, the premium may be higher than that implied by the profit criterion because of the stability and solvency constraints.

The expected profit criterion coupled with the safety first constraints form a supply function for insurance that relates required premiums to the number of policies written. Figure 1 illustrates the prices required under the safety first constraints. Here $\Theta = 0.10$, $m = 0.10$, $\ell = 1$, and $r^* = 1.0$. The survival constraint is graphed for $w = 0.5$, and 10. The safety first constraints are, respectively, $p_1 = 0.05$, $p_2 = 0.00001$.

It is also expected that the magnitude of $\pi$ is only important for very small values of $n$ (and for $w$ greater than $w^*$) while the stability constraint because needed for the loss ratio to be less than unity.

A function that results from the safety first constraint is graphed in Figure 1. This supply curve is shown in Figure 1: (i) Stability, (ii) Profit Objective. Specifically, the supply $\pi$ at each value of $n$.

Profit Objective.
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Figure 1
Premiums Under Safety First Constraints
With $\Theta = 0.10$ and $m = 0.10$

![Graph showing premium rates under different safety first constraints](image)

Number of Risks, $n$
(Natural Log-Scale)

For the supply function $n$, the number of risks insured, is the quantity supplied, and $\pi$ is the price per unit risk.

The supply function can be compared with that implied when the insurer sets premiums on the basis of expected utility. For an exponential utility function

$$u(x) = -e^{-\lambda x},$$

(5)
equation (1) implies

$$1 = \sum_{k=0}^{n} e^{-\lambda (n\pi - k)} \binom{n}{k} \Theta^k (1 - \Theta)^{n-k}.$$ 

Solving for $\pi$ yields

$$\pi = \frac{1}{\lambda} \ln(1 + \Theta(e^\lambda - 1))$$

so that $\pi$ is independent of $n$ and $w$. This property of exponential utility pricing, known as additivity, is desirable because the order in which
independent risks are taken do not affect the price of each risk.\(^3\) This property does not hold for logarithmic and quadratic utility functions.

We will show in the next section that for logarithmic and quadratic utility functions, the premiums are practically constant, increasing very slowly as \(n\) increases. Premiums will increase substantially with \(n\) under all of these utility functions, however, when there is ambiguity in the probability distribution associated with losses.

3 Ambiguity and Insurance Pricing

The literature on credibility theory provides the foundation for modeling the impact on premiums if there is ambiguity with respect to the parameters of probability distributions; see, for example, Heilmann\(^3\).

\(^3\) It is well-known that exponential utility yields premiums that are independent of \(n\) and \(w\); see Gerber (1979, Chapter 5).
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(1980) and Venter (1990) for more on credibility theory. Given the nature of credibility theory, it has been long recognized that Bayesian techniques can be utilized to replace the ad hoc formulae that actuaries have been using for pricing (Mayerson, 1964). Klugman (1992) gives an excellent treatment of the application of Bayesian techniques to credibility theory. Although research is being conducted on expected utility premium principles (Goovaerts and Taylor, 1987) and credibility theory, little work has been done on expected utility pricing under parameter uncertainty. ⁴

In the literature on ambiguity, uncertainty over the parameters of probability distributions often is characterized as disagreement among experts. ⁵ In such situations, however, we may be able to use mixing distributions⁶ for parameters in modeling such uncertainty. A uniform distribution, for instance, may depict a situation in which opinion is spread evenly over a range of values. A discrete mixing distribution, on the other hand, could be used to represent a case where there are substantial differences of opinion and the experts have specific values for the parameters. We will see that under extreme ambiguity insurers often will be unwilling to provide coverage at any price when they are following safety first principles. This will not be the case under expected utility pricing.

It is useful to contrast the concept of ambiguity, as defined in this paper, with that of process risk which often is used to characterize uncertainty. Ambiguity (also called parameter risk) refers to uncertainty in the parameters of the probability or outcome distribution, whereas process risk refers to the risk associated with the projection of future losses which are inherently random. ⁷ Actuaries often use mixtures of distributions to model situations where parameters vary over a population; see Panjer and Willmot (1992, Chapters 2.8 and 8). We are, however, applying mixtures in a different way. The distribution of the parameter now is used to characterize parameter risk. This approach differs from Bayesian analysis as there is no updating procedure; rather, the focus is the degree of uncertainty about the true value of the parameter.

