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Ambiguity aversion and an intertemporal equilibrium model of catastrophe-linked securities pricing

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ABSTRACT

To explain several stylized facts concerning catastrophe-linked securities premium spread, we propose an intertemporal equilibrium model by allowing agents to act in a robust control framework against model misspecification with respect to rare events. We have presented closed-form pricing formulas in some special cases and tested the model using empirical data and simulation.

1. Introduction

With the recent rise in catastrophic disaster losses and resulting effect on insurance industry solvency, the insurance market is increasingly calling for some new mechanisms to provide coverage against catastrophe risks. One form of solution is through catastrophe (CAT) risk securitization by bringing catastrophe exposures directly to the capital market. Such mechanisms include CAT-linked bonds, swaps, exchange-traded options and futures, etc. Through a careful study of the catastrophe-linked securities market, we may document several interesting stylized facts.

First, the spread premium of CAT-linked securities is very high relative to the expected principal loss. For example, Cummins et al. (2004) report an average spread yield of 9.09 times expected loss for 32 catastrophe bonds issued in the period of 1997–2000 (see Table 1). Since historical data suggests that catastrophe risk can usually be looked as uncorrelated with capital market, or more exactly, amounts to a small fraction of total wealth in the economy, the CAT loss securities premium should equal actuarial fair losses covered by the contract. The fact that CAT bond spread is far above the expected loss of bond principal is contradictory to traditional capital market theories. Though it seems CAT bonds spread has declined over the last decade, the spread magnitude is still quite large to be reconciled with standard economic models.

Second, it appears that premium spread is more pronounced for CAT securities with low probability that a contingent loss payment to the security issuers will be triggered. This may generate a kind of "smirk" pattern in the cross-sectional plot of the premium to $E[\text{loss}]$ ratio against the probability of first loss (see Fig. 1). As a comparison, there is no apparent relation between the ratio and the expected loss percentage conditional on a loss occurrence, which indicates that the premium implicit in CAT securities is mainly sensitive to the rareness of catastrophe events.

The third stylized fact of particular interest is about catastrophe security spread move after a large event. Cummins and Weiss (2009) find that the market price of windstorm risk increased on average by more than 50% after 2005 Hurricane Katrina and more interestingly the increase was even stronger for non-windstorm related risk. The key puzzle here is that: while a bad hurricane like Katrina might be expected to change the cost of hurricane-linked

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Table 1
Catastrophe bond issues (1997–2000 bonds data). The spread premium is the annual coupon rate above one-year LIBOR. The Prob of First Loss is the probability that a contingent payment will be trigged under the bond. The CE is the expected principal payment proportion to the issuing insurer, conditional on the occurrence of a loss that triggers payment under the bond, expressed as a percentage of the principal of the bond. The expected loss is the product of the probability of first loss and CE. Prem to E[loss] is the ratio of the spread premium to the expected loss of principal of the bond.

Date	Transaction sponsor	Spread premium (%)	Prob of first loss (%)	CE (%)	Expected loss (%)	Prem to E[Loss]
March-00	SCOR	2.70	0.19	57.89	0.11	24.55
March-00	SCOR	3.70	0.29	79.31	0.23	16.09
March-00	SCOR	14.00	5.47	59.23	3.24	4.32
March-00	Lehman Re	4.50	1.13	64.60	0.73	6.16
November-99	American Re	2.95	0.17	100.00	0.17	17.35
November-99	American Re	5.40	0.78	80.77	0.63	8.57
November-99	American Re	8.50	0.17	100.00	0.17	50.00
November-99	Gerling	4.50	1.00	75.00	0.75	6.00
June-99	Gerling	5.20	0.60	75.00	0.45	11.56
June-99	USAA	3.66	0.76	57.89	0.44	8.32
July-99	Sorema	4.50	0.84	53.57	0.45	10.00
July-98	Yasuda	3.70	1.00	94.00	0.94	3.94
March-99	Kemper	3.69	0.58	86.21	0.50	7.38
March-99	Kemper	4.50	0.62	96.77	0.60	7.50
May-99	Oriental Land	3.10	0.64	66.04	0.42	7.35
February-99	St. Paul/F&G Re	4.00	1.15	36.52	0.42	9.52
February-99	St. Paul/F&G Re	8.25	5.25	54.10	2.84	2.90
December-98	Center Solutions	4.17	1.20	64.17	0.77	5.42
December-98	Allianz	8.22	6.40	56.41	3.61	2.28
August-98	XL/MidOcean Re	4.12	0.61	63.93	0.39	10.56
August-98	XL/MidOcean Re	5.90	1.50	70.00	1.05	5.62
July-98	St. Paul/F&G Re	4.44	1.21	42.98	0.52	8.54
July-98	St. Paul/F&G Re	8.27	4.40	59.09	2.60	3.18
June-98	USAA	4.16	0.87	65.52	0.57	7.30
March-98	Center Solutions	3.67	1.53	54.25	0.83	4.42
December-97	Tokio Marine & Fire	2.09	1.02	34.71	0.35	5.90
December-97	Tokio Marine & Fire	4.36	1.02	68.63	0.70	6.23
July-97	USAA	5.76	1.00	62.00	0.62	9.29
August-97	Swiss Re	2.55	1.00	45.60	0.46	5.59
August-97	Swiss Re	2.80	1.00	46.00	0.46	6.09
August-97	Swiss Re	4.75	1.00	76.00	0.76	6.25
August-97	Swiss Re	6.25	2.40	100.00	2.40	2.60
					Average	9.09
					Median	6.77

Source: Cummins et al. (2004).

