Optimal Risk-Sharing under Dual Asymmetry of both Information and Market Power: A Unifying Approach*

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Abstract

The impact of information distribution on optimal risk-sharing between different parties has been an important field of research over the past 30 years. In the literature of insurance, however, the focus has remained essentially on one configuration: insurers (the principal) propose a contract to insured (the agent), but are unable to observe perfectly the risk of their insured who can. This asymmetry of information is the source of the well-known adverse selection effect that leads higher risk individuals to buy full or near-full insurance, while lower risk buy less complete coverage, if they buy at all. While the prediction seems to hold in some markets, there is growing empirical evidence it does not in a number of others (advantageous selection or no selection at all).

This paper proposes a model of risk-sharing between two risk-averse agents that expands the traditional approach by studying 4 different cases. It comprises two possible allocations of information (the insured knows the risk better, or the insurer knows the risk better) and of market power (the principal is the insurer, or the insured). We show that when the insurer is the principal and is less informed, we have the traditional adverse selection effect: high risks are optimally covered, low risks are not. When the insurer is the principal but knows more about the risk than the insured (*reversed* asymmetric information), the result is reversed: high risks are not optimally covered, but low risks are. These results are mirrored when the market power is reversed (i.e., the insured is the principal, not the insurer). Many economic situations are specific cases of our setting.

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1 Introduction

Starting with seminal work by Akerlof [1], Rothschild-Stiglitz [22], Stiglitz [23] and Wilson [25], the impact of information distribution on risk-sharing has been an important field of research over the past 30 years. In the insurance literature, one focus has been related to asymmetric information between the insurer and the insured, and its impact on adverse selection and moral hazard.

In a world where an agent is unable to observe perfectly the risk she covers from another who can, this asymmetry of information is the source of the well-known adverse selection effect in which low-risk people are induced to under-cover their assets. While insurance is the prime application we study here, our approach also applies to other research fields studying adverse selection and signaling models (e.g., finance, banking, health, education, to name a few).

When an insurer is unable to distinguish different types of risk among individuals in a group, selling the same contract (at an average price) to all its customers amounts to overcharging low-risk applicants (which decreases demand) and undercharging high-risk ones. In some sense, the presence of high-risk agents has a negative externality on low-risk individuals.

One of the key results of this theoretical field of research lies in the case when it does exist, the equilibrium is a separating one in which high risk people are perfectly insured whereas low-risk ones are only partially covered (compared to what would it be under perfect information about the risk type) (Rothschild and Stiglitz [22], Stiglitz [23], Wilson [25])[1].

New Perspectives from Empirical Studies

Recently, there has been a growing interest in investigating whether the results contained in theoretical litterature over the past twenty-five years were consistent with empirical data[2]. Beyond the difficulties inherent to correctly defining empirical tests[3], several recent papers show that there does not exist empirical evidence allowing one to conclude the presence of adverse selection in the studied markets (life insurance, long-term care insurance, automobile insurance);

[1] For a survey of adverse selection in insurance markets, see Dionne, Doherty and Fombaron [11]. See also Chiappori and Salanie [6] and Chiappori et al. [7].

[2] For a survey of recent empirical works see Culter, Finkelstein and McGarry (2006).

[3] See for instance Chiappori and Salanie [6] and Dionne, Gourieroux and Vanasse [12] on Puelz and Snow [21].

i.e. the presence of positive correlation between insurance coverage and risk occurrence[4].

This absence of positive correlation is shown by Cawley and Philipson [4] who study the covariance between contract size and risk in the U.S. life insurance market. This study actually demonstrates the existence of advantageous selection; a similar result is obtained by McCarthy and Mitchell who study the US, UK and Japanese life insurance markets [18].

Finkelstein and McGarry [14] find no evidence of positive correlation between insurance coverage and risk occurrence in the long-term care insurance market in the U.S. either. Cardon and Hendel [3] arrive at the same conclusion about the U.S. health insurance market they study.

Mitigate results were obtained on the automobile insurance market. Chiappori and Salanie [6] find no evidence in the French market of car insurance for young drivers. Cohen [8] obtains similar results for young drivers in Israel. In the same way, using a sample of Canadian automobile insurance policies, Dionne, Gourieroux and Vanasse [12,13] show that such a positive correlation is not present in the studied sample[5].

How should we interpret this absence of a positive coverage-risk correlation in these insurance markets? There is obviously not a single answer to that question, but we can imagine at least two possible explanations (non-exclusive).

(a) One possibility is that low risk individuals might be more risk averse and then more concerned with protecting themselves. They end up being more careful, investing more in protection measures and also being ones who buy more insurance. Converse to adverse selection results, low risk individuals would thus tend to hold *more* coverage than high risk ones[6].

[4]That is not saying that the presence of adverse selection is rejected in all markets. For example, Makki and Somwaru [16] analyze farmers choices of crop insurance contracts by selecting a sample of nearly 60,000 U.S. farm-level insurance records over ten years. They offer empirical evidence of adverse selection in this market. More precisely, they show that low-risk farmers are overcharged (pay more than their competitive rates) and high-risk farmers are undercharged (tables 1 and 2, p702). Culter [8] discusses a large selection of literature which suggests the presence of asymmetry of information in health insurance markets. Browne and Doerpinghaus [2] also find evidence of adverse selection in their sample of individual health insurance. Finkelstein and Poterba [13] find evidence of a positive coverage-risk correlation in the annuities markets in the U.K as well.

[5]In order to test the prediction of Rothschild–Stiglitz and Wilson that low-risk individuals will choose a higher deductible and higher-risk individuals will choose a smaller deductible, the paper analyses the deductible choices of nearly 4,700 policyholders of a large private insurer in Quebec. The authors show that there is no adverse selection on risk types in that portfolio.

[6]For example Finkelstein and McGarry [14] conclude that more cautious individuals are both more likely to have insurance and be less risky. See also Culter, Finkelstein and McGarry [10] for a discussion on the impact on

(b) Another possibility lies more directly in the distribution of information about the risks between the insured and the insurers. More specifically, it is not clear at all that in the insurance markets where there is not adverse selection the insured knows his/her risk better than the insurance company.

For example, in Chiappori and Salanie [6], insured are young drivers. We shall argue that at best these drivers have only prior beliefs about the real risk they are exposed to. Because of the statistical data most insurers collect over time, the insurance company might have a better knowledge about the risk any new driver is exposed to. This asymmetry of information in favor of the insurer over young drivers can decrease over time, and even be reversed as the driver learns more and more about driving conditions and risks associated to it (including her own behavior). Indeed, someone who has been driving for 20 years knows certainly the risk he is exposed to better than someone who just starts driving. In that perspective, the study by Cohen [8] is important because her sample is made of both experienced and inexperienced drivers, which allows comparison. Referring to correlation between risk and coverage, Cohen concludes that "the extent to which such a correlation exists (and indeed whether it exists) varies among subsets of the pool of all new [automobile insurance] policies. Such a correlation does not exist for policyholders with little or no driving experience. Such policyholders might have had relatively little opportunity to obtain private information about their risk type and thereby to gain an informational advantage over the insurer." (p.197)[7]

The reversed asymmetry of information is also suggested in the aforementioned study by Cawley and Philipson. "This evidence is consistent with our more disaggregate analysis which controls for many factors such as income yet still indicates a negative association between insurance coverage and risk ([4], p.830). One potential interpretation of the negative relationship we find between risk and quantity [of insurance] is that insurers can distinguish risks through underwriting and observing systematic patterns in claims over time and then limit coverage to high-risk, instead of low-risks, individuals. (...) Producers in this market, as in many others, may know their costs of production better than customers." ([4], p.842).

Reconsidering Basic Assumptions

The possibility of a reversed asymmetry of information - This possibility of reversed asym-

preference heterogeneity on insurance decision (U.S. life insurance market).

[7] However, the positive correlation between high coverage (low deductible) and risk occurrence does exist for new customers who have had three or more years of driving experience (Cohen [8]).

metric information in favor of the insurance company over their applicants has been somewhat overlooked by literature. Only a few recent papers started analyzing the possibility that in a principal-agent relationship as a non-cooperative game, the principal, who proposes the contract, may have *ex ante* relevant private information at the contracting date.

This structure of information corresponds to the general framework studied by Maskin and Tirole [20] in which the principal is informed about a common value parameter. Such an assumption has also been more recently considered in an insurer-insured relationship by Villeneuve [24] for a monopolistic insurance market. The model we introduce below captures these two situations: the insured being more informed about her type than the insurer *or* the insurer being more informed.

