“Auctioning the Provision of an Invisible Public Good”

90-01-04

Paul Kleindorfer and Murat Sertel
Auctioning the Provision of an Indivisible Public Good

Paul R. Kleindorfer*

The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6366

Murat R. Sertel
Department of Economics
Bogazici University
Istanbul 80815, Turkey

*For useful discussions, we are thankful to participants during Spring 1989 in our Game Theory course, DS901 (University of Pennsylvania), as well as to Ronald Harstad, Richard Kihlstrom, Howard Kunreuther and Marc Knez, the associate editor and referee for JET, and to participants of the XVth Bosphorous Workshop on Economic Design, August 1992. This research was supported in part by Grant SES 88-09299 from the National Science Foundation, by the Scientific and Technical Research Council of Turkey (TUBITAK), and by the Bogazici University Research Fund.
LIST OF SYMBOLS

1. Standard Arabic

2. Standard Greek

3. Mathematical Symbols
   \( \times \) (times)
   \( \in \) (belongs to)
   \( \mathcal{N} \) (calligraphic N, The Natural Numbers)
   \( \mathbb{R} \) (Re, The Real Line)
   \( \emptyset \) (Empty Set)
   \( \prod \) (Product)
   \( \sum \) (Sum)
Figure for Theorem 1 of Kleindorfer and Sertel, "Auctioning the Provision of an Indivisible Public Good"
Running Head: Auctioning a Public Good

Corresponding author:

Paul R. Kleindorfer
Operations and Information Management Department
The Wharton School
University of Pennsylvania
Philadelphia, PA 19104-6366
A "community" of \( n \) agents must determine which of its members should provide an indivisible public good. Each of the agents can provide the public good, but the provision cost varies among the agents. We identify here an efficient solution concept for such problems and design a class of auction-like mechanisms, each of which Nash implements (and under appropriate domain restrictions fully Nash implements) this. The mechanisms are in the spirit of "\( k^{th} \) lowest bidder" auctions.
The context we have in mind is one where an indivisible public good or service is to be provided by one of the members of a community, subject to monetary compensation by all of the other members. It is understood that a certain, say unit, amount of this good or service is to be procured from among the populace, and there is no question of whether or how much of it is to be provided, just the question of which member should be the provider and how much the provider should be paid by the other members for the provision. Typically, each member of the community is able to provide the good or service, but at a cost which may vary from one member to the other. (The quality of the good or service does not vary from one possible provider to another.)

A common household example is where someone has to take out the garbage every morning it is collected. Every member of the household is just as competent at the task, as a consequence of which everyone is indifferent as to who among the rest of the members performs it, but some may be willing to pay more than others to get out of the chore. A blow-up of this example is found among communities, where a locally undesirable land use (LULU) facility, such as a prison or a hazardous waste incinerator, has to be located in one of several neighboring towns or states. Each community may differently reckon the nuisance value of acting as host to the undesirable object. A site must, however, be provided for the LULU facility by some neighbor. Also, either it is technically infeasible or it simply does not pay to chop up the facility and use more than one site, in which sense the good or service being provided is indivisible. It is public in the usual sense, as all members benefit, without the possibility of excluding any member, from the provision of the good or service in question. Other instances of such indivisible public good provision problems include choosing the location of possibly desirable facilities or events, such as locating the Capitol
of Europe\(^1\) or the next Olympics or the next meeting of a scientific society.

We identify here an efficient solution concept for such problems and design a class of auction-like mechanisms, each of which Nash implements (and, with a domain restriction to the effect of there being enough competition among the less efficient, fully Nash implements) this. The mechanisms are in the spirit of the "\(k\)-th lowest bidder" auctions. However, as will become plain below, the auction procedures we propose are not Vickrey-type auctions. Among other differences, in our auctions, all losing bidders end up paying the winner of the auction some portion of the cost of providing the public good.

This paper lies at the crossroads of three important currents in the modern literature of economic design: the provision of public goods, auctions, and implementation. Specifically, we are concerned with the provision of an indivisible public good by means of auction mechanisms which implement an efficient, balanced budget, nominally egalitarian and envy free solution (social choice correspondence) in Cournot-Nash equilibrium. Some straightforward necessary and sufficient conditions are indicated for the outcome to be individually rational. The question of whether the public good should be provided is understood to have been resolved affirmatively beforehand and is not left to be addressed by our mechanisms.\(^2\) The examples provided above display a common and widespread application area where there is no question that the public good should be provided and for which, therefore, our mechanisms would be appropriate. Being auctions, the mechanisms we propose are rather natural and familiar in both practice and economic theory. They employ a simple message space, namely the real line, for each participant bidder. This is in contrast with the more complex message spaces we find in the "implementability" literature (e.g., Maskin [9], Moore and Repullo [10,11], Abreu and Sen [1], and Dutta and Sen [3]\(^3\)), where the concern is with
demonstrating general existence theorems for implementing various classes of social choice functions.