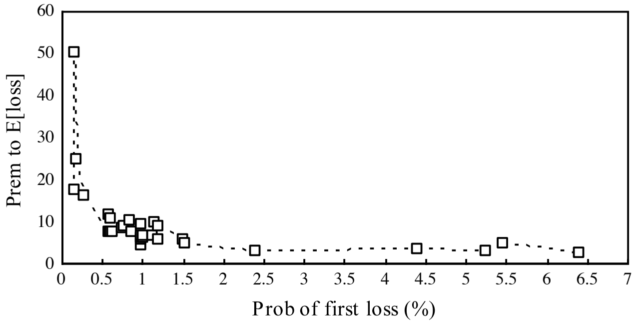

Fig. 1. The "smirk" curve of empirical premium spreads against probabilities of first loss (1997–2000 bonds data).

contracts, there is no reason for prices of non-windstorm linked contracts to have shifted a great deal. Another interesting point is that the CAT bonds of lower occurrence frequencies have seen more increases than those of higher occurrence frequencies.

Among the existing literature, Bantwal and Kunreuther (2000) may be the first to point out the high CAT bond spread phenomena and they suggest using behavior economics to explain this puzzle. But they have not tried to explain the other two facts mentioned above. In a closely related paper concerning empirical facts about reinsurance similar to those for CAT securities, Froot (1999) examines eight potential explanations and concludes that the most compelling explanation is capital supply restrictions associated with capital market imperfections. Since catastrophe securitization may help solve the capital scarcity problem, this reason seems not so suitable for explaining stylized facts with catastrophe-linked securities. In a recent working paper, Dieckmann (2008) adapts the external habit model of Campbell and Cochrane (1999) to explain high level and cyclical behavior of CAT bonds spread. But Dieckmann's model also does not explicitly explain the "smirk" pattern in the cross-sectional plot of the premium spread against the probability of first loss.

The objective of this paper is to explain more about the empirical facts concerning catastrophe-linked securities pricing by admitting aversion to ambiguity as well as to risk, we establish an intertemporal equilibrium model by allowing agents to act in a robust control framework against model specification with respect to rare events in the sense of Hansen and Sargent (2001) and Liu et al. (2005). The intuition behind our specific model is that the lack of clarity or uncertainty about insurance event occurrence will lead agents to set premiums high and since the uncertainty results partly from the scarcity of statistical information, the lower the catastrophe occurrence frequency is, the higher ambiguity aversion of agents and thus the higher the multiple of premium over expected loss for the related catastrophe-linked securities will be. Moreover, we will assume the ambiguity aversion magnitude of agents depends on the prior aggregate catastrophe losses performance, thus all catastrophe-linked securities will be significantly affected by a large catastrophe event occurrence.

The rest of the paper is organized as follows. Section 2 sets up the equilibrium model by allowing agents to be ambiguity averse in the robust control framework. We present explicit pricing formula in some special cases and Section 3 demonstrates how to use the derived formula in pricing catastrophe-linked securities. This section also presents estimation and simulation of the model using empirical CAT bonds data and gives a further discussion comparing our model with several previous related literature. Section 4 concludes. Technical details are collected in the Appendices.

2. Economic model

Our CAT market is based on a partial equilibrium model and we make the following assumptions to characterize our economy:

Assumption 1. The economy faces risks of catastrophe events and the catastrophe loss Y_s follows $\mathrm{d}Y_s = L_s \mathrm{d}N_s$, where N_s is a non-homogeneous Poisson process with intensity $\lambda_s > 0$ and L_s is the random loss amount if a loss occur at time s, which is independent of N_s. We assume $0 \le s \le T$ and $Y_0 = 0$ at the initial time 0.

Assumption 2. The economy also faces other catastrophe events and the aggregate catastrophe losses can be described by a stochastic process Z_s.

Assumption 3. There is a catastrophe risk market where catastrophe loss Y_s is traded and the market exchanges bundles of state-contingent contracts that can be analyzed as catastrophe-linked securities. The catastrophe loss Y_s is totally shared by a representative agent who can be considered as an investor in the insurance market. For simplicity here all loss payments are assumed to occur at the maturity time T.