The possibility of a reversed market power - Here we also like to expand our model on another assumption traditionally made in the litterature. We would like to consider a more general case than the traditional principal-agent relationship where the covering part (insurer) is the principal offering a set of contracts to the agent (insured). Although this assumption might first appear somewhat counter-intuitive, there are actually many situations where it exists. Imagine for instance a very large industrial group negotiating its insurance contract with an insurer. If that group constitutes an important part of the insurer's portfolio, it is not unrealistic to consider the possibility that the insured is the party with more market power, and can thus control the negotiation. In that case, the insurer can be put in a position in which a specific menu of contracts (payment of a certain premium, deductible and total coverage) is suggested by the large industrial group rather then the insurer. The insurer has to chose one of these contracts, or refuse all of them. In the latter case, the group will make the deal with another insurer. In other words, in a principal-agent framework, this is a situation where insurer is not the principal anymore but the agent. The model we suggest below captures also these two mirror configurations (principal-agent relationship reversed).

Risk Aversion of the Insurer - Another assumption asserted in most of the previously quoted theoretical works lies in seeing the insurer as the risk-neutral principal and the insured as the risk-averse agent. That assumption might go unquestioned for insurers with a large financial base and a highly diversify portfolio. But this risk neutrality assumption may fall for many others. And even large insurers are certainly concerned with risks which potential losses are large and highly correlated. Consider catastrophic risks for which the insuring part may be extremely exposed to serious problems of liquidity or even bankruptcy[8]. [9]. Without further specification, the model introduced here deals with a general case of two risk-averse agents. Final section of this paper presents an application of our model with a risk-neutral insurer.

Main Findings

We propose hereafter a model of risk-sharing between two risk-averse agents. Depending on the configuration we study, one will be the risk-covering party, the other one the risk-covered party. As mentioned before, our setting is very general so the results of the model ought to apply to a number of other situations than just insurance. The model allows covering 4 possible cases: 2 scenarios of information distribution (the insurer knows the risk better, or the insurer knows the risk better) and 2 allocations of market power (the insurer is the principal, or the insured is the principal). When the principal has better information we have a signaling game, and when the agent has better information we have an adverse selection model. We now summarize our main findings (see Figure 1).

- When the insurer (as the principal) has *no* or *less* information than the insured (as the agent), the model shows that there is *no* distortion for a *high* risk individual who is fully covered at the equilibrium ($\overline{H}$, upper-bar for optimality). The model also shows that there is *sub*-optimal risk-sharing arrangements for *low* risk (L, lower-bar for sub-optimality). The risk-sharing equilibrium is thus such that we have ($\underline{L}$,$\overline{H}$) in Figure 1. Case (a) at the North-West corner [10].

- When the insurer (as the principal) is *more* informed than the insured (the agent), the results are reversed: there is *no* distortion for *low* risk as optimal risk-sharing is achieved ($\overline{L}$), but only a *sub*-optimal risk-sharing arrangement can be obtained for *high* risk ($\underline{H}$). This risk-type equilibrium is represented by ($\overline{L}$,$\underline{H}$) in Figure 1; Case (b) at the South-West corner.

[8] Terrorism risk certainly illustrates the risk aversion of insurers. The terrorist attacks of September 11, 2001 inflicted 35 billion of dollars of insured losses, and plausible scenarios of attacks in the US and in Europe would have much larger economic consequences. (Kunreuther and Michel-Kerjan [17]).

[9] Ambiguity aversion is also something impooirtant, although we do not discuss that in this paper). Indeed, there is evidence from behavioral economic literature that insurers would charge a much higher price for ambiguous risk than for risk they have good knowledge of (Kunreuther, Meszaros, Hogarth, and Spranca [16]). Low probability/high consequences events offer a good illustration of insurers' ambiguity and risk aversion.

[10] These results are consistent with Stiglitz (1977).

	Principal	
	Insurer	**Insuree**
Not informed (Adverse selection game)	$(\underline{L}, \overline{H})$	$(\overline{L}, \underline{H})$
Informed (Signaling game)	$(\overline{L}, \underline{H})$	$(\underline{L}, \overline{H})$

Figure 1. Impact of Asymmetry of Information and Market Power on Optimal Risk Sharing.

Interestingly enough, we find mirror results to cases (a) and (b) when the principal-agent relationship is reversed. Indeed, when the principal is the insured who proposes a menu of contracts and the insurer is the agent who choses one of them (if any), we obtain the following results:

- When the insured (now the principal), has *no* or *less* information than the insurer (now the agent), there is *no* distortion for *low* risk; optimal risk-sharing is achieved and the low risk are then fully covered at the equilibrium $(\overline{L})$. But only a *sub*-optimal risk-sharing arrangement can be obtained for *high* risk. This risk-type equilibrium is represented by $(\overline{L},\underline{H})$ in Figure 1; This case (c) at the North-East corner mirrors case (a).
- Finally, when the insured (now the principal) has *more* information than the insurer (the agent), there is *no* distortion for *high* risk as optimal risk-sharing is achieved; only a sub-optimal risk-sharing arrangement can be obtained for low risk: $(\underline{L},\overline{H})$; This case (d) at the South-East corner in Figure 1 mirrors case (b) that the South-West .

The reminder of the paper proceeds as follows. Section 2 sets up the basic model, introduces important definitions and discusses the first best contracts (perfect and symmetrical information). Section 3 introduces imperfect information between the insured and the insurer and the possibility of having the asymmetry of information benefiting either the principal or the agent. In this section we also characterize Perfect Bayesian Equilibria (PBE) in terms of risk-sharing and suggest a simple topological approach for determining the PBE which dominates all the others. Using this approach, we derive the results for each of the four cases described in Figure 1. Section 4 is an application of our general model in which the insurer is risk-neutral. Section 5 concludes the paper. An appendix is offered in section 6.

2 The basic model

2.1 Setting

The setting is the following. Consider an exchange economy with two agents (A and B), one good, and two states of nature (1 and 2). Let $\overline{x}$ denote the total quantity of good in state 1 and $\overline{y}$ in state 2. We suppose, without further loss of generality that $\overline{x} < \overline{y}$. In state 1 the economy bears a total loss of $\overline{y} - \overline{x}$ such that state 2 can be called the "normal state" and the state 1 the "bad state of nature". The initial endowments are (x_0, y_0) for agent A and $(\overline{x} - x_0, \overline{y} - y_0)$ for agent B. A feasible arrangement between A and B is simply a vector $z \equiv (x, y) \in [0, \overline{x}] \times [0, \overline{y}]$ which gives (x, y) to agent A and $(\overline{x} - x, \overline{y} - y)$ to agent B.

As there are only two agents in the economy and the total quantity of good is limited on both states, most of our argument can be made by analysis in an Edgeworth box. In Figure 2, the horizontal and vertical axes (X, Y) represent quantities of good in state 1, and state 2, respectively. The point z_0 with coordinates (x_0, y_0) is the typical agent's endowment without exchange.

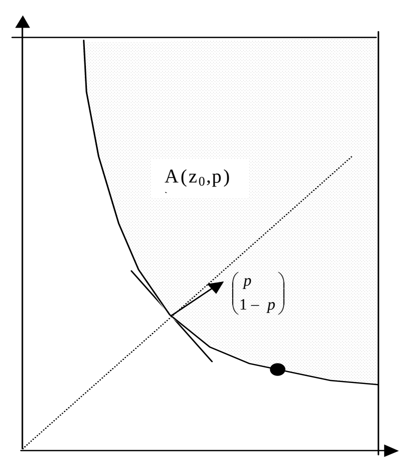

Figure 2. Acceptable contracts

Agent A (respectively B) has a von Neumann strictly concave increasing continuous utility function $u(.)$ (resectively $v(.)$). The probability of state 1 is p, the probability of state 2 is $(1-p)$. p is *a priori* distributed on a compact subset K of $[0,1]$ with a lower bound $\underline{p} = \inf K > 0$ and an upper bound $\overline{p} = \sup K < 1$. The *a priori* distribution of p is a measure f on $[0,1]$.

The final arrangement will be the result of the game. First, we will suppose that A is the principal. She plays first by proposing a contract to agent B who then reacts to agent A's proposal.

The paper considers two main possibilities concerning the nature of the information about p. In the first one, agent A is informed but agent B is not. In the second one, agent B is informed (A is not).

Going forward the following notations are adopted.