A detailed discussion of auction procedures for indivisible public good provision problems can be found in Kunreuther et al. [8], which may be regarded as an immediate predecessor of the present paper. Jackson and Moulin [4] also consider the provision of an indivisible public good, addressing a question which is complementary to ours, namely whether the good should be provided or not. They emphasize the allocation of cost for the provision of a public good whose method and cost of provision is already known. In our framework, the mechanism decides on which agent is to provide the public good, allocating the resulting cost of provision so that the remaining participants' burdens are equal. (The providing agent's burden may fall below those of the others to the extent of his competitive advantage within the community as a provider.) Like Jackson and Moulin, and in contrast with Abreu and Matsushima [2], we do not consider lotteries as solutions, nor do we regard lotteries as feasible strategies for participants, thus confining ourselves to implementation of sure outcomes in pure strategies.

We denote \( \mathcal{N} \) for the set \( \{1, 2, \ldots, n\} \) of positive integers, \( N \) for the set \( \{1, \ldots, n\} \) whenever \( n \in \mathcal{N}, \) and \( \mathbb{R} \) for the set of real numbers. Given \( n \in \mathcal{N} \) and a family \( \{X_i|i \in N\} \) of sets \( X_i, \) we denote \( X = \Pi_{i \in N} X_i, \) and for any \( j \in N \) we denote \( X_{-j} = \Pi_{i \notin \{j\}} X_i, \) writing \( x_j \in X_j, x_{-j} \in X_{-j} \) and \( x \in X \) for generic elements.

We consider a “community” \( N \) with \( n \geq 2, \) where an indivisible public good is to be provided by some member \( i \in N, \) who will be compensated by his fellow members for his services. Each member \( i \) has preferences over \( N \times S, \) where \( S = \{s \in \mathbb{R}^n | \sum s_i \leq 0\} \) is the set of feasible compensation schemes, and \( (j, s) \in N \times S \) indicates \( j \) as the provider.
of the public good, receiving \( s_j \), which cannot exceed the sum of the payments \(-s_i\) of his fellow community members \( i \in N \setminus \{j\} \). The \( i \)th member's preference over \( N \times S \) is represented by a real-valued "utility function" \( u_i : N \times S \rightarrow \mathbb{R} \) of the form \( u_i(j, s) = s_i - c_i(j) \) with \( c_i(i) \geq c_i(j) = c_i(k) \) for any \( i \in N \) and any \( j, k \in N \setminus \{i\} \). Here \( c_i(j) \) (respectively, \(-c_i(j)\)) expresses the net "cost" (respectively, the net "benefit") to member \( i \) of member \( j \)'s providing the public good. Thus, each member is indifferent as to which of his fellow members is to provide these services and, if anything, would rather have one of them rather than himself be chosen as the provider. Furthermore, each member is the happier the greater the compensation that he receives and is otherwise unconcerned with the other coordinates of a feasible compensation scheme.

What we mean by an **indivisible public good provision problem** is any \( u = (u_1, \ldots, u_n) \) for some \( n \in \mathcal{N} \) where \( u_i \) for each \( i \in N \) is a utility function of the form just described. We let \( U \) stand for the space of all such problems. By a **solution** for a problem \( u \in U \) we mean any \((i, s) \in N \times S\), and by a **solution concept** (for such problems) we mean any map \( \sigma : U \rightarrow 2^{N \times S \setminus \{0\}} \). A solution \((i, s)\) to a problem \( u \) is efficient for \( u \) iff it so happens that whenever \( u_j(h, r) > u_j(i, s) \) for some \( j \in N \) and \((h, r)\), then also \( u_k(i, s) > u_k(h, r) \) for some \( k \in N \). A solution concept \( \sigma \) is efficient iff at each problem \( u \in U \) every element of \( \sigma(u) \) is efficient for \( u \).

Here we identify a solution concept for indivisible public good provision problems and we design a class of mechanisms which implement it in Nash equilibrium. We first record a well-known characterization (see, e.g., Kleindorfer and Sertel [7]) of efficiency which will apply to the solution concept we immediately identify.
1. Lemma: A solution \((i, s)\) to a problem \(u \in U\) is efficient for \(u\) iff

\[ i \in \arg \min \{ \sum_{j \in N} c_j(k) \mid k \in N \} \quad \text{and} \quad \sum_{j \in N} s_j = 0. \]

From the above Lemma, the efficient solutions to a problem \(u \in U\) are those for which the provider of the public good is any \(i\) maximizing the sum \(G(i) = \sum_N -c_j(i)\) of net benefits. Clearly, the efficient solutions to any problem \(u \in U\) would be unchanged if we added, for each \(i\), a constant \(K_i\) to \(u_i\), in particular the constant \(K_i = c_i(j)\) for some \(j \neq i\). For \(i\) maximizes \(G(i)\) iff it maximizes \(G(i) + K\), where \(K\) is the constant \(K = \sum_N K_i\), with \(K_i = c_i(j(i))\) for any \(j(i) \in N \setminus \{i\}\) at each \(i \in N\). Thus, the interesting data of a problem \(u \in U\) are the \(n\) values \(c_i = c_i(i) - c_i(j) \geq 0, \ j \neq i,\) measuring in monetary units the nuisance value to member \(i\) of his providing the good rather than a fellow member \(j\) doing so. This motivates and justifies our restricting attention from here on to problems \(u \in U\) of the canonical form with \(c_i(j) = 0\) for all \(j \in N \setminus \{i\}\) at each \(i \in N\). Without loss of generality, for each \(u \in U\), we also assume henceforth that \(c_1 \leq c_2 \leq \ldots \leq c_n\). We also assume that this is commonly known by all \(i \in N\).