Assumption 4. Besides underwriting catastrophe loss Y_s by selling catastrophe-linked securities, the representative agent also allocates his wealth in stocks and risk-free bonds. The stock is assumed to pay no dividends and the stock price S_s is modeled as a geometric Brownian motion: $\mathrm{d}S_s = S_s(\mu \mathrm{d}s + \sigma \mathrm{d}B_s)$ where the process B_s is a standard Brownian motion independent of Y_s and Z_s, and the coefficients μ and σ are respectively the *expected rate of return* and the *volatility* of the stock. The risk-free bond value is accumulated at a continuously compounded interest rate r. We assume the stock and risk-free bond markets are exogenously given and the wealth allocation decision of the representative agent does not affect the stock and bond markets.

Assumption 5. There is no information asymmetry in the catastrophe market and the market is "perfect", i.e., there are no transaction costs and taxes.

Assumption 6. The representative agent acts in a robust control framework and the objective function of the representative agent at time $s < T$ is defined recursively to satisfy the following stochastic integral equation

$$U(s) = \inf_{\tilde{P}\in P} \left\{ \tilde{E}_s \left[\mathrm{e}^{-\rho(T-s)} \left(u(W_T) - \int_s^T \psi(U(\tau), \tau)\mathrm{d}\tau \right) \right] \right\}, \quad (2.1)$$

with $u(w) = -(1/\gamma)\mathrm{e}^{-\gamma w}$ and $\gamma > 0$ being the risk aversion coefficient. P denotes the entire collection of alternative probability measures. ρ is a discount rate; W_T denotes the accumulated wealth of the agent at time T. $\int_s^T \psi(U(\tau), \tau)\mathrm{d}\tau$ is a penalty term that controls the ambiguity aversion magnitude of the agent.

We will focus on the effect of ambiguity aversion only to catastrophe occurrence. Let P be the reference probability measure associated with Y_s in Assumption 1. The alternative models are specified by restricting the associated probability measure $\tilde{P}$ by its Radon–Nikodym derivative with respect to P as $\xi_T = \mathrm{d}\tilde{P}/\mathrm{d}P$, and $\{\xi_s, 0 \le s \le T\}$ is a Doleans-Dade exponential defined by $\xi_s = 1 + \int_0^s \xi_{\tau-}(\mathrm{e}^{h_\tau} - 1)(\mathrm{d}N_\tau - \lambda_\tau \mathrm{d}\tau)$ with h_τ controlling the distortion magnitude. Note that if N_τ is a homogeneous Poisson process with $\lambda_\tau = \lambda$ and if $h_\tau = h$, then $\xi_T = \exp(hN_T)/E[\exp(hN_T)]$. We will discuss more about this simplified case in the next section.

Unlike that in Hansen and Sargent (2001)and Liu et al. (2005), we define the penalty term ψ at time τ in a more general form as:

$$\psi(U(\tau), \tau) = \gamma U(\tau)\lambda_\tau g_\tau(\mathrm{e}^{h_\tau} h_\tau - \mathrm{e}^{h_\tau} + 1), \quad (2.2)$$

where g_τ is assumed to be an increasing and convex function with $g_\tau(0) = 0$ so that $\int_s^T \psi(U(\tau), \tau)\mathrm{d}\tau = \int_s^T \gamma U(\tau)\lambda_\tau g_\tau(\mathrm{e}^{h_\tau} h_\tau - \mathrm{e}^{h_\tau} + 1)\mathrm{d}\tau$ is indeed a penalty term. The penalty term (2.2) is related to the concept of 'relative entropy' (see e.g., Appendix B of Liu et al., 2005) and in Appendix D we show that it is also related to Gerber–Shiu penalty function.

Example 2.1. If we define $g_\tau(x) = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } x > 0, \end{cases}$ the objective function is then reduced to the traditional expected utility function form $E_s[\mathrm{e}^{-\rho(T-s)}u(W_T)]$, in other words, no ambiguity at all.

Example 2.2. If we define $g_\tau(x) = x/a_\tau$ with $a_\tau > 0$, the objective function in (2.1) is then similar to that defined in Hansen and Sargent (2001) and Liu et al. (2005) with $a_\tau > 0$ represents an ambiguity aversion parameter.

For the purpose of describing the dynamic behavior of the catastrophe risk pricing, we assume that given an initial function $g_s(x)$ at time s, the forward function $g_\tau(x)$ evolves as follows.

Assumption 7. The dynamic behavior of $g_\tau(x)$ is modeled as:

$$\mathrm{d}g_\tau(x) = \eta_1(\tau, x)\mathrm{d}\tau + \eta_2(\tau, x)\mathrm{d}Z_\tau. \quad (2.3)$$

Example 2.3. If we define $g_\tau(x) = x/a_\tau$, the dynamic behavior of g_s is then reduced to the dynamic behavior of a_τ. For example, we may assume it as $\mathrm{d}a_\tau = \rho_1(\tau, a_\tau)\mathrm{d}\tau + \rho_2(\tau, a_\tau)\mathrm{d}Z_\tau$.

Remark 2.1. The parameters η_1, η_2 should be chosen so that g_τ keeps being an increasing and convex function with $g_\tau(0) = 0$ (this condition for Example 2.3 is that ρ_1, ρ_2 are chosen so that a_τ is kept as positive).