A contract (x, y) is denoted z, $z_0 = (x_0, y_0)$ corresponding to the initial situation. For a given contract $z = (x, y)$ the utility achieved by agent A (considering the two possible states of nature) is described by:

$$U(p, z) \equiv pu(x) + (1 - p)u(y) \tag{1}$$

the utility achieved by agent B is:

$$V(p, z) \equiv pv(\overline{x} - x) + (1 - p)v(\overline{y} - y) \tag{2}$$

Rewriting $V(p, z)$ as $V(p, z) = v(\overline{y} - y) - p(v(\overline{y} - y) - v(\overline{x} - x))$ allows the interpretation of the term $v(\overline{y} - y) - v(\overline{x} - x)$ as the impact on agent B (in utility terms) of the total loss she will occur from the bad state of nature. A final arrangement between A and B can increase or decrease this impact with respect to the initial situation. It is hence useful to define two distinct sets of arrangements.

Definition 1.

$L(z_0) = \{z/v(\overline{y} - y) - v(\overline{x} - x) \leq v(\overline{y} - y_0) - v(\overline{x} - x_0)\}$ is the set of contracts which decreases the impact of total loss on agent B (L for loss).

$R(z_0) = \{z/v(\overline{y} - y) - v(\overline{x} - x) \geq v(\overline{y} - y_0) - v(\overline{x} - x_0)\}$ is the set of contracts which increases the impact of total loss on agent B (R for revenue).

Obviously, both parties will not agree on arrangements which do not improve their utility. It is then useful to define the sets of contracts which are rational for each party.

Definition 2. The set of contracts that gives more utility to agent A (resp. B) than the initial endowment is $A(p, z_0) \equiv \{z, U(p, z) \geq U(p, z_0)\}$ (resp. $B(p, z_0) \equiv \{z, V(p, z) \geq V(p, z_0)\}$). As the utility functions of both agents are concave, $A(p, z_0)$ and $B(p, z_0)$ are convex subsets of $[0, \overline{x}] \times [0, \overline{y}]$ (see Figure 2).

The following lemmas show the relationship between the two previous concepts. The first lemma simply means that if an arrangement, which increases the impact of loss on B, is acceptable (by B) for high probability of being in the bad state of nature, it will be also acceptable for low probability. In the same way, if an arrangement which decreases this impact is acceptable for low probability, it is also acceptable for high probability.

Lemma 1 $\forall (p, p') \in K^2$, *with* $p < p'$, *we have:*

$$B(p, z_0) \cap L(z_0) \subset B(p', z_0) \cap L(z_0)$$
$$B(p', z_0) \cap R(z_0) \subset B(p, z_0) \cap R(z_0).$$

Proof. See Appendix. ■

For a given p, $A(p, z_0) \cap B(p, z_0)$ is the set of mutual improving arrangements. Because of the concavity of u and v, $z \in A(p, z_0)$ implies $(x - x_0)u'(x_0) + (y - y_0)u'(y_0) \geq 0$ and $z \in B(p, z_0)$ implies $(x - x_0)v'(\overline{x} - x_0) + (y - y_0)v'(\overline{y} - y_0) \leq 0$.

This leads to the second Lemma:

Lemma 2 $z \in A(p, z_0) \cap B(p, z_0) \Longrightarrow sign(x - x_0) = -sign(y - y_0) = sign\left(\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)}\right)$.

(i) if $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} < 0$, *then* $\forall p \in K$, *we have* $A(p, z_0) \cap B(p, z_0) \subset L(z_0)$ *and* $A(p, z_0) \cap B(p, z_0) \cap R(z_0) = \{z_0\}$

(ii) if $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} > 0$, *then* $\forall p \in K$, *we have* $A(p, z_0) \cap B(p, z_0) \subset R(z_0)$ *and* $A(p, z_0) \cap B(p, z_0) \cap L(z_0) = \{z_0\}$.

Proof. See Appendix. ■

When $\frac{u'(x_0)}{u'(y_0)} < \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)}$, mutual improving arrangements are in $L(z_0)$, as the impact of loss is decreased for B and increased for A. In other words, mutual improving arrangements are such that A *covers* B . In that case we call A the *insurer*, B the *insured*.

When $\frac{u'(x_0)}{u'(y_0)} > \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)}$, the final burden of risk for agent A is decreased by exchange with B; in that case party A is covered by party B.

In that case we call A the *insured*, B the *insurer*. (see Figure 3).

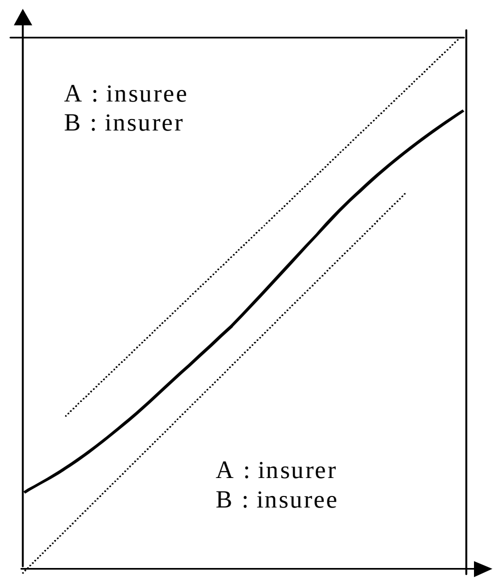

Figure 3. First-best line and zoning

In the next section, we analyze the first best contract under perfect information on risk.

2.2 First best contracts (FB)

In this paragraph, we assume perfect and symmetric information on p. The first best contract is hence defined as follows.

Definition 3. The first best contract $z^*(p, z_0)$ *associated to* z_0 *is the unique one that maximizes* $U(p, z)$*, with respect to* z *in* $B(p, z_0)$*.*

One of the main properties of the von Neumann Morgenstern assumption is that the set of first best contracts when z_0 varies in $[0, \overline{x}] \times [0, \overline{y}]$ does not depend on p.

Proposition 1.

$\forall (p, p') \in K^2$, *the sets* $FB(p) = \{z^*(p, z_0),\ z_0 \in [0, \overline{x}] \times [0, \overline{y}]\}$ and $FB(p') = \{z^*(p', z_0),$ $z_0 \in [0, \overline{x}] \times [0, \overline{y}]\}$ are identical.

Proof. See Appendix. ■

$FB(.)$ shares the set of initial endowments $[0, \overline{x}] \times [0, \overline{y}]$ in two areas: the first one is on the right side and contains the 45° line $\{x = y\}$. If z_0 is in this area, mutual arrangements imply that A plays the role of insurer. The second zone is on the left and contains the 45° line $\{\overline{x} - x = \overline{y} - y\}$. If z_0 belongs to this area, then A is the insured and mutual arrangements will be in $R(z_0)$.

Any first best contract when the probability is p is also a first best contract when the probability is p' for a well chosen initial endowment. This leads to the following important corollary:

Corollary 1:

$$\forall p, p' \; z^*(p, z_0) = \arg\max(U(p, z), V(p, z) \geq V(p, z_0))$$
$$= \arg\max(U(p', z), V(p', z) \geq V(p', z^*(p, z_0)))$$

Proposition 2. Principal may send inaccurate signal.

(i) if $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} < 0$, *, i.e. A is the insurer, then* $\forall p \in K$, $U(p, z^*(p', z_0))$ is strictly increasing with p'.

(ii) if $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} > 0$, *i.e. A is the insured, then* $\forall p \in K$, $U(p, z^*(p', z_0))$ is strictly decreasing with p'.

Proof. See Appendix. ■

Proposition 2 is important because it illustrates how adverse selection can occur. Suppose for instance that B has no information on risk, whereas A is informed. If A is the insurer, that is $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} < 0$, then she might be induced to fool B, by convincing him that the probability of state 1 (the bad state), p, is very high.

Conversely if A is the insured she might be induced to convince B that p is low.

Figure 4 gives a graphical representation of the proposition 2 when A is the insured.

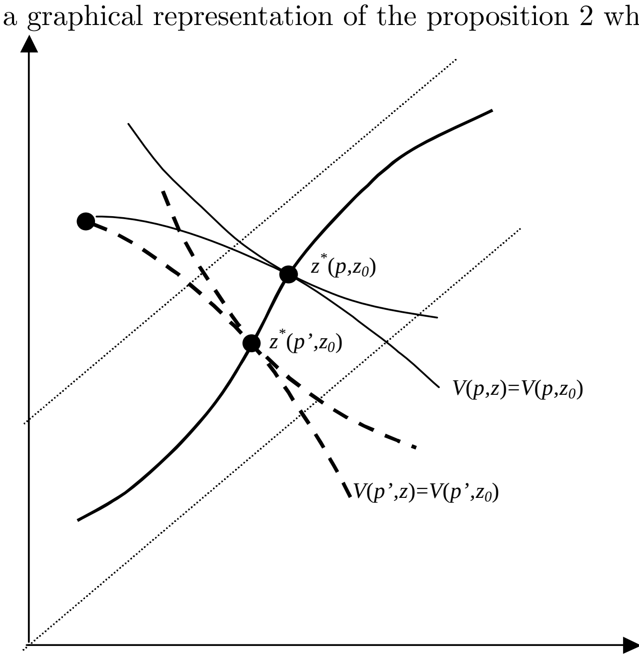

Figure 4. Monotonicity on the first-best line

We now turn to a more detailed analysis of the case of asymmetric information and how it impacts the result of the game.