We identify a solution concept \(\alpha : U \to \pi^{N \times S \setminus \{\emptyset\}}\) as follows:

\[ \alpha(u) = \{(i, s) \in I(u) \times S \mid \rho_1(u) \leq s_i \leq \rho_2(u) \text{ and } s_j = -s_i/(n - 1) \text{ whenever } j \in N \setminus \{i\} \}, \]

where

\[ I(u) = \{ h \in N | c_h = \min \{c_j \mid j \in N\} \} \]

and

\[ \rho_i(u) = (n - 1)c_i / n \text{ for each } u \in U \text{ and } i \in N. \]
We will see (Lemma 6), when the context below permits, that \( \rho_i(u) \) is the “prudent” (i.e. maximin) bid for \( i \) in a “(first) lowest bidder auction.” Thus, \( \alpha(u) \) appoints any agent \( i \) with minimal cost \( c_i \) as provider of the indivisible public good, pays this agent some \( s_i \in [\rho_1, \rho_2] \) and evenly spreads the burden of mustering \( s_i \) among the other agents. By direct application of Lemma 1, the reader will note that \( \alpha \) is an efficient solution concept.

2. Example: Consider a 3-agent community \( N = \{1, 2, 3\} \), where agent \( i \in N \) receives a benefit \( \beta_i(j) \) when agent \( j \in N \) provides a certain indivisible public good as shown in Table 1 below. The cost for agent \( i \in N \) to provide the good is shown as \( \gamma_i \) in the Table.

<table>
<thead>
<tr>
<th>Agent ( i )</th>
<th>( \beta_i(1) )</th>
<th>( \beta_i(2) )</th>
<th>( \beta_i(3) )</th>
<th>( \gamma_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 1: Benefits and Provision Costs

Note that an agent \( i \) may benefit more or less from his own providing the public good than from another’s providing the good (e.g., due to a perceived honor or stigma of being the provider), although every agent \( i \) is indifferent as to which of his fellow members is to provide the good so long as it is not himself.

From these data, we derive the (net) costs
to any agent \( i \in N \) of the public good being provided by any agent \( j \in N \), as tabulated in

Table 2. (The reader should feel welcome to think in terms of (net) benefits \( b_i(j) = -c_i(j) \).)

The parameter values \( \{c_i(j) \mid i, j \in N\} \) determine the public good provision problem

\( u = \{u_i \mid i \in N\} \) as explained above through \( u_i(j, s) = s_i - c_i(j) \). To express this problem

in canonical form, we need only regard the parameter values \( c_i = c_i(i) - c_i(j), j \neq i \), as

tabulated in Table 3. This table also displays the prudent bids \( \rho_i \) which characterize the

solution \( \alpha(u) \) defined above. Thus, the solution \( \alpha(u) \) to the problem \( u \) specified above is

\( \alpha(u) = \{1\} \times \{s \in \mathbb{R}^3 \mid 4/3 \leq s_1 \leq 8/3, s_2 = s_3 = -s_1/2\}. \)

\[ c_i(j) = \begin{cases} \gamma_i - \beta_i(i), & i = j \\ -\beta_i(j) & i \neq j \end{cases} \]

<table>
<thead>
<tr>
<th>Agent ( i )</th>
<th>Provider ( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>-3</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>-3</td>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td>- \sum_{i \in N} c_i(j)</td>
<td></td>
<td>5</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>Agent ( i )</th>
<th>( c_i )</th>
<th>( \rho_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>( \frac{4}{5} )</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>( \frac{8}{3} )</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>( \frac{10}{3} )</td>
</tr>
</tbody>
</table>

Table 3

Net Costs \( c_i(j) \)  Canonical Costs and Prudent Bids

Now for the solution concept \( \alpha \) we design a class of "auction" mechanisms \( \mu^k(k \in N) \)

which implement \( \alpha \) in Nash equilibrium. Setting \( X = \mathbb{R}^n \), for each \( k \in N \) the function

10
\( \mu^k: X \to N \times S \), which we refer to as the "\( k^{th} \) lowest bidder auction", is defined by
\[
\mu^k(x) = (i^1(x), s^k(x)),
\]
where
\[
s^k_i(x) = \begin{cases} 
M^k(x) & \text{if } i = i^1(x) \\
-M^k(x)/(n-1) & \text{if } i \in N\setminus\{i^1(x)\}
\end{cases}
\]
and the functions \( i^k: X \to N \) and \( M^k: X \to \mathbb{R} \) are recursively defined through
\[
\begin{align*}
M^k(x) &= \min\{x_j \mid j \in N^k(x)\} \\
i^k(x) &= \min\{j \in N^k(x) \mid x_j = M^k(x)\} \\
N^{k+1}(x) &= N^k(x) \setminus \{i^k(x)\}
\end{align*}
\]
with \( N^1(x) = N \). For any bid vector \( x \in X \), denote the set of \( k \) lowest bidders as
\[
W^k(x) = \{i^1(x), \ldots, i^k(x)\}.
\]

For any "bid" vector \( x \in X \), \( i^k(x) \) thus identifies the \( k^{th} \) lowest bidder, with ties broken by resort to the natural order of indices in \( N \), and \( M^k(x) \) is just the bid of \( i^k(x) \). Note that the bid \( M^k(x) \) of the \( k^{th} \) lowest bidder \( i^k(x) \) is generally distinct from the \( k^{th} \) lowest bid.