Now we consider the wealth process of the representative agent. We assume the representative agent starts with an initial wealth w at time 0 and receives premium of mc at that time to underwrite a proportion of catastrophe loss Y_s, with c denoting the price of the insurance covering total catastrophe loss between time 0 and time T and m denoting the proportion the agent underwrites in the full insurance. All loss payments are assumed to occur at time T. At any time $0 \le s < T$, the agent is assumed to invest π_s in stock and the remaining asset in risk-free bond respectively.

The agent's wealth process W_s $(0 \le s < T)$ hence follows

$$\begin{cases} \mathrm{d}W_s = [rW_s + (\mu - r)\pi_s]\mathrm{d}s + \sigma\pi_s \mathrm{d}B_s \\ W_0 = w + mc, \end{cases}$$

and we have $W_T = W_{T-} - mY_T$.

With (2.2) and other above specifications, we may rewrite (2.1) as

$$\begin{aligned} U(s) &= U(W, s, y, m, g_s) \\ &= \inf_{\{h_\tau \ge 0\}} \left\{ \tilde{E}_s \left[\mathrm{e}^{-\rho(T-s)} \left(u(W_T) - \int_s^T \gamma\lambda_\tau U(\tau) g_\tau \right.\right.\right. \\ &\quad \left.\left.\left. \times\ (h_\tau \mathrm{e}^{h_\tau} - \mathrm{e}^{h_\tau} + 1)\mathrm{d}\tau \right) \middle| W_s = W, Y_s = y \right] \right\}, \end{aligned} \quad (2.4)$$

and define the indirect objective function J by

$$J(s) = J(W, s, y, m, g_s) = \sup_{\{\pi_\tau, s \le \tau \le T\}} U(W, s, y, m, g_s).$$

The corresponding HJB equation for J is then given as follows:

$$\sup_{\pi_s}\Bigg\{J_s-\rho J+(\mu-r)\pi_s J_W+\frac{1}{2}\sigma^2\pi_s^2 J_{WW}+rWJ_W$$
$$+\inf_{h_s}\Bigg\{\tilde{E}_s\Bigg[J\left(W,s,y+\frac{\mathrm{d}Y_s}{\mathrm{d}s},m,g_s+\eta_1+\eta_2\frac{\mathrm{d}Z_s}{\mathrm{d}s}\right)$$
$$-J(W,s,y,m,g_s)-\gamma\lambda_s g_s(h_s\mathrm{e}^{h_s}-\mathrm{e}^{h_s}+1)J\Bigg]\Bigg\}\Bigg\}=0, \qquad (2.5)$$

where J_s is the derivative of J with respect to s; J_W, J_{WW} are its first and second derivatives with respect to W; the terminal condition is $J(T)=-\frac{1}{\gamma}\exp(-\gamma(W-my))$.

In general, there is no explicit solution to Eq. (2.5) and we are unable to get a general analytic pricing formula for catastrophe-linked securities. We will give an approximating numerical illustration of the dynamic behavior of CAT bonds prices in the next section. For that purpose we make simplifying assumptions here that g_s is the same during the time period $[0,T]$ with $g_s = g$ (i.e., we assume η_1, η_2 in Example 2.3 as zero). In this special case the equilibrium price can be given by the following closed-form formula:

Proposition 1. *In equilibrium and under Assumptions 1–6 and above assumption about g_s, the price of catastrophe risk $\{Y_s\}$ between time 0 and T is given by*

$$c=\mathrm{e}^{-rT}\int_0^T \mathrm{e}^{h_s^*}\lambda_s E[L_s\mathrm{e}^{\gamma L_s}]\mathrm{d}s \qquad (2.6)$$

with h_s^* satisfying $\frac{M_{L_s}(\gamma)-1}{\gamma}=h_s^* g'(h_s^*\mathrm{e}^{h_s^*}-\mathrm{e}^{h_s^*}+1)$, in which M_{L_s} denotes the moment generating function of L_s and g' denotes the derivative of g.

See Appendix A for proof of Proposition 1.

Remark 2.2. In the above discussion, we have used exponential utility function for convenience, i.e., we assume the absolute risk aversion coefficient as a constant. As is well known, the absolute risk aversion coefficient γ of a representative agent should be decreasing with the agent's wealth w. For later empirical test, we assume $\gamma = \tilde{\gamma}/w$ with $\tilde{\gamma}$ being a risk aversion coefficient with more sensible meaning.

Remark 2.3. From the pricing formula (2.6), we find that it is independent of the stock return. In other words, the behavior of stock market has no effect on CAT risk market. The rationale of this phenomenon is due to the assumption that catastrophe risk is independent of the stock risk and the behavior of our representative agent has no effect on the stock and risk-free bond market. Cummins and Weiss (2009) recently gave evidence that CAT bond returns are significantly correlated with other financial market during the subprime crisis period. One possible explanation of this phenomenon by our model is that a subprime crisis may change the ambiguity aversion magnitudes of all agents in the economy, including the representative agent in our pure catastrophe risk market.