With ambiguity, Θ is the uncertain parameter in the specification of the cumulative distribution function (cdf) X, FΘ(x). In this case, Θ is

⁴But see Freifeld (1976, 1979). Goovaerts, De Vylder, and Haendendick (1984) study the effect of parameter uncertainty on premium principles such as the Escher, but not utility theory.

⁵For example, in assessing risks such as underground storage tank, earthquake, and satellite, the scientific community is divided due to the lack of data and causal models.

⁶The mixing distribution is the probability distribution of the uncertain parameter.

⁷This definition of process risk follows McClernahan (1990, page 61).
viewed as a random variable with cdf $G(\theta)$. Throughout the rest of this paper, the probability distribution function (pdf) (if $\Theta$ is continuous) or probability function (if $\Theta$ is discrete) of $\Theta$ is denoted by $g(\theta) = dG(\theta)$. The pricing equation for expected utility theory is

$$u(w) = E_\theta[E[u(w + n\pi - X) \mid \Theta]]$$

$$= \int_{\theta = -\infty}^{\infty} \int_{X = 0}^{\infty} u(w + n\pi - x) dF_\theta(x) dG(\theta).$$

(6)

Because the mean of $\Theta$ is assumed to be a known constant, ambiguity will not influence pricing under risk neutrality.

To contrast the differences between the effect of ambiguity on expected utility and safety first pricing we consider the same case discussed above where the firm is assumed to be insuring identical independent Bernoulli risks. The premium per risk, $\pi$, as a function of $n$ is computed under the following conditions:

- No ambiguity;
- The uniform mixing distribution,

$$g_1(\theta) = \begin{cases} 1 & \text{for } 0 \leq \theta \leq 1 \\ 0 & \text{otherwise}; \end{cases}$$

- The discrete mixing distribution,

$$g_2(\theta) = \begin{cases} 0.5 & \text{for } \theta = 0 \text{ or } 1 \\ 0 & \text{otherwise}. \end{cases}$$

Note that both $g_1(\theta)$ and $g_2(\theta)$ yield $E[\Theta] = 0.5$.

Under each of the above conditions, we will use the exponential utility function defined in equation (5) and the following utility functions:

$$u_1(x) = \ln(201 - x) \quad \text{logarithmic: with } x > -201; \text{ and}$$

$$u_2(x) = -(10 - x)^2 \quad \text{quadratic: with } -\infty < x \leq 10.$$  

(7)

(8)

Using equation (6), the expressions for the premiums can easily be calculated.

Table 1 depicts the resulting premiums for these different utility functions and mixing distributions. Although the premium for exponential utility remains constant with no ambiguity, the fact that premiums increase as $n$ increases for all of the mixing distributions means
that additivity does not hold under parameter uncertainty. The premiums for the logarithmic utility function for the no ambiguity and uniform mixing distribution cases are determined by numerical methods. As is the case for the exponential utility function, the premium under ambiguity increases with \( n \) and is higher for the discrete than for the uniform mixing distribution. The same results hold for the quadratic utility function. For the discrete mixing distribution, the premiums are higher than under the uniform mixing distribution due to the concentration of mass at probability zero and one. One interpretation of such behavior is a split in expert opinion: one group believes an event is certain to occur, while another group believes it will not. If the event does occur, all risks will suffer losses. This kind of extreme ambiguity, such as that given by a discrete mixing distribution, thus translates into a perfect correlation of risks.

### Table 1

<table>
<thead>
<tr>
<th>Utility</th>
<th>( n )</th>
<th>NAB</th>
<th>( g_1(\theta) )</th>
<th>( g_2(\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>1</td>
<td>0.6200</td>
<td>0.6200</td>
<td>0.6200</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.6200</td>
<td>0.8060</td>
<td>0.9310</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.6200</td>
<td>0.8710</td>
<td>0.9650</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.6200</td>
<td>0.9580</td>
<td>0.9930</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>1</td>
<td>0.5006</td>
<td>0.5006</td>
<td>0.5006</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.5006</td>
<td>0.5025</td>
<td>0.5062</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.5006</td>
<td>0.5046</td>
<td>0.5124</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.5006</td>
<td>0.5212</td>
<td>0.5613</td>
</tr>
<tr>
<td>Quadratic</td>
<td>1</td>
<td>0.5125</td>
<td>0.5125</td>
<td>0.5125</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.5126</td>
<td>0.5513</td>
<td>0.6340</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.5127</td>
<td>0.6021</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.5134</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Note: NAB = No Ambiguity.*