For example, regarding a bid vector \( x = (1, 2, 1, 3) \), we have \( i^1(x) = 1 \), \( i^2(x) = 3 \), \( i^3(x) = 2 \), \( i^4(x) = 4 \), \( M^1(x) = M^2(x) = 1 \), \( M^3(x) = 2 \), \( M^4(x) = 3 \), whereas the first, second and third lowest bids registered in \( x \) are 1, 2, and 3, respectively.

3. Note: We note a couple of further facts concerning the behavior of the functions \( M^k \). First, the \( M^k \) are all monotonically nondecreasing, i.e. for any \( i \in N \) we have
\[
0 \leq M^k(x_i, x_{-i}) - M^k(y_i, x_{-i}) \text{ whenever } (x_i, x_{-i}) \in X \text{ and } y_i \in X_i \text{ with } x_i \geq y_i.
\]
Second,
the change in $M^k(x)$ due to a change in any coordinate of $x$ is bounded by the change in that coordinate, so that for every $x = (x_i, x_{-i}) \in X$ and $y_i \in X_i$ with $x_i \geq y_i$ we actually have

$$0 \leq M^k(x_i, x_{-i}) - M^k(y_i, x_{-i}) \leq x_i - y_i.$$ 

4. **Lemma:** Given any $x = (x_i, x_{-i}) \in X$ and $y_i \in X_i$,

$$M^k(x) \geq y_i \text{ implies } M^k(y_i, x_{-i}) \geq y_i.$$ 

**Proof:** Write $y = (y_i, x_{-i})$. If $y_i \geq x_i$, then the monotonicity of $M^k$, as noted above, yields $M^k(y) \geq M^k(x)$, and the result follows. So assume $y_i < x_i$ and that $M^k(x) \geq y_i$. Now there are just two possibilities: either (a) $i \in W^k(y)$ or (b) $i \notin W^k(y)$. In case (a) $y_i$ is the bid of an $h^{th}$ lowest bidder for some $h \leq k$, so clearly $M^k(y) \geq y_i$. Otherwise, i.e. in case (b), it follows from $x_i > y_i$ that $i$ does not belong to $W^k(x)$ either. Thus, since $x$ and $y$ differ only in the $i^{th}$ coordinate, $W^k(y) = W^k(x)$ and $M^k(x) = M^k(y) = x_j$ with $j = i^k(x) = i^k(y)$. Thus, again $M^k(y) \geq y_i$, as desired, completing the proof. Q.E.D.

Now each pair $(u, \mu^k)$ with $u \in U$ and $k \in N$ defines an auction game (in normal form) $\Gamma^k[u] = (X, u^k)$ with $u^k = u \circ \mu^k : X \to \mathbb{R}^n$ evaluating the resulting outcome $\mu^k(x)$ of each bid $x \in X$ for each bidder $i \in N$ according to $u^k_i(x) = u_i(\mu^k(x))$. Our main theorem is that each of the mechanisms $\mu^k$ implements the (efficient) solution concept $\alpha$ in Cournot-Nash equilibrium.\footnote{Denoting the set of Cournot-Nash equilibria of any game $\Gamma^k$ by $\sigma_0(\Gamma^k)$, we can formally state this fact as follows:}
5. Implementation Theorem: Let \( u = (u_1, \ldots, u_n) \) be any indivisible public good provision problem in canonical form, and take any \( k \in N \).

1. \( \mu^k(\sigma_0(\Gamma^k[u])) \subset \alpha(u) \);

2. If \( \#(N \setminus I(u)) \geq \max[1, k - 1] \), then in fact \( \mu^k(\sigma_0(\Gamma^k[u])) = \alpha(u) \), and so the following diagram commutes.

\[
\begin{array}{ccc}
u & \overset{\alpha}{\rightarrow} & \alpha(u) \\
\downarrow \Gamma^k & & \downarrow \mu^k \\
\Gamma^k[u] & \overset{\sigma_0}{\rightarrow} & \sigma_0(\Gamma^k[u])
\end{array}
\]

In plain English, our theorem says that whichever of the mechanisms \( \mu^k \) with \( k \in N \) is instituted, whenever a collection \( N \) of agents with utility functions \( u_i \) posing a problem \( u \in U \) play the auction game \( \Gamma^k[u] \) determined by \( u \) and \( k \), the set of bids which are non-cooperative (Cournot-Nash) equilibria of \( \Gamma^k[u] \) lead through \( \mu^k \) to efficient solutions in the set \( \alpha(u) \) of solutions for the problem \( u \). Moreover, if there are a sufficient number of agents with non-minimal costs (i.e., costs \( c_i \) greater than \( c_1 \)), then every solution in \( \alpha(u) \) can be realized through \( \mu^k \) as such a non-cooperative equilibrium. Thus, each \( k^{th} \) lowest bidder auction implements (fully implements if \( \#(N \setminus I(u)) \) is large enough), in Cournot-Nash equilibrium, the set of efficient solutions identified by \( \alpha \).