Remark 2.4. The above discussion has assumed that the contract exchanged covers a proportion of total risk but by augmenting new types of contingent claim contracts in our model we will have the same equilibrium pricing formula as given in Proposition 1. See Proposition 3 in the next section for a valuation formula of more general catastrophe-contingent securities.

3. Catastrophe-linked securities pricing

3.1. More pricing formulas

We make further simplifying assumptions that the catastrophe risk process Y_s is a homogeneous compound Poisson process with λ_s being a constant λ and L_s being described by identical probability distribution function, $\Pr(L_s \le x) = F_L(x)$ where $F_L(x)$ denotes the probability distribution function of a random variable L, Proposition 1 can then be simplified as follows:

Proposition 2. *In equilibrium and under above assumptions, the pricing formula (2.6) then becomes*

$$c=\mathrm{e}^{-rT}\lambda\mathrm{e}^{h}E[L\mathrm{e}^{\gamma L}]T, \qquad (3.1)$$

with h satisfying $\frac{M_L(\gamma)-1}{\gamma} = hg'(h\mathrm{e}^h-\mathrm{e}^h+1)$. The pricing formula (3.1) can also be rewritten as an expectation with respect to an equivalent martingale measure (EMM) Q in the following equivalent form:

$$c=E_0^Q[\mathrm{e}^{-rT}Y_T]=E\left[\mathrm{e}^{-rT}Y_T\frac{\exp(\gamma Y_T+hN_T)}{E[\exp(\gamma Y_T+hN_T)]}\right], \qquad (3.2)$$

with the Radon–Nikodym derivative of Q with respect to P being given by $\mathrm{d}Q/\mathrm{d}P=\exp(\gamma Y_T+hN_T)/E[\exp(\gamma Y_T+hN_T)]$.

See Appendix B for the proof of Proposition 2.

Remark 3.1. Notice that if h becomes zero, i.e., the market shows no ambiguity aversion, the form of $\mathrm{d}Q/\mathrm{d}P$ is reduced to the traditional Esscher transform, which has been used for a long time in insurance and finance pricing (Bühlmann, 1980; Gerber and Shiu, 1994, 1996). The difference between the pricing kernel here and the traditional one is that we augment an Esscher transform applying only to Poisson process N_s to represent the ambiguity aversion of the agents.

Remark 3.2. It is well known that the traditional Esscher transform can be derived by a kind of relative entropy minimizing martingale measure (see, e.g. Esche, 2004, for a thorough introduction). Proposition 2 shows that Esscher transform is also related to the robust control theory when the penalty term is defined by a function of relative entropy of the ambiguity distorted probability measure with respect to the original one. The difference is that with the entropy minimizing martingale measure the non-singleton set of equivalent martingale measures arises from the market incompleteness while in the setting of robust control theory, the multiplicity of measures is due to the ambiguity of agents.

Now we apply the pricing kernel derived above to the pricing of more general catastrophe-linked securities. These kinds of securities may be looked as derivatives contingent on catastrophe losses. Let φ be a measurable function that specifies the payoff at maturity to the buyer of the catastrophe-linked securities, i.e., at time T the buyer receives $\varphi(Y_T)$. The following proposition states the valuation formula for catastrophe-linked securities.

Proposition 3. *Under above assumptions, the price of a contingent claim at time 0 with payoff $\varphi(Y_T)$ at time T is given by*

$$c=E_0^Q[\mathrm{e}^{-rT}\varphi(Y_T)]=E\left[\mathrm{e}^{-rT}\varphi(Y_T)\frac{\exp(\gamma Y_T+hN_T)}{E[\exp(\gamma Y_T+hN_T)]}\right]. \qquad (3.3)$$

The above result can be derived either by augmenting the new types of contingent claim contracts in the equilibrium model in Section 2 or by using no arbitrage principle directly.

3.2. Catastrophe bonds pricing and empirical test

We now apply formula (3.3) to catastrophe bonds pricing. We assume for convenience of discussion that the form of CAT bond is described as follows:

The CAT bond is priced at p, which denotes the bond principal. If the loss for a single CAT event in the period $(t, t+1)$ is less than a loss trigger B_1, the agent will get back his principal p, and spread premium plus interest $p(\mathrm{e}^r - 1 + \tilde{l})$ at maturity time $t+1$, in which $\tilde{l}$ denotes the spread premium rate. Once a CAT loss exceeds the trigger, the agent will forfeit some or all the principal at time $t+1$. The loss fraction f_{12} of the principal is in proportion to the conditional loss L in the range between the trigger B_1 and a cap B_2 and is given by $f_{12} = \mathrm{Max}[0, \mathrm{Min}(L - B_1, B_2 - B_1)]/(B_2 - B_1)$. We assume the spread premium plus the risk-free interest is guaranteed no matter whether any principal loss occurs. The expected value of the bond principal loss proportion is denoted as l.

The ratio of the spread premium $\tilde{l}$ to the expected principal loss l is then given by the following proposition.