In general, probability uncertainty introduces correlation into portfolios that otherwise would consist of independent risks if the value of \( \Theta \) were known. The relation between ambiguity and correlation is
particularly clear when both the risks and the parameters are normally distributed. In this case, if the normal mixing distribution for $\Theta$ has variance $\tau^2$ then the correlation between the risks is also $\tau^2$ (Heilmann, 1989, p. 81). Thus, parameter uncertainty translates directly into correlation between risks that are conditionally independent.

Premiums under ambiguity are calculated in Table 2 using exponential utility and the safety first stability constraint. In each case we use three different mixing distributions:

\[
\begin{align*}
g_1(\theta) &= \begin{cases} 
5 & \text{for } 0 \leq \theta \leq 0.2 \\
0 & \text{otherwise}; 
\end{cases} \\
g_4(\theta) &= \begin{cases} 
0.5 & \text{for } \theta = 0 \text{ or } 0.2 \\
0 & \text{otherwise}; 
\end{cases} \\
g_5(\theta) &= \begin{cases} 
0.9 & \text{for } \theta = 0 \\
0.1 & \text{for } \theta = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

For each of these three mixing distributions $E[\Theta] = 0.1$.

<table>
<thead>
<tr>
<th>Premiums for Exponential Utility and Safety First</th>
<th>NAB</th>
<th>$g_1(\theta)$</th>
<th>$g_4(\theta)$</th>
<th>$g_5(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>1</td>
<td>0.159</td>
<td>0.159</td>
<td>0.159</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.159</td>
<td>0.188</td>
<td>0.231</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.159</td>
<td>0.211</td>
<td>0.261</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.159</td>
<td>0.263</td>
<td>0.288</td>
</tr>
<tr>
<td>Safety First&lt;br&gt;(with stability constraint)</td>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0.300</td>
<td>0.300</td>
<td>0.400</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.200</td>
<td>0.250</td>
<td>0.300</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.150</td>
<td>0.210</td>
<td>0.250</td>
</tr>
</tbody>
</table>

*Note: NAB = No Ambiguity.*

For the exponential utility function defined in equation (5) and no ambiguity, the insurer's premium is independent of the number of risks and is given by $\pi = 0.159$. Probability ambiguity causes premiums to
increase rapidly as \( n \) increases. For the discrete mixing distribution \( g_3(\theta) \), the premium is as high as 0.977 when \( n = 100 \).

For the safety first model, the story is completely different, as indicated in Table 2. The premiums required under the stability constraint given by equation (2) decline as \( n \) increases. In the most extreme case, the discrete mixing distribution over \([0, 1]\), the constraint never can be met because the probability of \( n \) losses is 0.10 and we have assumed that \( p_1 = 0.05 \) in equation (2).

Table 3 shows the impact of the survival constraint (given by equation (4) with \( p_2 = 0.00001 \)) on premiums for relatively small values of \( n \) for the non-ambiguous case and when the probability distribution of losses is ambiguous using \( g_3 \). Consider the non-ambiguous case. In order to understand how the firm's surplus and number of policies \( n \) affect \( \pi_r \), let

\[
k(n) = \min\{k : \Pr[K > k] < p_2\}.
\]

Then for \( \ell = 1 \),

\[
\pi_r = \max\left\{0, \frac{k(n)}{n} - \frac{w}{n}\right\}.
\]

Thus the insurer's current surplus on hand, \( w \), can be viewed as a measure of its capacity to accept risks and has its greatest impact for small values of \( n \). Insurers with larger capacity are able to charge lower premiums. Note that as \( n \) increases (starting from 1), premiums may increase or decrease depending on the behavior of \( k(n)/n \). But, from the law of large numbers, \( k(n)/n \) goes to \( E[\Theta] \) as \( n \) goes to infinity. So the premium \( \pi_r \) eventually will approach the expected loss. (See Figure 1.) Like the stability constraint, the required premiums increase under this mixing distribution compared to the non-ambiguous case, but the premiums also decline as the number of risks increases.