To illustrate the Theorem, consider Example 2. For the data \( u \) given in Table 3, the
equilibria for $\Gamma^k[u], k = 1, 2, 3$, give rise to payoffs which are precisely those corresponding to the efficient outcomes:

$$\alpha(u) = \mu^k(\sigma_0(\Gamma^k[u])) = \{1\} \times \{x \in \mathbb{R}^3 | 4/3 \leq s_1 \leq 8/3, s_2 = s_3 = -s_1/2\}.$$ 

The key intuition driving this result is that no player wants to lose the auction to a winning bid greater than his security bid. Since ties are won by the lowest index player, this leads players 1 and 2 to lock in at a bid between $\rho_1(u)$ and $\rho_2(u)$, with all other players bidding as low as possible while still losing. For example, consider $\Gamma^k[u]$ for the data of Example 2. For simplicity, suppose that player 3’s bid is no less than that of players 1 or 2, i.e., $x_3 \geq \text{Max}[x_1, x_2]$ (the reader should find it straightforward to establish that $x_3 < \text{Max}[x_1, x_2]$ cannot belong to an equilibrium bid vector). If $x_1 < \rho_2(u) = 8/3$, then player 2’s earnings are maximized if he bids $x_1$, losing the auction to player 1. Indeed, the reader can check that the resulting bids ($x_1 = x_2 \leq x_3$) are an equilibrium in $\Gamma^k[u]$ as long as $x_1 \geq \rho_1(u)$. On the other hand, suppose that $x_1 > \rho_2$. Then player 2 will improve his earnings if he wins the auction by bidding just below $x_1$. But such a bid will not remain unchallenged by player 1, forcing the winning bid down to $\rho_2(u)$.

The proof of Theorem 5 will use several lemmata, recorded below.

6. **Lemma**: Given any $u \in U$, the prudent strategy (bid) $\rho_i$ of Player $i \in N$ in $\Gamma^1[u]$, defined by

$$\min_{x_{-i}} \{x_i \in \mathbb{R} | u_i(\rho_i, x_{-i}) = \max_{x_i \in \mathbb{R}^3} \min_{x_{-i}} u_i(\rho, x_{-i})\},$$

is unique and is given by
\[ \rho_i = \frac{n-1}{n}c_i. \]

**Proof:** First, regard the left hand side of (\(\ast\)): When \(x_{-i} \in X_{-i}\) is such that \(i = i^1(\rho_i, x_{-i})\), i.e. \(i\) "wins," then \(u_i^1(\rho_i, x_{-i}) = \rho_i - c_i = -c_i/n\). Otherwise, when \(i \neq j = i^1(\rho_i, x_{-i})\) that is when \(M^k(\rho_i, x_{-i}) = x_j < \rho_i\) for some \(j \in N \setminus \{i\}\) or \(M^1(\rho_i, x_{-i}) = x_j \leq \rho_i\) for some \(j < i\), we have

\[ u_i^1(\rho_i, x_{-i}) = -M^1(\rho_i, x_{-i})/(n-1) \geq -\rho_i/(n-1) = -c_i/n. \]

Thus, the least utility \(u_i^1\) for \(i\) when \(i\) bids \(x = \rho_i\) is \(u_i^1(\rho_i, x_{-i}) = -c_i/n\), attained when the rest of the bidders bid \(x_{-i}\) so as to let \(i\) "win" (\(i = i^1(\rho_i, x_{-i})\)).

Now we regard the right hand side to see that the least utility for \(i\) is less than \(-c_i/n\) when \(x_i \neq \rho_i\). This is clear when \(x_i < \rho_i\), for then taking \(x_j = x_i + 1\) for each \(j \in N \setminus \{i\}\), we have \(i = i^1(x_i, X_{-i})\) and \(u_i^1(x_i, x_{-i}) = x_i - c_i < \rho_i - c_i = -c_i/n\). When \(x_i > \rho_i\), take \(x_i = (x_i + \rho_i)/2\) for each \(j \in N \setminus \{i\}\), so that \(i\) loses and pays \((x_i + \rho_i)/2\) with resulting utility \(u_i^1(x_i, x_{-i}) = -(x_i + \rho_i)/2(n-1) < -\rho_i/(n-1) = -c_i/n\). Thus, the least utility for \(i\) when \(i\) is bidding \(x_i \neq \rho_i\) is less than \(-c_i/n\) and just reaches this value when \(x_i = \rho_i\) as to be shown.

Q.E.D.

7. **Lemma:** Given any \(u \in U\) and \(k \in N\), consider the game \(\Gamma^k[u]\), and take \(x \in X\) and any \(i \in N\).

1. If \(M^k(x) < \rho_i\) and \(i = i^1(x)\), then \(u_i^k(\rho_i, x_{-i}) > u_i^k(x)\).
2. If $M^k(x) > \rho_i$ and $i \neq i^1(x)$, then there exists $y_i < x_i$ such that $u^k_i(y_i, x_{-i}) > u^k_i(x).