Proposition 4. *Under above assumptions, the ratio of the spread premium to the expected principal loss proportion (i.e., the ratio of the modified expected principal loss to the expected loss) of the above CAT bond is given by*

$$\frac{\tilde{l}}{l} = \frac{(1 - \mathrm{e}^{-\lambda \mathrm{e}^h M_L(\gamma)})E[f_{12}\mathrm{e}^{\gamma L}]}{(1 - \mathrm{e}^{-\lambda})E[f_{12}]M_L(\gamma)}. \tag{3.4}$$

See Appendix C for proof of Proposition 4.

With convexity assumption of the function g, h is always positive and the ratio derived in Proposition 4 has following properties:

Property 1. *The ratio is larger than* 1*, i.e., there is a risk and ambiguity aversion loading for taking the catastrophe risk.*

Property 2. *When λ is very small, the following approximation formula holds:*

$$\frac{\tilde{l}}{l} = \frac{(1 - \mathrm{e}^{-\lambda \mathrm{e}^h M_L(\gamma)})E[f_{12}\mathrm{e}^{\gamma L}]}{(1 - \mathrm{e}^{-\lambda})E[f_{12}]M_L(\gamma)} \approx \frac{\lambda \mathrm{e}^h M_L(\gamma) E[f_{12}\mathrm{e}^{\gamma L}]}{\lambda E[f_{12}]M_L(\gamma)} = \frac{\mathrm{e}^h E[f_{12}\mathrm{e}^{\gamma L}]}{E[f_{12}]}. \tag{3.5}$$

From Eq. (3.5), we can find that the spread premium emerging from risk aversion is relatively small and is not important for CAT bond pricing. For example, if we assume the risk aversion coefficient $\tilde{\gamma} = 2$ and catastrophe loss L as a constant with its shock to the wealth of the representative agent L/w being about 5%, the spread premium ratio emerging from risk aversion in (3.5) may approximately be estimated as

$$\frac{E[f_{12}\mathrm{e}^{\gamma L}]}{E[f_{12}]} = E[\mathrm{e}^{\gamma L}] = \mathrm{e}^{\tilde{\gamma}\frac{L}{w}} \approx 1.1. \tag{3.6}$$

So in later paragraphs, we will focus on the effect of ambiguity aversion and assume the market is risk neutral to the catastrophe risk, in other words, we will assume $\gamma = 0$ later. Under this assumption, the formula (3.5) then reduces to $\frac{\tilde{l}}{l} \approx \mathrm{e}^h$ with h satisfies $E[L] = hg'(h\mathrm{e}^h - \mathrm{e}^h + 1)$.

We now assume the solution of h for the equation $E[L] = hg'(h\mathrm{e}^h - \mathrm{e}^h + 1)$ can be expressed as $\beta = \phi_1(E[L])$ and with the convexity assumption about the function g, it can be proved that ϕ_1 is an increasing function.

Example 3.1. If we choose $g(x) = x/a$ as that in Example 2.2, we then have $h = \phi_1(E[L]) = aE[L]$.

We find the traditional penalty function in Example 3.1 is too weak to fit empirical test. In the following empirical test we choose another specific function g so that

$$h = \phi_1(E[L]) = \ln(1 + A_1(E[L])^{A_2}),$$

where $A_1 > 0$ and $A_2 > 0$ are two parameters dependent on g that measure the ambiguity aversion of the representative agent. The approximating formula for the spread premium to the expected loss ratio then becomes:

$$\frac{\tilde{l}}{l} \approx \mathrm{e}^h = 1 + A_1(E[L])^{A_2}. \tag{3.7}$$

Given the lack of empirical data for $E[L]$, we are unable to directly apply the formula (3.7) and thus are forced to assume a functional relationship between $E[L]$ and the probability of first loss $1 - \mathrm{e}^{-\lambda}$ because the later data is usually available. It is well known that for catastrophe risk, there is a converse relationship between the conditional expected loss and catastrophe occurrence probability. The relationship is usually called power law in natural disaster science (see, e.g. Newman, 2005) and is written as $E[L] = \varsigma_1(1 - \mathrm{e}^{-\lambda})^{-\varsigma_2}$ with ς_1, ς_2 being two positive constants given by the nature of the catastrophe. With this relationship, the approximation spread premium ratio (3.7) then becomes:

$$\frac{\tilde{l}}{l} \approx \mathrm{e}^h = 1 + A_1(\varsigma_1(1 - \mathrm{e}^{-\lambda})^{\varsigma_2})^{-A_2}. \tag{3.8}$$

We can find that the parameter A_1 affects the agent's ambiguity aversion for all kinds of catastrophe risks in the same way, whereas the parameter A_2 affects ambiguity aversion for lower frequency risks more than for higher frequency ones. Since both A_1 and A_2 are positive, we have the following property for the approximation ratio:

Property 3. *The approximation ratio of the spread premium to the expected principal loss* e^h *is a decreasing function of the probability of first loss* $(1 - \mathrm{e}^{-\lambda})$.