3 Asymmetric information

This section is divided into two subsections. In the first one, agent A who proposes a contract is *informed* of the true probability p. In the second one, agent A who proposes a contract is *not informed*: she has only *a priori* belief of p. Agent B will accept or refuse the offer taking into account the knowledgeof the risk sha might have or not.

In the model, we assume that both agents know not only whether they are informed but also whether the other party is informed about the probability of the bad state of nature 1. For sake of simplicity in the analysis, only these two symmetrical cases will be studied" A knows the risk, B has only prior beliefs; or B knows the risk, A has only prior beliefs..

Hence, the paper describes four cases depending on both the nature of the information asymmetry between A and B and on whether $\frac{u'(x_0)}{u'(y_0)}$ is higher or lower than $\frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)}$.

3.1 Agent A is informed

Agent B is not informed when receiving the proposition of contract by agent A.

Seeing the proposition $z = (x, y)$, made by A, agent B will update his belief on the *a priori* distribution of p. Finally, he accepts or refuses the offer taking into account his updated belief on probability.

Definition 4. An updated belief is a mapping that assigns to each contract z a probability distribution $g(z)$ on K. $E_{g(z)}(p)$ denotes the expected probability of state 1, for agent B.

Agent B will accept the offer z only if $V(E_{g(z)}(p), z) \geq V(E_{g(z)}(p), z_0)$, that is: $z \in B(E_{g(z)}(p), z_0)$. To each $g(.)$, one can hence associate the family of contracts that would be accepted by B if proposed by agent A.

Definition 5. Let $F(g)$ be the family of accepted contracts when the belief is $g(.)$. Formally,

$$F(g) = \{z / V(E_{g(z)}(p), z) \geq V(E_{g(z)}(p), z_0)\}).$$

It is easy to see that $F(g)$ is a closed subset of $[0, \overline{x}] \times [0, \overline{y}]$

Reciprocally, the question is to determine under which conditions a given (closed) family of contracts is associated to a belief such that all these contracts would be accepted ? The answer is given by proposition 3.

Proposition 3: *For a given closed family F, there exists at least one belief g such that $F = F(g)$ if and only if:*

$$B(\underline{p}, z_0) \cap L(z_0) \subset F \cap L(z_0) \subset B(\overline{p}, z_0) \cap L(z_0)$$
$$\text{and } B(\overline{p}, z_0) \cap R(z_0) \subset F \cap R(z_0) \subset B(\underline{p}, z_0) \cap R(z_0).$$

We will say that such families are "rationalizable".

Proof. See Appendix. ■

Definition 6. In the set of rationalizable families we define the subset Σ of "consistent families" as follows: a family F belongs to Σ if and only if $\forall p \in K$, $\arg\max\{U(p,z),\ z \in F\} \in B(p, z_0)$.

A rationalizable family (i.e. there exists an implicit belief under which each contract in the family is accepted by B), is consistent if the optimal contract in F for A , who knows the true value p, is accepted by B if he believes that the true probability is p. In other words, the true probability p is an implicit belief associated to the best contract for A in F.

It can be remarked that the subset Σ is not empty: the family $F_0 = B(\overline{p}, z_0) \cap B(\underline{p}, z_0)$,that is the family of contracts that are always accepted, whatever the risk, belongs to Σ. Indeed, for all p, $\arg\max\{U(p,z),\ z \in F_0\}$ lies in $B(\overline{p}, z_0) \cap B(\underline{p}, z_0)$ and either in $L(z_0)$ or in $R(z_0)$, that is either in $B(\underline{p}, z_0) \cap L(z_0)$ or in $B(\overline{p}, z_0) \cap R(z_0)$. Using lemma 1, $\arg\max\{U(p,z),\ z \in F_0\}$ is either in $B(p, z_0) \cap L(z_0)$ or in $B(p, z_0) \cap R(z_0)$. This implies that $\arg\max\{U(p,z),\ z \in F\}$ belongs to $B(p, z_0)$. We are now in position to examine some kind of perfect Bayesian equilibria of the game.

Definition 7. A weak-separating perfect bayesian equilibrium, is a mapping $\widehat{z}(p)$ and a belief g^* such that:

(i) $\widehat{z}(.)$ is almost injective : $\widehat{z}(p) = \widehat{z}(p') \Rightarrow \{p = p' \text{or } \widehat{z}(p) = \widehat{z}(p') = z_0\}$

(ii) $\widehat{z}(p) \neq z_0 \Rightarrow g^*(\widehat{z}(p)) = \delta_p$ (that is the Dirac mass on p)

(iii) *$\widehat{z}(p)$ is the best contract for A in $F(g^*)$,*

In such a PBE, pooling can only occur on z_0, that is only in the case of no exchange. The set of weak-separating perfect bayesian equilibria obviously contains the set of separating equilibria.

We have the following proposition:

Proposition 4: *Given a consistent family F in Σ, if there exists an almost injective selection $\widetilde{z}(p)$ of* $\arg\max\{U(p,z),\ z \in F\}$, *then $\widetilde{z}(p)$ will be the equilibrium contract of an weak-separating perfect Bayesian equilibrium.*

Reciprocally, given an weak-separating PBE, $F(g^*)$ is obviously in Σ .

The proof of this proposition is straightforward and hence omitted. We can now state our first theorem:

Theorem 1:

(i) *if* $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} > 0$, *for all weak-separating perfect bayesian equilibrium* $(\widehat{z}(p), g^*)$ *we have:*

$$\widehat{z}(\overline{p}) = \arg\max\{U(\overline{p},z),\ z \in B(\overline{p},z_0)\} = z^*(\overline{p},z_0)$$
$$\text{and } \forall p \in K,\ p < \overline{p}\ ,\ U(p,\widehat{z}(p)) < U(p,z^*(p,z_0)).$$

(ii) *if* $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} < 0$, *for all weak-separating perfect bayesian equilibrium* $(\widehat{z}(p), g^*)$ *we have:*

$$\widehat{z}(\underline{p}) = \arg\max\{U(\underline{p},z),\ z \in B(\underline{p})\} = z^*(\underline{p},z_0)$$
$$\text{and } \forall p \in K,\ p > \underline{p}\ ,\ \ U(p,\widehat{z}(p)) < U(p,z^*(p,z_0)).$$

Proof. See Appendix. ■

Theorem 1 offers several results.

As shown by (i), when the informed principal A is the insured, in all separating PBE, when $p = \overline{p}$ (that is when the risk is high), the equilibrium contract is the first best one. Hence, there is no distortion for a high-risk individual: optimal risk sharing is achieved (first best). However, only sub-optimal risk-sharing arrangements can be obtained for a lower risk individual.

As shown by (ii), when the informed principal A is the insurer, the equilibrium contract is the first best one when $p = \underline{p}$ (low risk). Here, there is no distortion for a low-risk individual and sub-optimal arrangements for a high-risk individual.

However, as usual, there exists several weak-separating perfect bayesian equilibria (PBEs).

Some refinements could be introduced to select the equilibria which are the more plausible. One approach in that case could consist on using the Intuitive Criterion of Cho and Kreps [4] to refine PBEs.

We adopt here another (but rather similar) point of view. We will show that there exists a particular PBE which dominates (from A view point) all the weak-separating PBEs. The idea is quite natural and uses some remarkable features of the set of consistent families Σ.

- First, let notice that Σ is stable for the set union operation, that is if $F \in \Sigma$, and $F' \in \Sigma$ then $F \cup F' \in \Sigma$. Indeed, $\arg\max\{U(p,z),\ z \in F \cup F'\}$ lies either in $\arg\max\{U(p,z),\ z \in F\}$ or in $\arg\max\{U(p,z),\ z \in F'\}$, and hence, as F and F' belong to Σ , lies in $B(p, z_0)$.

- Second, the utility achieved by A, $\max\{U(p,z),\ z \in F\}$, increases by set union: for all F and F' in Σ and for all p, we have:

$$\max\{U(p,z),\ z \in F \cup F'\} = \max\left(\max\{U(p,z),\ z \in F\}, \max\{U(p,z),\ z \in F'\}\right)$$

We can now suggest the following definition.