Proof: (ad 1): Assume the hypothesis of (1). Compute that

$$u^k_i(x) = M^k(x) - c_i < \rho_i - c_i = -c_i/n.$$  

Set $y = (\rho_i, x_{-i})$. Now there are just two cases: (a) $i^1(y) = i$ (i "wins" at y), or (b) $i^1(y) \neq i$. In case (a),

$$u^k_i(y) = M^k(y) - c_i \geq M^1(y) - c_i = \rho_i - c_i > u^k_i(x).$$

and there is nothing left to prove. In case (b),

$$u^k_i(y) = -M^k(y)/(n-1),$$

and to complete the proof, we argue that $M^k(y) \leq \rho_i$, for in that case,

$$u^k_i(y) \geq -\rho_i/(n-1) = -c_i/n > u^k_i(x).$$

To see $M^k(y) \leq \rho_i$, note that the only difference between $x$ and $y$ is that $y_i = \rho_i$ while $x_i = M^1(x) \leq M^k(x) < \rho_i$. Thus, it is clear that $\{j \in N \mid y_j \leq \rho_i\} = \{j \in N \mid x_j \leq \rho_i\}$ and this set has at least $k$ members since $M^k(x) < \rho_i$, hence $M^k(y) \leq \rho_i$, completing the proof of (1).

(ad 2): Assume the hypothesis of (2). Now there are just two cases: (I) there exists $z_i < M^k(x)$ such that $M^k(z_i, x_{-i}) = M^k(x)$, or (II) $z_i < M^k(x) \Rightarrow M^k(z_i, x_{-i}) < M^k(x)$. 

16
In case (I), writing \( z = (z_i, x_{-i}) \) by the non-decreasing monotonicity of \( M^k \), we have \( z_i < M^k(z) \) and so \( i \in W^k(z) \), but also \( i \neq i^k(z) \). Thus, for any \( y_i \leq z_i, W^k(y_i, x_{-i}) = W^k(z) \) and \( M^k(y_i, x_{-i}) = M^k(z) = M^k(x) \). In particular, taking any \( y_i < M^1(x) \), and writing \( y = (y_i, x_{-i}) \), we have \( M^k(y) = M^k(x) \) and \( i^1(y) = i \), so

\[
u_i^k(y) = M^k(x) - c_i > \rho_i - c_i = -c_i/n = -\rho_i/(n - 1) > M^k(x)/(n - 1) = u_i^k(x),
\]
as desired. Assuming (II) and recalling Lemma 4, we have \( \rho_i \leq M^k(\rho_i, x_{-i}) < M^k(x) \), and either (a) \( i = i^1(\rho_i, x_{-i}) \) or (b) \( i \neq i^1(\rho_i, x_{-i}) \). In case (a),

\[
u_i^k(\rho_i, x_{-i}) = M^k(\rho_i, x_{-i}) \geq \rho_i - c_i,
\]
which we already saw strictly to exceed \( u_i^k(x) \). In case (b), we have directly

\[
u_i^k(\rho_i, x_{-i}) = M^k(\rho_i, x_{-i})/(n - 1) > -M^k(x)/(n - 1) = u_i^k(x).
\]

This completes the proof of (2) and of the Lemma. Q.E.D.

What Lemma 7 states is, in plain English:

1. If the winner's compensation falls below the winner's (first) lowest-bidder-auction prudent bid, then the winner is better off bidding this "prudent" bid.

2. If the winner's compensation exceeds a non-winning bidder's lowest-bidder-auction prudent bid, then this non-winning bidder is better off decreasing his bid by some.
Lemma 7 also makes clear why the cost burdens distributed by the \( k^{th} \) lowest bidder auction still do not exceed the equal split (among the \( m - 1 \) non-providers) of \( \rho_2 \), the prudent bid \((\Gamma^1[u])\) of the second lowest cost bidder. In particular, effective competition is really among the first two players, no matter what the order of the auction is. The role of the remaining agents (see 5(2)) is to enforce this competition. Agents 1 and 2 profit directly from any competitive advantage they may enjoy relative to other agents. But no agent is worse off, and all agents may be better off at the equilibrium outcome of \( \Gamma^k[u] \), if any of their members becomes more efficient in providing the public good.

8. **Corollary:** Given any \( u \in U \) and any \( k \in N \), regard any Cournot-Nash equilibrium \( x \in \sigma_0(\Gamma^k[u]) \) of the game \( \Gamma^k[u] \), and take any \( i \in N \).

1. If \( i = i^1(x) \), then \( M^k(x) \geq \rho_i \).

2. If \( i \neq i^1(x) \), then \( M^k(x) \leq \rho_i \).

**Proof of Theorem:** ad (1): Let us first note that \( \sigma_0(\Gamma^k[u]) \) is nonempty since \( \{x_i = \rho_1(u) | i \in N \} \in \sigma_0(\Gamma^k[u]) \) for every \( k \in N \). Thus, take any \( x \in \sigma_0(\Gamma^k[u]) \) and regard \( \mu^k(x) = (i^1(x), s^k(x)) \). Writing \( i = i^1(x) \) and \( s^k_j = s^k_i(x) \), to prove (1), we just need to show that \( i \in I(u) \) and \( s^k_j \in [\rho_1, \rho_2] \). (The function \( \mu^k \) sets, in any case, \( s^k_j = -s^k_i/(n - 1) \) for each \( j \in N \setminus \{i\} \).)