Property 3 shows that our model does help explain the second stylized fact in the introduction. With the above approximating formula (3.8), we now turn to calibrate our model to match empirical data. The data is from Table 1 with 32 CAT bonds issued between 1997 and 2000. Without loss of generality we will assume $\varsigma_1 = 1$, $\varsigma_2 = 1$ for convenience later and our following empirical test has already adjusted the risk aversion effect with the spread premium ratio magnitude of about 1.1 emerging in Eq. (3.6).

Lacking enough data for a single year, we will use data of all CAT bonds issued in the period 1997–2000 for empirical fitting. By using intertemporal empirical data we have neglected the bonds' intra-year pricing shift. But if we use secondary market data at a fixed time instead to get a contemporaneous view, we will be at the risk that the occurrence probability data (which are estimated when the bonds were issued) have shifted. We may use these intertemporal data here because there is no large price fluctuation in catastrophe insurance market during this period.

The in-sample regression shows that $A_1 = 0.1801$ and $A_2 = 0.7558$. Fig. 2 shows the calibrated result estimated by our model as well as the adjusted empirical ratio of premium to expected loss against the probabilities of first loss. The R-square of the regression is about 0.81 and it can be seen from the graph that the calibrated result provides an adequate fit to the observed data.

We now turn to examine the model's dynamic performance. We will add a time indicator or subscript to the letters later to represent their dynamic properties. Since we have been unable to get a closed-form solution for the general continuous time

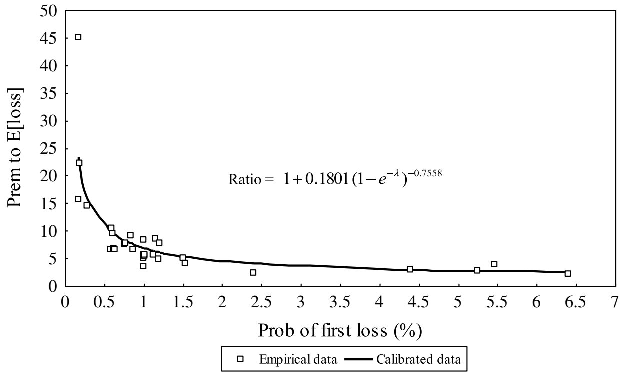

Fig. 2. The calibrated and empirical ratios of expense-adjusted premium spreads to expected loss against probabilities of first loss (In-sample fitting to 1997–2000 CAT bonds data).

case, we will approximate the continuous time Eq. (2.3) by its corresponding discrete time model. Assuming an appropriate discrete dynamic behavior for g_t (which denotes the corresponding g at time t), we may write the discrete stochastic dynamic behavior of $A(t) = (A_1(t), A_2(t))'$ as follows:

$$A(t) = \vartheta_0 + \vartheta_1 A(t-1) + \vartheta_2 \Delta Z_{t-1}, \tag{3.9}$$

where ϑ_0 and ϑ_2 are two constant 2×1 vectors, ϑ_1 is a constant 2×2 diagonal matrix, and $\Delta Z_{t-1} = Z_t - Z_{t-1}$.

With constant assumption of ambiguity aversion parameters over the interval $(t, t+1)$, the approximation ratio of spread premium to the probability of first loss of the CAT bond for the interval is then given by $e^{h_t} = 1 + A_1(t)(1 - e^{-\lambda})^{-A_2(t)}$. We thus have the following Property 4 to describe the dynamic behavior of the CAT bonds pricing:

Property 4. *The dynamic behavior of the approximation ratio of spread premium to the expected principal loss is described as below:*

$$\begin{aligned} e^{h_t} - e^{h_{t-1}} &= A_1(t)(1-e^{-\lambda})^{-A_2(t)} \\ &\quad - A_1(t-1)(1-e^{-\lambda})^{-A_2(t-1)}, \end{aligned} \tag{3.10}$$

in which the dynamic behaviors of $A_1(t)$ and $A_2(t)$ are described by Eq. (3.9).

Lacking enough data, we still cannot apply the above dynamic model to empirical analysis. So we will only do some simulation by assuming that the vectors ϑ_0 and ϑ_2 and matrix ϑ_1 are as below:

$$\vartheta_0 = \begin{pmatrix} 0.03 \\ 0.09 \end{pmatrix}, \qquad \vartheta_1 = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.7 \end{pmatrix}, \qquad \vartheta_2 = \begin{pmatrix} 0.01 \\ 0.01 \end{pmatrix}.$$

We further assume that the aggregate catastrophe loss is described by

$$\Delta Z_t = \tilde{L}_0 + \sum_{i=1}^{\tilde{N}} \tilde{L}_i,$$

where $\tilde{L}_0$ is lognormal distributed with parameters (0, 1), $\tilde{N}$ is Poisson distributed with intensity $\tilde{\lambda} = 0.03$ and $\tilde{L}_i$, $i \geq 1$ are identically lognormal distributed random variables with parameters (2.5, 0.7). At original time 0 we assume the ambiguity aversion parameters are $A_1(0) = 0.1801$ and $A_2(0) = 0.7558$.