Definition 8. Define $\overline{F}$ as *the smallest closed family containing all consistent families. Formally,* $\overline{F}$ = *closure* $\left(\underset{\Sigma}{\cup} F\right)$.

By a continuity argument and using the stability of Σ, it is easy to show that $\overline{F}$ is itself a consistent family. We can now state the following lemma that leads to our main theorem.

Lemma 3: Let $\widehat{z}^*(p)$ be equal to $\arg\max\{U(p,z),\ z \in \overline{F}\}$, with $\overline{F}$ = *closure* $\left(\underset{\Sigma}{\cup} F\right)$. $\widehat{z}^*(p)$ *is almost injective.*

Proof. See Appendix. ■

We can now state our main theorem.

Theorem 2: $\widehat{z}^*(p) = \arg\max\{U(p,z),\ z \in \overline{F}\}$ is a weak-separating PBE which dominates all the other weak-separating PBEs.

Proof. >From Proposition 4 and Lemma 3. ■

To illustrate the theorem, it is interesting to identify $\widehat{z}^*$ when $K = \{\underline{p}, \overline{p}\}$ or $K = [\underline{p}, \overline{p}]$.

- In the first configuration, when there are only two possible levels of risk, the result is very simple.

If A is the insurer then the two equilibrium contracts are the contract $\widehat{z}^*\left(\underline{p}\right) = z^*(\underline{p}, z_0)$ and the contract $\widehat{z}^*(\overline{p})$ solution of $U(\underline{p}, z) = U(\underline{p}, \widehat{z}^*(\underline{p}))$ and $V(\overline{p}, z) = V(\overline{p}, z_0)$. Hence, for a low-risk individual, the PBE contract which dominates all the other is precisely the first best one (see Figure 5).

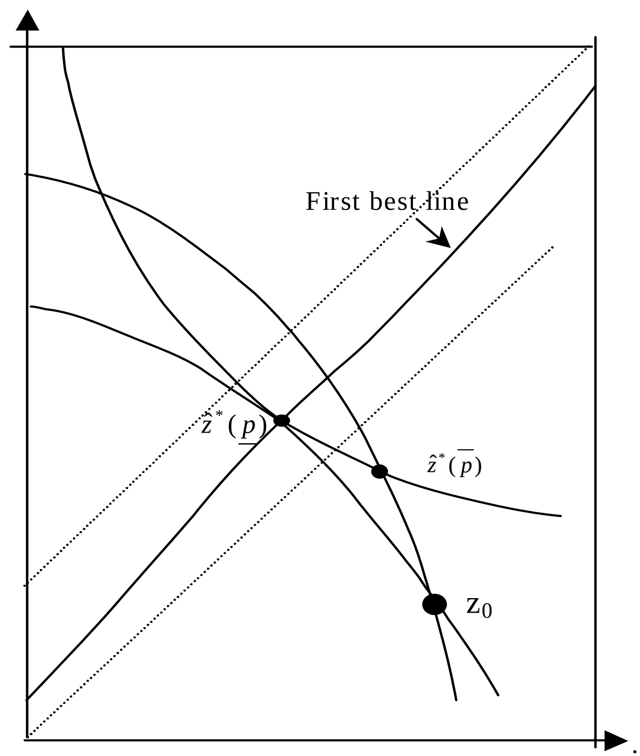

Figure 5. Best PBE

Therefore, from a descriptive viewpoint, the fact that the insurer (the principal) is better informed than the insured (the agent) leads to a equilibrium that deeply differs from those of the Stiglitz's model (1977) as simply being the opposite[11]: optimal risk sharing is achieved (first best) for a low-risk individual and not for high-risk as in the Stiglitz's seminal paper.

If A is the insured the two contracts are: $\widehat{z}^*(\overline{p}) = z^*(\overline{p}, z_0)$ (the PBE which dominates all the other is precisely the first best one for a high-risk individual) and the contract $\widehat{z}^*(\underline{p})$ solution of $U(\overline{p}, z) = U(\overline{p}, \widehat{z}^*(\overline{p}))$and $V(\underline{p}, z) = V(\underline{p}, z_0)$.

From a descriptive viewpoint, the fact that the insured, who is the informed part, is the principal instead of being the agent does not change the well-known result of risk-sharing second best optimality: optimal risk-sharing is achieved for a high-risk individual when the low-risk one is only partially covered by the other the insurer.

When the insured is the principal, one obtains first best for the high risk and inefficient coverage for the low risk.

[11]That result is consistent with those obtained in Villeneuve (2000).

- When there is a continuum of risks the family of equilibrium contracts $\widehat{z}^*(p) = (x(p), y(p))$ is given by the solution of the following differential system :

$$\begin{cases} pu'(x)x' + (1-p)u'(y)y' = 0 \\ pv(\overline{x} - x) + (1-p)v(\overline{y} - y) = pv(\overline{x} - x_0) + (1-p)v(\overline{y} - y_0) \end{cases}$$

and we still have $\widehat{z}^*(\underline{p}) = z^*(\underline{p}, z_0)$ if A is the insurer and $\widehat{z}^*(\overline{p}) = z^*(\overline{p}, z_0)$ if A is the insured.

3.2 Agent B is informed

When B is informed, the problem is a traditional adverse-selection one. A proposes a closed family G of contracts, and B chooses in $G \cup \{z_0\}$. This closed family G can be seen as a price curve along which different types of B auto-select themselves.

Let $\widetilde{z}_G(p) = \arg\max(V(p,z) - V(p,z_0),\ z \in G \cup \{z_0\})$,and $W_G(p) = \max(V(p,z) - V(p,z_0),$ $z \in G \cup \{z_0\})$.

Obviously, the mapping $\widetilde{z}_G(.)$ fulfills incentive compatibility, and participation constraints:

$$\begin{cases} \forall p, p' \in K\ V(p, \widetilde{z}_G(p)) \geq V(p, \widetilde{z}_G(p')) \\ V(p, \widetilde{z}_G(p)) \geq V(p, z_0) \end{cases}$$

The problem of the principal is hence to find a family G such that his objective function is maximized:

$$\max \int_K U(p, \widetilde{z}_G(p)) f(p) dp$$

Lemma 4. *For each closed family G, $W_G(.)$ is a non negative convex function.*

Proof. As a function of p, $V(p,z) - V(p,z_0)$ is a convex function. $W_G(.)$, as a maximum of convex functions is itself convex. It is obviously non negative (by definition). ■

$W_G(.)$ is hence decreasing on some interval $[\underline{p}, p_1]$, constant on $[p_1, p_2]$ and increasing on $[p_2, \overline{p}]$ with $\underline{p} \leq p_1 \leq p_2 \leq \overline{p}$.

Proposition 5.

(i) *if* $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} > 0$, *for each closed family* G, *there exists a family* H *in* $R(z_0)$ *that dominates* G *from the principal's point of view. This family is such that* $V(\overline{p}, \widetilde{z}_H(\overline{p})) = V(\overline{p}, z_0)$

(ii) *if* $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} < 0$, *for each closed family* G, *there exists a family* H *in* $L(z_0)$ *that dominates* G *from the principal's point of view. This family is such that* $V(\underline{p}, \widetilde{z}_H(\underline{p})) = V(\underline{p}, z_0)$.

Proof. See appendix. ■

This proposition allows us restricting our attention in $R(z_0)$ (resp. $L(z_0)$)if A is the insured (resp. insurer), with price curves such that high risk (resp. low) stay at their initial level of utility.

Using the envelop theorem we can replace incentive compatibility constraints by:
$\overset{\bullet}{W}(p) = v(\overline{x} - x(p)) - v(\overline{x} - x_0) - v(\overline{y} - y(p)) + v(\overline{y} - y_0)$

When A is the insurer her program is hence:

$$\max_{z(.),W(.)} \int_K U(p, z(p)) f(p) dp$$

$$u.c. \begin{cases} W(p) = V(p, z(p)) - V(p, z_0) \\ \overset{\bullet}{W}(p) = v(\overline{x} - x(p)) - v(\overline{x} - x_0) - v(\overline{y} - y(p)) + v(\overline{y} - y_0) \leq 0 \\ W(\underline{p}) = 0 \end{cases}$$

When A is the insured:

$$\max_{z(.),W(.)} \int_K U(p, z(p)) f(p) dp$$

$$u.c. \begin{cases} W(p) = V(p, z(p)) - V(p, z_0) \\ \overset{\bullet}{W}(p) = v(\overline{x} - x(p)) - v(\overline{x} - x_0) - v(\overline{y} - y(p)) + v(\overline{y} - y_0) \geq 0 \\ W(\overline{p}) = 0 \end{cases}$$

Theorem 3.