Thus a distinction emerges between the predictions of safety first and utility theory under ambiguity. With extreme ambiguity, coverage will be denied under the safety first criteria, while under more moderate conditions premiums will increase but eventually will decline for large \( n \). On the other hand, in utility theory, increased ambiguity results in higher premiums and the failure of the law of large numbers to have any influence as the number of risks increases.
Table 3
Premiums ($\pi_r$) Under Survival Constraint

<table>
<thead>
<tr>
<th>$w$</th>
<th>$n$</th>
<th>No Ambiguity</th>
<th>$g_4(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>0.60</td>
<td>0.70</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0.45</td>
<td>0.55</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>0.32</td>
<td>0.42</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>0.25</td>
<td>0.35</td>
</tr>
<tr>
<td>200</td>
<td></td>
<td>0.20</td>
<td>0.30</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>0.10</td>
<td>0.20</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>0.20</td>
<td>0.30</td>
</tr>
<tr>
<td>50</td>
<td></td>
<td>0.22</td>
<td>0.32</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>0.20</td>
<td>0.30</td>
</tr>
<tr>
<td>200</td>
<td></td>
<td>0.18</td>
<td>0.28</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>20</td>
<td></td>
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<tr>
<td>50</td>
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<td>0.22</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>0.15</td>
<td>0.25</td>
</tr>
<tr>
<td>200</td>
<td></td>
<td>0.15</td>
<td>0.25</td>
</tr>
<tr>
<td>20</td>
<td>50</td>
<td>0.00</td>
<td>0.02</td>
</tr>
<tr>
<td>100</td>
<td></td>
<td>0.05</td>
<td>0.15</td>
</tr>
<tr>
<td>200</td>
<td></td>
<td>0.10</td>
<td>0.20</td>
</tr>
</tbody>
</table>
4 Empirical Results

Hogarth and Kunreuther (1990) conducted a survey to test the various theories of insurance pricing. Actuaries were asked to price warranties on the performance of a component of a new line of microcomputers. They were told that the cost of repair is $100, and there can be at most one breakdown per period. Experimental variations concern the number of units insured, ambiguous and non-ambiguous probabilities of breakdown, and probability levels of 0.001, 0.01, and 0.10. In the ambiguous versions of the scenario, respondents were told there is considerable disagreement among experts regarding the probability of a breakdown of any given unit, while in the non-ambiguous versions they were told that the experts all agree on the chances of a breakdown.

The results are listed in Table 4 in terms of the ratios of the prices proposed by the actuaries to the expected losses. It is evident in all cases that ambiguity results in higher prices. What is difficult to explain, however, is that even in the absence of ambiguity, prices increase as risks are added for $\Theta = 0.001$ and $\Theta = 0.01$ while they decline for $\Theta = 0.10$. Utility theory implies that premiums increase as risks are added under decreasing absolute risk aversion, and premiums decline under increasing absolute risk aversion (Goovaerts and Taylor, 1987; see note 5). The above empirical results seem to suggest that the actuaries' utility functions exhibit decreasing and increasing absolute risk aversion over the appropriate ranges.

<table>
<thead>
<tr>
<th>Table 4</th>
<th>Actuaries' Price-Expected Loss Ratios for Warranties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = 0.001$</td>
<td>$\theta = 0.01$</td>
</tr>
<tr>
<td>$n = 10,000$</td>
<td></td>
</tr>
<tr>
<td>Non-ambiguous</td>
<td>1.266</td>
</tr>
<tr>
<td>Ambiguous</td>
<td>2.439</td>
</tr>
<tr>
<td>$n = 100,000$</td>
<td></td>
</tr>
<tr>
<td>Non-ambiguous</td>
<td>1.538</td>
</tr>
<tr>
<td>Ambiguous</td>
<td>3.333</td>
</tr>
</tbody>
</table>

It is possible, however, to explain the actuaries' pricing from the perspective of safety first theory. We noted earlier that the premiums implied by the survival constraint can rise for smaller values of \( n \) followed by declines. The observed patterns of pricing with no ambiguity can be explained if premiums are increasing for \( \Theta = 0.001 \) and \( \Theta = 0.01 \) and decreasing for \( \Theta = 0.10 \) over the range of \( n \) being considered.