To see that \( i \in I(u) \), recall that \( c_1 \leq \ldots \leq c_n \). If \( i = 1 \), then directly \( i \in I(u) \). Now 8(1) tells us that \( M^k(x) \geq \rho_1 \), and, if \( i \neq 1 \), then 8(2) tells us furthermore that \( \rho_1 \geq M^k(x) \), so we form the inequality chain

\[
\frac{(n - 1)}{n} c_i = \rho_1 \geq M^k(x) \geq \rho_i = \frac{n - 1}{n} c_i
\]
to see that \( c_1 \geq c_i \). Thus, in any case, we have \( i \in I(u) \).

Next, regarding \( s_i^k \), first observe that, as \( \rho_i \geq \rho_1 \) (since \( c_j \geq c_1 \) for all \( j \in N \) ) and \( s_i^k = M^k(x) \geq \rho_1 \), we actually have \( s_i^k \geq \rho_1 \). Now, if \( i = 2 \), then \( i \neq 1 \) and \( 8(2) \) gives us \( M^k(x) \leq \rho_1 \); while, if \( i \neq 2 \), then \( 8(2) \) directly gives \( M^k(x) \leq \rho_2 \). Thus, in any case, we have \( s_i^k \in [\rho_1, \rho_2] \).

So much shows that \( \mu^k(x) \in \alpha(u) \), thereby proving (1).

ad(2): Taking any \( k \in N \) satisfying the hypothesis of (2), to see (2), now pick any \( (i, t) \in \alpha(u) \). Thus, \( i \in I(u) \) and we have \( t_i \) in \( [\rho_1, \rho_2] \) with \( t_j = -t_i/(n - 1) \) for all \( j \in N \setminus \{i\} \). We set \( x = \{x_j\}_{j \in N} \) with

\[
x_j = \begin{cases}
t_i + 1 & \text{if } j < i \\
(j \in N) & \\
t_i & \text{if } j \geq i.
\end{cases}
\]

Now \( x_i = t_i \) and \( i = i^1(x) \), while the hypothesis of (2), namely that there are enough (at least \( \max\{1, k - 1\} \)) agents \( j \in N \) with non-minimal \( c_j \) and \( j \geq i \), guarantees that the bid \( M^k(x) \) or the \( k \)-th lowest bidder is \( t_i \). Thus, \( (i, t) = \mu^k(x) \). It remains only to check that \( x \) is a Cournot-Nash solution of the game \( \Gamma^k[u] \).

We begin this exercise by considering the plight of bidder \( i \). As the winner of the auction, this agent reaps utility

\[
u_i^k(x) = t_i - c_i = M^k(x) - c_i \geq \rho_i - c_i = -c_i/n,
\]

where we have used \( 8(1) \) to place the weak inequality sign. Now regard \( u_i(y) \) for any \( y = (y_i, s_{-i}) \) with \( y_i \in \mathbb{R} \). If \( y - i < x_i \), then \( i \) is still the winner \( (i = i^1(y)) \) at \( y \) and the
non-decreasing monotonicity of \(M^k\) yields \(u_i^k(y) = M^k(y) - c_i \geq M^k(x) - c_i = u_i^k(x)\). If instead \(y_i > x_i\), then \(i\) is no longer winner \((i \neq i^1(y))\), while \(M^k(y) \geq M^k(x) \geq \rho_i\), and so

\[u_i^k(y) = -M^k(y)/(n-1) \leq -M^k(x)/(n-1) \leq -\rho_i/(n-1) = u_i^k(x)\.

Thus, \(x_i = t_i\) is optimal for \(i\) in response to \(x_{-i}\) in the game \(\Gamma^k[u]\).

We run through similar reasoning for the other agents \(j \in N\setminus\{i\}\) to see that \(x_j\) is an optimal response to \(x_{-j}\). First consider any such \(j > i\). Now decreasing \(x_j\) below \(t_j\) will turn \(j\) into winner and from (7(1)) we know that this will hurt (the "winner's curse"). On the other hand, increasing \(x_j\) will leave \(i\) as the winner and, if anything, hurt \(j\) by increasing the payment to be made to \(i\). [For either the bid \(M^k\) of the \(k^{th}\) lowest bidder will be unaltered, causing no welfare changes, or \(M^k\) will rise (when \(i = n-1, j = n\) and \(k = 2\)) and with it the utility \(u_j^k = -M^k/(n-1)\) will fall.] Thus, \(x_j\) is \(j\)'s best response to \(x_{-j}\) when \(i < j \in N\).

Finally, if \(i = 1\) there is nothing left to prove, so assume that \(i \geq 2\) and take any \(j \in N\) with \(j < i\). Now \(c_j \leq c_j \leq c_i\) and \(i \in I(u)\), so in fact \(\rho_j = \rho_i = \rho_1 = t_i\). In this case, a small decrease in \(x_j\) (to a level exceeding \(x_i\)) will leave both the winner \((i^1 = i)\) and the bid \((M^k)\) of the \(k^{th}\) lowest bidder unchanged, thus bringing about no increase in the utility \(u_j^k = -M^k/(n-1)\), but decreasing \(x_j\) to any \(y_j \leq x_i = t_i\) will turn \(j\) into the winner and bring \(u_j^k\) from \(-\rho_1/(n-1)\) to \(y_j - c_j \leq \rho_j - c_j = \rho_1 - c_1 = -\rho_1/(n-1)\). As to increasing \(x_j\), clearly it will alter neither the winner, nor payments, nor then welfare. Thus, \(x_j\) is again \(j\)'s best response to \(x_{-j}\).