When A is not informed, her best offer H^* is a price curve such that:

(i) if she is insured, H^* lies in $R(z_0)$, when high risk, insurer stays at his initial level of utility, there is an efficient risk sharing for low risk, i.e. $\widetilde{z}_H(\underline{p}) = \arg\max \left(U(\underline{p}, z), V(\underline{p}, z) \geq V(\underline{p}, \widetilde{z}_H(\underline{p})\right)$.

(ii)if she is insurer, H^* lies in $L(z_0)$, when low risk, insured stays at his initial level of utility, there is an efficient risk sharing for high risk, i.e. $\widetilde{z}_H(\overline{p}) = \arg\max \left(U(\overline{p}, z), V(\overline{p}, z) \geq V(\overline{p}, \widetilde{z}_H(\overline{p})\right)$.

Proof. See Appendix. ■

Those results go in the same direction than results obtained by the Stiglitz's model [20] in which the insured (the agent) is the best informed party: optimal risk-sharing is achieved for high risks when only partially coverage is obtained for low risks. In that case, the insurer who has no information can only believe insured who tells being hisk-risk. It is a traditional adverse selection result. The opposite result should be obtained when the principal is the insured and is not the informed part: low-risks are entirely covered whereas the high-risks are only partially covered.

4 Application with a risk-neutral insurer

In order to compare our results with some traditional approaches considering risk-neutral insurers, we can draw the four following graphs. They correspond to the four cases analyzed above and obtained by our theorems and propositions.

In Figure 6, when the agent is informed, there is a possible adverse selection (denoted AS). In those cases, there are a lot of possible solutions: they depend on the proportion of low and high risks.

When the principal is informed, there is a signaling game (denoted SG in the graph): the non-informed agent is at z_0 whereas the principal benefits from her information.

In both graphs depicted on the left, the risk-sharing is optimal for a high-risk individual whereas a low-risk individual is only partially covered. In both graphs depicted on the right, the risk-sharing is optimal for a low-risk individual whereas a high-risk individual is only partially covered.

To sum up: when the insurer is better informed than the insured about risk, then whatever his position (principal or agent), that is whatever his bargaining power, low risk is better insured than high risk. Conversely, when the insured is better informed than insurer, high risk is better insured that low.

The results are drawn in traditional (coverage, premium) graphs.

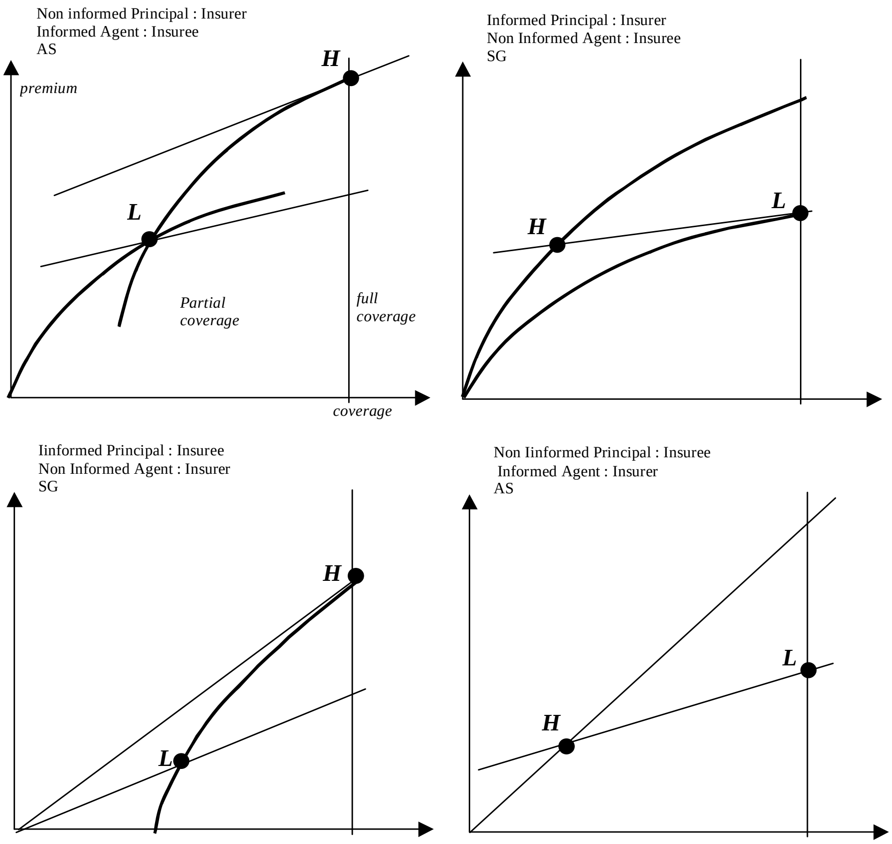

Figure 6. Application with a risk-neutral insurer

5 Conclusion

Using a simple model of risk-sharing, this note derives implications of the theory of insurance under asymmetric information. The general model introduced here is consistent with well-known results of the economic literature which consider either that the insurer offers contracts to insured who are more informed, as well as more recent works that deal with the insurer offering contracts and being more informed on the level of risk than the insured.

This note presents a mustering approach that does not constraint the insurer to be risk-neutral nor to be the principal in the traditional principal-agent framework.

Perhaps, the recent and increasing debate discussed in the introduction on the existence of adverse selection in insurance markets may also be analyzed at the light of the main theoretical results obtained here. Whether these results may occur in market structures other than the four polar ones analyzed here may constitute a starting point for future research.

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6 APPENDIX

Proof. Lemma 1.

Take z in $B(p, z_0) \cap L(z_0)$. We have:

$V(p,z) \geq V(p,z_0) \Longrightarrow v(\overline{y}-y)-v(\overline{y}-y_0)+p\left(v(\overline{x}-x)-v(\overline{x}-x_0)-v(\overline{y}-y)+v(\overline{y}-y_0)\right) \geq 0$

$z \in L(z_0) \Longrightarrow v(\overline{x}-x)-v(\overline{x}-x_0)-v(\overline{y}-y)+v(\overline{y}-y_0) \geq 0.$

Hence we have for all $p' \geq p$:

$v(\overline{y}-y)-v(\overline{y}-y_0)+p'\left(v(\overline{x}-x)-v(\overline{x}-x_0)-v(\overline{y}-y)+v(\overline{y}-y_0)\right) \geq 0.$ ■

Proof. Lemma 3.

Assume that $\frac{u'(x_0)}{u'(y_0)} > \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)}$. Consider a contract z in $A(p, z_0) \cap B(p, z_0)$. According to Lemma 2 we have $x > x_0$ and $y < y_0$. As v is strictly inscreasing, we have $v(\overline{x}-x)+v(\overline{y}-y_0) \leq v(\overline{x}-x_0)+v(\overline{y}-y)$. which exactly means that, z belongs $R(z_0)$.

Let now consider a contract $z = (x; y)$ in $A(p, z_0) \cap B(p, z_0)$, it has been proved that z is also in $R(z_0)$, that is $v(\overline{y}-y_0)-v(\overline{y}-y) \leq v(\overline{x}-x_0)-v(\overline{x}-x)$. For being also in $L(z_0)$, z has also to solve $v(\overline{y}-y_0)-v(\overline{y}-y) \geq v(\overline{x}-x_0)-v(\overline{x}-x)$. The only contract that solve both inequalities is $(x_0, y_0) = z_0$. (i) is proved.

(ii) is proved by similar arguments. ■

Proof. Proposition 1.

The first order conditions for an interior solution are:

$$\begin{cases} \frac{u'(x)}{u'(y)} = \frac{v'(\overline{x}-x)}{v'(\overline{y}-y)} \\ V(p,z) = V(p,z_0) \end{cases}$$

Let $S(p, z_0)$ be the solution of this system. It is easy to see that $z^*(p, z_0)$ is simply the projection of $S(p, z_0)$ on $[0, \overline{x}] \times [0, \overline{y}]$.

Let $S^*$the projection on $[0, \overline{x}] \times [0, \overline{y}]$ of the set defined by $\frac{u'(x)}{u'(y)} = \frac{v'(\overline{x}-x)}{v'(\overline{y}-y)}$. This set does not depend neither on p nor on z_0. For every z_0 and p, $z^*(p, z_0)$ belongs to S^*.

Reciprocally, given a contract z in S^*, $z = z^*(p, z)$ for all p. ■

Proof. Proposition 2.

Assume that $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} > 0$, then because of Lemma 2, $\forall p$, $A(p,z_0) \cap B(p,z_0) \subset R(z_0)$.