Consider, for example, the results exhibited in Table 5 for \( w = 15 \). For \( \Theta = 0.001 \) prices peak at \( n = 40,000 \), while for \( \Theta = 0.10 \) prices decline over the entire range of values for \( n \). When \( n \) increases from 10,000 to 100,000, prices thus increase for \( \Theta = 0.001 \) and decline for \( \Theta = 0.10 \). This occurs because the relative magnitude of the initial surplus is higher for small values of \( \Theta \). The capacity goes further for smaller values of \( \Theta \) and, therefore, the premiums rise for larger values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \Theta = 0.001 )</th>
<th>( \Theta = 0.100 )</th>
<th>( \Theta = 0.001 )</th>
<th>( \Theta = 0.100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10,000</td>
<td>0.00110</td>
<td>0.1115</td>
<td>1.10</td>
<td>1.115</td>
</tr>
<tr>
<td>20,000</td>
<td>0.00135</td>
<td>0.1084</td>
<td>1.35</td>
<td>1.084</td>
</tr>
<tr>
<td>30,000</td>
<td>0.00137</td>
<td>0.1070</td>
<td>1.37</td>
<td>1.070</td>
</tr>
<tr>
<td>40,000</td>
<td>0.00138</td>
<td>0.1061</td>
<td>1.38</td>
<td>1.061</td>
</tr>
<tr>
<td>50,000</td>
<td>0.00136</td>
<td>0.1055</td>
<td>1.36</td>
<td>1.055</td>
</tr>
<tr>
<td>60,000</td>
<td>0.00135</td>
<td>0.1050</td>
<td>1.35</td>
<td>1.050</td>
</tr>
<tr>
<td>70,000</td>
<td>0.00133</td>
<td>0.1046</td>
<td>1.33</td>
<td>1.046</td>
</tr>
<tr>
<td>80,000</td>
<td>0.00133</td>
<td>0.1044</td>
<td>1.33</td>
<td>1.044</td>
</tr>
<tr>
<td>90,000</td>
<td>0.00131</td>
<td>0.1041</td>
<td>1.31</td>
<td>1.041</td>
</tr>
<tr>
<td>100,000</td>
<td>0.00130</td>
<td>0.1039</td>
<td>1.30</td>
<td>1.039</td>
</tr>
</tbody>
</table>

The fact that for the survey of actuaries relative prices are higher for smaller values of \( \Theta \) is explained readily in the presence of expenses that do not vary with the probability of loss.
5 Coverage Limits

The consideration of ambiguity can shed light on another question of interest. The fact that insurance companies offer liability policies with specified coverage limits is a puzzle from the perspective of utility theory. Models of risk sharing under expected utility maximization invariably conclude that the entire risk should be split according to the risk preferences of the parties to the exchange. The optimal contractual forms do not include coverage limits but involve deductibles and coinsurance above some level. Only in the case of a regulatory constraint requiring insurers to sell a policy with a prescribed actuarial value has it been shown that there will be policy limits (Raviv, 1979). Huberman, Meyers, and Smith (1983) derive coverage limits when demand is influenced by limited liability under specific assumptions about the nature of the risk, but this does not explain insurers' reluctance to offer policies with unlimited exposure. In fact, consideration of limited liability of insurers would suggest that they would be more than willing to sell such policies if there were relatively low costs of bankruptcy.

The above analysis assumes a fixed loss size of one, but it easily is extended to a severity of loss size \( l \). The survival constraint may be written as

\[ Pr[K \geq \frac{w}{l} + n \pi] \leq 0.00001, \tag{10} \]

where \( \pi \) is now the premium per dollar of coverage. Note that capacity is now \( w/l \) instead of \( w \). As \( l \) increases, capacity approaches zero, which limits the ability of firms to write small numbers of large risks. In this case the supply function with \( w = 0 \) in Figure 1 is an appropriate representation of the insurer's ability to provide coverage.