This shows that \(x \in o_0(\Gamma^k[u])\), completing the proof of (2) and of the theorem. Q.E.D.

9. **Closing Remarks:** In closing this paper we wish to remark on a number of aspects of our
mechanism.

**Balanced Budget:** All the mechanisms considered generate no deficit or surplus of funds collected for the provision of the public good, once the provider has been compensated. This is true in and out of equilibrium.

**Fairness:** Fairness properties of our mechanisms may be noted in two separate veins. For one, our solution is nominally egalitarian in the common parlance sense that all nonproviders pay equal sums for the provision of the public good. Second, it is fair in the sense that it is envy free: no agent \( i \in N \) would strictly prefer to be in the position of any agent \( j \in N \setminus \{i\} \).

To see this, the reader can first check that no two nonproviders can envy one another, since their "positions" are the same: they are both nonproviders and they both pay \( s_i/(n-1) \).

Neither would a nonprovider prefer to be in the position of the providing winner, since by Lemma 7 every nonprovider is less well off providing the public good at any bid less than his prudent bid. Lemma 7 also tells us that our solution \( \alpha \) permits no "winner's curse" to occur, i.e. under \( \alpha \), the winner of the auction never envies a nonprovider's position.

**Individual Rationality (IR):** Regarding IR, let us first spell out what that would mean here, done best perhaps with resort to our example. It is easy to see that if everyone would have done the project by himself, then he certainly would find it IR to join the community. Suppose this is not the case; for instance, in our example, agent 1 would not undertake the public project if he were all by himself, because his benefit from undertaking the project would be \( \beta_1(1) = 1 \), while his cost would be \( \gamma_1 = 2 \), exceeding \( \beta_1(1) \), so that \( c_1(1) = 1 \), i.e., the net cost is positive and it stands uncompensated in a singleton society. On the other hand, agent 1 receives at least \( u_1(1, \rho_1) = \rho_1 - c_1(1) = 1/3 \) by participating in the community under our mechanism.
Regarding IR, take any outcome \((i^*, s^*)\) from any of the auction mechanisms above. By the Theorem, \((i^*, s^*) \in \alpha(u)\). But as in the proof of 6, it follows that the \(u_i(i^*, s^*) \geq -c_i/n\), for each \(i \in N\). Thus, participating in the community with the indicated mechanism is IR when, for each agent \(i \in N\), the default option defining IR entails net benefits no larger than \(-c_i/n\). This would be the case, for example, if the following two conditions hold:

i. Each \(c_i \geq 0\) so that each agent weakly prefers someone else to provide the public good, unless he is compensated for so doing.

ii. The default option for each agent is to undertake the project himself, incurring cost \(c_i\) and leading to the utility at default option of \(u_i(\text{default}) = -c_i\).

From conditions (i-ii) we see that IR is likely to be satisfied for public good provisioning problems which entail a nuisance factor or a net cost for the service provider, as would be the case, for example, with noxious facilities.

It is interesting to contrast the "noxious facility" case just described with a case like the "Capitol of Europe" problem for which it seems likely that \(c_i < 0\) for some \(i \in N\), i.e., some country would be willing to pay something in order to host the capitol of Europe. In the Capitol of Europe problem, it is obvious that some host country must be selected, and the default option of not joining in the decision procedure to select a host country for the capitol would not seem to be available to any country (i.e., the default option of non-cooperation would entail a large negative payoff). In this situation, the mechanisms described here would therefore still yield an IR solution, even though the solution itself would likely yield payments (probably in kind, e.g., in new embassy buildings) by the "winner" of \(-c_i/(n - 1) > 0\) to each non-host country.
In general, however, where \( c_i < 0 \), the default option may be preferable to paying (a total of) \(-c_i\) to other cooperating agents. Thus, when \( c_i < 0 \), the mechanisms/procedures described here may not be IR. When they are not IR, they are also unlikely to attract agents who can provide the public good efficiently.

**Successfulness:** A necessary and sufficient condition that the mechanisms described here will select a solution that should be implemented, as opposed to the default option, is that the sum of benefits exceeds the sum of costs at some (and therefore also at the efficient) solution \( j \in N \). Since the outcome of any of these mechanisms is in the efficient set \( \alpha(u) \), successfulness is implied by IR.

**Information:** We do not intend to enter the usual debate as to the informational requirements for Nash equilibrium to make sense. We assume that \( u \in U \) is commonly known to all participants \( i \in N \). The mechanism designer knows only that the profile \( u \) comes from \( U \). Some initial theoretical and experimental results on the performance of these auction mechanisms under imperfect information are given in Kunreuther et al. [8] and Knez [6].
REFERENCES


FOOTNOTES

1. We thank Jacques Drèze for suggesting this application.

2. Thus, the matter of whether our mechanisms are "successful" in the terminology of Jackson and Moulin [4] is not an issue here.

3. For a simple account of Dutta and Sen and Moore and Repullo, see Koray and Sertel [7]. For a dashing summary of some pertinent aspects of the implementation literature in the context of public good provision, see Jackson and Moulin [4].

4. For implementation theory, we use the terminology of Maskin [9].

5. Note that we are using here the fact that $X = \mathbb{R}^N$, so that bids are unbounded below.