This implies that $\forall p$, $z^*(p,z_0) \in R(z_0)$, which means:

$v(\overline{y}-y_0) - v(\overline{y}-y^*(p,z_0)) \leq v(\overline{x}-x_0) - v(\overline{x}-x^*(p,z_0))$

- Take $p \leq p'$, we know that

$$U(p', z^*(p,z_0)) = \max(U(p',z), V(p',z) \geq V(p', z^*(p,z_0))$$

$U(p', z^*(p',z_0)) = \max(U(p',z), V(p',z) \geq V(p',z_0))$

Let us compare $V(p', z^*(p,z_0))$ with $V(p',z_0)$. We know that $V(p, z^*(p,z_0)) = V(p,z_0)$, hence the difference $V(p', z^*(p,z_0)) - V(p',z_0)$ can be rewritten as follows:

$$V(p', z^*(p,z_0)) - V(p, z^*(p,z_0)) - \{V(p',z_0) - V(p,z_0)\}.$$

By (2), that expression is equal to:

$$[p'v(\overline{x}-x^*) + (1-p')v(\overline{y}-y^*)] - [pv(\overline{x}-x^*) + (1-p)v(\overline{y}-y^*)]$$
$$- \{[p'v(\overline{x}-x_0) + (1-p')v(\overline{y}-y_0)] - [pv(\overline{x}-x_0) + (1-p)v(\overline{y}-y_0)]\}.$$

That is:

$$(p'-p)\left[v(\overline{x}-x^*(p,z_0)) - v(\overline{x}-x_0) + v(\overline{y}-y_0) - v(\overline{y}-y^*(p,z_0)) + v(\overline{y}-y_0)\right].$$

As $z^*(p,z_0) \in R(z_0)$, the second term of the above expression is negative. As $(p'-p)$ is positive, the expression is negative. Hence, $V(p', z^*(p,z_0)) \leq V(p',z_0)$.

Therefore, $U(p', z^*(p,z_0)) \geq U(p', z^*(p',z_0))$.

- We can also prove that $U(p, z^*(p,z_0)) \geq U(p, z^*(p',z_0))$ by using similar arguments.

  Indeed, we have:

$U(p, z^*(p,z_0)) = \max(U(p,z), V(p,z) \geq V(p,z_0))$

$U(p, z^*(p',z_0)) = \max(U(p,z), V(p,z) \geq V(p, z^*(p',z_0)))$.

Let us compare $V(p,z_0)$ with $V(p, z^*(p',z_0))$. We know that $V(p',z_0) = V(p', z^*(p',z_0))$,hence the difference $V(p,z_0) - V(p, z^*(p',z_0))$ can be rewritten as follows:

$$\{V(p', z^*(p',z_0)) - V(p, z^*(p',z_0))\} - \{V(p',z_0) - V(p,z_0)\}.$$

Let $z^*(p',z_0) = (x^*(p',z_0), y^*(p',z_0)) = (x'^*, y'^*)$. By (2), that expression is equal to:

$$[p'v(\overline{x}-x'^*)+(1-p')v(\overline{y}-y'^*)]-[pv(\overline{x}-x'^*)+(1-p)v(\overline{y}-y'^*)]$$
$$-\{[p'v(\overline{x}-x_0)+(1-p')v(\overline{y}-y_0)]-[pv(\overline{x}-x_0)+(1-p)v(\overline{y}-y_0)]\}.$$

That is:

$$(p'-p)\left[v(\overline{x}-x^*(p',z_0))-v(\overline{x}-x_0)+v(\overline{y}-y_0)-v(\overline{y}-y^*(p',z_0))\right].$$

As $z^*(p',z_0)\in R(z_0)$, the second term of the above expression is negative. As $(p'-p)$ is positive, the expression is negative. Hence, $V(p,z^*(p',z_0))\leq V(p,z_0)$.

Therefore, $U(p,z^*(p,z_0))\geq U(p,z^*(p',z_0))$.

The same kind of arguments can be used to prove the second part of Proposition 2, i.e. when $\frac{u'(x_0)}{u'(y_0)}-\frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)}<0$. ■

Proof. Proposition 3.

Let prove the proposition in two steps.

- First, let prove that z belongs to $F(g)$ implies that:

$$B(\underline{p},z_0)\cap L(z_0)\subset F\cap L(z_0)\subset B(\overline{p},z_0)\cap L(z_0)$$
$$\textit{and }\ B(\overline{p},z_0)\cap R(z_0)\subset F\cap R(z_0)\subset B(\underline{p},z_0)\cap R(z_0).$$

Given g, z belongs to $F(g)$ if and only if $E_{g(z)}(p)v(\overline{x}-x)+(1-E_{g(z)}(p))v(\overline{y}-y)\geq E_{g(z)}(p)v(\overline{x}-x_0)+(1-E_{g(z)}(p))v(\overline{y}-y_0)$, that is:

$$E_{g(z)}(p)\left\{[v(\overline{y}-y_0)-v(\overline{y}-y)]-[v(\overline{x}-x_0)-v(\overline{x}-x)]\right\}\geq v(\overline{y}-y_0)-v(\overline{y}-y)$$

If $z\in F\cap L(z_0)$, as $z\in L(z_0)$, this implies $[v(\overline{y}-y_0)-v(\overline{y}-y)]-[v(\overline{x}-x_0)-v(\overline{x}-x)]>0$. Thus, as $E_{g(z)}(p)\leq\overline{p}$, we can write:

$$\overline{p}\left\{[v(\overline{y}-y_0)-v(\overline{y}-y)]-[v(\overline{x}-x_0)-v(\overline{x}-x)]\right\}\geq v(\overline{y}-y_0)-v(\overline{y}-y)$$

Such an inequality implies that $z\in B(\overline{p})$. Hence $F\cap L(z_0)\subset B(\overline{p})\cap L(z_0)$

We can now take z in $B(\underline{p})\cap L(z_0)$. As $z\in B(\underline{p})$, we have:

$$\underline{p}\left\{[v(\overline{y}-y_0)-v(\overline{y}-y)]-[v(\overline{x}-x_0)-v(\overline{x}-x)]\right\}\geq v(\overline{y}-y_0)-v(\overline{y}-y)\ .$$

But we know that $E_{g(z)}(p)\geq\underline{p}$. So as $z\in L(z_0)$, the previous inequality implies:

$$E_{g(z)}(p)\left\{[v(\overline{y}-y_0)-v(\overline{y}-y)]-[v(\overline{x}-x_0)-v(\overline{x}-x)]\right\}\geq v(\overline{y}-y_0)-v(\overline{y}-y)$$

Hence, $B(\underline{p})\cap L(z_0)\subset F\cap L(z_0)$.

Therefore,we have proved that z belongs to $F(g)$ implies that:

$$B(\underline{p},z_0)\cap L(z_0)\subset F\cap L(z_0)\subset B(\overline{p},z_0)\cap L(z_0).$$

- By using definition of $R(z_0)$, a similar demonstration allows concluding that z belongs to $F(g)$ also implies that:

$$B(\overline{p}, z_0) \cap R(z_0) \subset F \cap R(z_0) \subset B(\underline{p}, z_0) \cap R(z_0).$$

Let's now go to the second part of the proof.

Reciprocally, consider a family F such that:

$B(\underline{p}) \cap L(z_0) \subset F \cap L(z_0) \subset B(\overline{p}) \cap L(z_0)$

and $B(\overline{p}) \cap R(z_0) \subset F \cap R(z_0) \subset B(\underline{p}) \cap R(z_0)$

It is easy to prove that $B(\underline{p}) \cap L(z_0) \subset F \cap L(z_0) \subset B(\overline{p}) \cap L(z_0)$ implies that there exists a distribution g on K such that $z \in F(g)$.

Indeed, consider $(x', y') \in B(\underline{p}) \cap L(z_0)$, we have:

$$\underline{p}\left\{[v(\overline{y} - y_0) - v(\overline{y} - y')] - [v(\overline{x} - x_0) - v(\overline{x} - x')]\right\} \geq v(\overline{y} - y_0) - v(\overline{y} - y'),$$

and this contract is in $F \cap L(z_0)$. Moreover, take $z = (x, y)$ in $F \cap L(z_0)$.

$F \cap L(z_0) \subset B(\overline{p}) \cap L(z_0)$ implies that $z \in B(\overline{p})$. Hence, we can write:

$$\overline{p}\left\{[v(\overline{y} - y_0) - v(\overline{y} - y)] - [v(\overline{x} - x_0) - v(\overline{x} - x)]\right\} \geq v(\overline{y} - y_0) - v(\overline{y} - y).$$

It means that there exists at least a real $\lambda \in [0, 1]$ such that:

$$(\lambda\overline{p} + (1 - \lambda)\underline{p})\left\{[v(\overline{y} - y_0) - v(\overline{y} - y)] - [v(\overline{x} - x_0) - v(\overline{x} - x)]\right\} \geq v(\overline{y} - y_0) - v(\overline{y} - y).$$

So there exists a distribution g on K (with two atoms on $\underline{p}$ and $\overline{p}$) such that $z \in F(g)$.