Large values of \( l \) together with ambiguity further will act to raise prices and limit availability. From equations (10) and (9),

\[ \pi = \frac{k(n)}{n} - \frac{w}{nl}, \tag{11} \]

so for large \( l \) the ratio \( k(n)/n \) becomes the key determinant of price. In the absence of ambiguity \( k(n)/n \) will approach \( \Theta \) (recall that \( \Theta \) is known) as \( n \) gets large. This is generally not the case when ambiguity is present. For example, when \( F(\Theta) = 0.5 \)

- Under the uniform mixing distribution, the probability of all outcomes is \( 1/(n + 1) \), i.e.
\[
\Pr[K = k] = \int_0^1 \binom{n}{k} \theta^k(1-\theta)^{n-k} d\theta = \frac{1}{n+1}.
\]

Hence the limit of \(k(n)/n\) as \(n \to \infty\) is 1;

- When the ambiguity is characterized by the discrete mixing distribution over the \([0, 1]\) interval, the probability of \(n\) losses is \(\Theta\), so unless \(n\ell \leq w + n\ell\pi\) the survival constraint always will be violated for \(\Theta > 0.00001\). Therefore, the law of large numbers is ineffective in cases such as these. Hence, only a relatively small number of risks may be underwritten.

Contrast these observations with the case of no ambiguity and independent risks: for \(\Theta = 0.1\) the number of risks \(n\) need only be six for the probability of \(n\) losses to be less than 0.00001.

The company has three alternatives in meeting the survival constraint under extreme ambiguity—it can reduce \(n\), raise \(\pi\), or reduce \(\ell\). Solving the constraint \(n\ell \leq w + n\ell\pi\) for \(n\), we get \(n \leq w/(1-\pi)\ell\). In general, the constraint on the number of risks will be

\[
n \leq \frac{w}{(k/n) - \pi}\ell.
\]

With ambiguity, \(k(n)/n\) will be close to one for small values of \(n\), so the same analysis goes through under these conditions. This means that the capacity \(w\) relative to the severity \(\ell\) determines how many risks can be underwritten. A natural way to increase the number of risks that can be underwritten is to reduce \(\ell\) by way of coverage limits. Capacity also is increased by increasing \(\pi\), but \(\pi\) cannot get too close to one, especially when expenses are considered. When the constraint is reached, \(n\ell = w + n\ell\pi\) and the capacity to assume new risks is exhausted. In order to assume a new risk, the premium collected must be increased 100 cents to the dollar\(^8\) in order to prevent violation of the constraint; that is, \(\pi\) must approach one and the insurance will not be purchased.

We see that ambiguity and safety first results in a focus on the worst possible outcome \((n\ell)\) that a portfolio of risks may suffer. Empirical studies of managerial behavior in the face of uncertainty suggest that managers do tend to focus on the severity of the worst possible

\(^8\)See Kunreuther (1989) for an interview with an actuary who uses this expression to explain his reaction to extreme ambiguity.
outcomes, paying less attention to the probability of their occurrence (March and Shapira, 1987, p. 1407). Not only does this describe managerial behavior, but such a focus is prescribed by the leading texts on risk management (e.g., Williams and Heins, 1989; Vaughan, 1992). Risk management procedures call for prioritizing risks according to their potential severity followed by an estimation of the probability of their occurrence. The ambiguity inherent in many organizational risks calls for a managerial focus on potentially catastrophic risks regardless of the probability of their occurrence.

6 Discussion

The safety first model of insurance pricing has been shown to provide significantly different predictions from those of expected utility theory. While the utility functions (exponential, quadratic and logarithmic) examined here exhibit nondecreasing premiums as risks are added to the portfolio, under a safety first model premiums may increase or decline for small values of \( \pi \) according to the predominance of the survival or stability constraints. Under safety first, the value of \( \pi \) declines as the number of insured risks becomes relatively large.

When ambiguity is present, price increases are exaggerated under utility theory, whereas for safety first the results depend on the extent of the ambiguity. Under extreme ambiguity insurers often will be reluctant to provide coverage at any price except dollar-for-dollar, while for more moderate ambiguity, premiums will be higher even though they eventually may decline. Safety first theory also has been shown to yield a definition of capacity as referring to the ability to underwrite relatively small numbers of risks and to provide explanations for the existence of coverage limits in liability insurance. It appears that Stone's characterization of insurer behavior has considerable merit.

References