- By definition of $R(z_0)$ and similar arguments, one proves that

  $B(\overline{p}) \cap R(z_0) \subset F \cap L(z_0) \subset B(\underline{p}) \cap R(z_0)$ implies existence of a distribution g on K such that $z \in F(g)$.

Proof. Lemma 3. We prove by contradiction the almost-injectivity of $\widehat{z}^*(p)$. Suppose that there could exist $p_1$and p_2 in K $(p_1 < p_2)$ such that $\widehat{z}^*(p_1) = \widehat{z}^*(p_2) \neq z_0$.

This implies that $\forall\ p \in [p_1, p_2] \cap K$, $\widehat{z}^*(p) = \widehat{z}^*(p_1) = \widehat{z}^*(p_2) = (\alpha, \beta)$.

Indeed, for all $(x, y) \in \overline{F}$, $p_1 u(\alpha) + (1 - p_1)u(\beta) \geq p_1 u(x) + (1 - p_1)u(y)$ and $p_2 u(\alpha) + (1 - p_2)u(\beta) \geq p_2 u(x) + (1 - p_2)u(y)$.

This implies that for all $\lambda \in [0, 1]$, the expression $(\lambda p_1 + (1 - \lambda)p_2)\, u(\alpha) + (1 - \lambda p_1 - (1 - \lambda)p_2)u(\beta)$ is higher than $(\lambda p_1 + (1 - \lambda)p_2)\, u(x) + (1 - \lambda p_1 - (1 - \lambda)p_2)u(y)$.

That is $\forall\ p \in [p_1, p_2]$, $pu(\alpha) + (1 - p)u(\beta) \geq pu(x) + (1 - p)u(y)$.

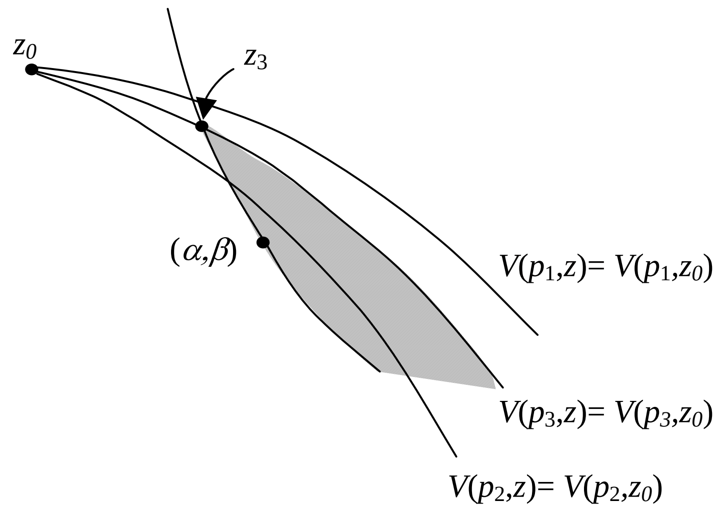

Figure 1: Figure A1.

Suppose first that $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} > 0$.

As $(\alpha, \beta) \in A(p_1, z_0) \cap B(p_1, z_0)$, by Lemma 2 we know that $(\alpha, \beta) \in R(z_0)$.

Take $p_3 \in]p_1, p_2[\cap K$ (if non empty) or $p_3 = p_1$ if $]p_1, p_2[\cap K = \{\emptyset\}$.

As $\overline{F}$ is consistent, we have:

$(\alpha, \beta) \in B(p_2, z_0) \cap R(z_0) \subset B(p_3, z_0) \cap R(z_0) \subset B(p_1, z_0) \cap R(z_0)$

Consider now the set $A(p_3, (\alpha, \beta)) \cap B(p_3, z_0)$ (shaded area in the following picture). This is a non empty convex subset of $R(z_0)$ containing (α, β).

Take z_3 the solution of $P = \min(U(p_2, z),\ z \in A(p_3, (\alpha, \beta)) \cap B(p_3, z_0))$.

We have: $(\alpha, \beta) \neq z_0, \Rightarrow z_3 \neq (\alpha, \beta)$. Indeed, it is easy to see that z_3 must simultaneously verify $U(p_3, z_3) = U(p_3, (\alpha, \beta))$ and $V(p_3, z_3) = V(p_3, z_0)$, that is the two constraints in P are binding. $z_3 = (\alpha, \beta)$ would then imply that $V(p_3, (\alpha, \beta)) = V(p_3, z_0)$. As $V(p_2, (\alpha, \beta)) \geq V(p_2, z_0)$ and $p_2 > p_3$, we would hence have $(\alpha, \beta) \in L(z_0)$, that is $(\alpha, \beta) = z_0$. Then we have:

for all $p \in [p_1, p_3[\cap K,\ U(p, z_3) > U(p, (\alpha, \beta))$,

for all $p \in [p_3, p_2] \cap K, U(p, (\alpha, \beta)) \geq U(p, z_3)$.

For all $p \leq p_3$ $z_3 \in B(p, z_0)$.

If we now take the family $\overline{F} \cup \{z_3\}$. This family is coherent.

This is impossible since it is not contained in $\overline{F}$.

Similar arguments apply for $\frac{u'(x_0)}{u'(y_0)} - \frac{v'(\overline{x}-x_0)}{v'(\overline{y}-y_0)} < 0$. ■

Proof. Proposition 5.

By the envelop theorem we have: $\dot{W}_G(p) = v(\overline{x} - \widetilde{x}(p)) - v(\overline{x} - x_0) - v(\overline{y} - \widetilde{y}(p)) + v(\overline{y} - y_0)$

Hence, $\widetilde{z}_G(p) \in R(z_0)$ for $p \in [\underline{p}, p_1]$, $\widetilde{z}_G(p) \in R(z_0) \cap L(z_0)$ for $p \in [p_1, p_2]$ and $\widetilde{z}_G(p) \in L(z_0)$ for $p \in [p_2, \overline{p}]$.

For all $p \in [p_1, p_2]$, $\widetilde{z}_G(p) = \widetilde{z}_G(p_1) = \widetilde{z}_G(p_2)$. Indeed, for $p \in [p_1, p_2]$, $W_G(p) = v(\overline{y} - \widetilde{y}(p)) - v(\overline{y} - y_0) + p(v(\overline{x} - \widetilde{x}(p)) - v(\overline{x} - x_0) - v(\overline{y} - \widetilde{y}(p)) + v(\overline{y} - y_0))$

$= v(\overline{y} - \widetilde{y}(p)) - v(\overline{y} - y_0)$ which implies that $\widetilde{y}(p)$ and hence $\widetilde{x}(p)$ is constant.

For all family G, for each contract (x, y) in G, define the contract $(\alpha(x, y), \beta(y))$ by:

$v(\overline{y} - \beta(y)) - v(\overline{y} - y_0) = v(\overline{y} - y) - v(\overline{y} - \widetilde{y}(p_1))$

$v(\overline{x} - \alpha(x, y)) - v(\overline{x} - x_0) - v(\overline{y} - \beta(y)) + v(\overline{y} - y_0) = v(\overline{x} - x) - v(\overline{x} - x_0) - v(\overline{y} - y) + v(\overline{y} - y_0)$

The utility achieved with the contract (α, β) is :

$p\,(v(\overline{x} - \alpha) - v(\overline{x} - x_0)) + (1 - p)\,(v(\overline{y} - \beta) - v(\overline{y} - y_0)) = V(p, z) - V(p, z_0) + v(\overline{y} - y_0) - v(\overline{y} - \widetilde{y}(p_1))$

$$= V(p, z) - V(p, z_0) - W_G(p_1)$$

Let H be the obtained set of contracts when (x, y) varies in G. It is easy to show that $\alpha \geq x$ and $\beta \geq y$.

Obviously, we have:

$(\alpha(\widetilde{x}(p), \widetilde{y}(p)), \beta(\widetilde{y}(p))) = \arg\max(V(p, (\alpha, \beta)), (\alpha, \beta) \in H \cup \{z_0\})$ and for $p \in [p_1, p_2]$, we have :

$\max(V(p, (\alpha, \beta)), (\alpha, \beta) \in H \cup \{z_0\}) = V(p, z_0)$. ■

Proof. of theorem 3

Write the FOC (Euler Lagrange Equation) for both programs. ■