Fig. 3 shows a simulated path of the aggregate catastrophe losses ΔZ_t from time 0 to time 17. It shows that there is a large catastrophe occurrence at time 5 and at other times the aggregate losses are relatively small.

Fig. 4 shows time series of the ratios e^{h_t} for the CAT bonds with probability of first loss $(1 - e^{-\lambda}) = 0.01$ based on the model and the above specified parameters and we can see that there is a price jump at time 6, which indicates that the prices are significantly affected by the large catastrophe occurrence at time 5. Price trend after the price jump at time 6 is slowly down. The reason is that our dynamic model incorporates a mean reversion assumption.

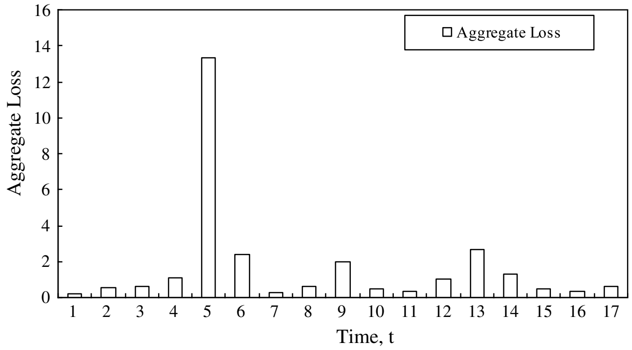

Fig. 3. Simulated aggregate loss process ΔZ_t based on the assumption that $\Delta Z_t = \tilde{L}_0 + \sum_{i=1}^{\tilde{N}} \tilde{L}_i$, in which $\tilde{L}_0$ is lognormal distributed with parameters (0, 1), $\tilde{N}$ is Poisson distributed with intensity $\tilde{\lambda} = 0.03$ and $\tilde{L}_i$, $i \geq 1$ is identically lognormal distributed with parameters (2.5, 0.7).

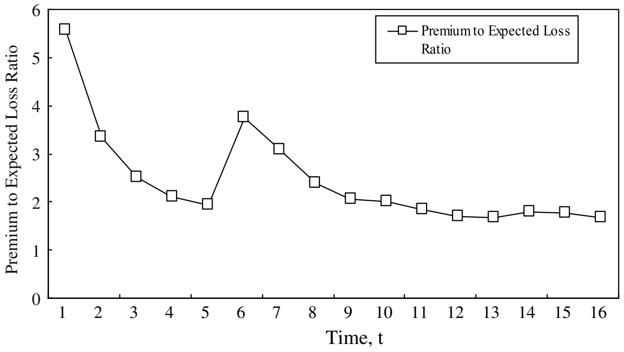

Fig. 4. The time series behavior of the premium spread to expected loss ratios for the CAT bonds with probability of first loss $(1 - e^{-\lambda}) = 0.01$ based on the model assumption and aggregate catastrophe loss simulation in Fig. 3.

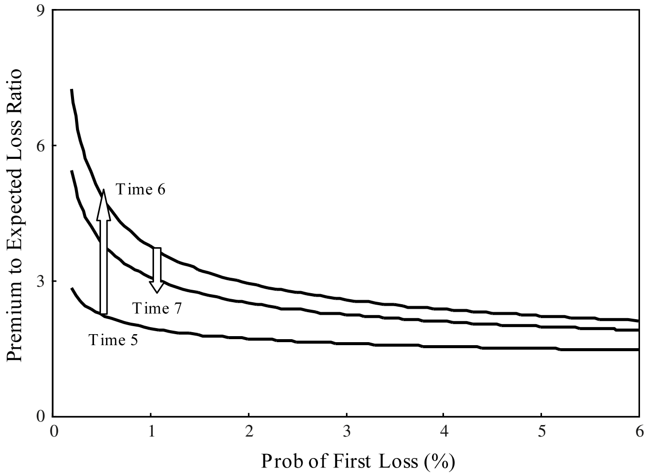

Fig. 5. The premium to expected loss ratios against probabilities of first loss based on the model assumption and aggregate catastrophe loss simulation in Fig. 3: Time 5 vs. Time 6 vs. Time 7.

Fig. 5 illustrates the price changes during the time 5 to time 7 based on the model and parameters assumption. As the graph shows, there is a price jump for catastrophe bonds with all kinds of occurrence frequencies from time 5 to time 6. An interesting fact is that the CAT bonds of lower occurrence frequencies have increased by more than those of higher occurrence frequencies. The reason is that by our model assumption, the ambiguity aversion magnitudes have increased more for catastrophe risks of lower occurrence probabilities than for those of higher occurrence probabilities and the curve of spread premiums to expected loss ratios is getting slightly steeper around a large catastrophe event. The above specific model simulation does illustrate the third stylized fact mentioned in the introduction.

3.3. Further discussion

Our specific model tries to apply ambiguity aversion and robust control theory to explain empirical facts concerning catastrophe securities pricing. The concept of ambiguity aversion is not new to actuarial professionals. The literature on credibility theory, for example, provides a framework examining the effect of learning the imprecise knowledge about insurance fundamentals by *successive approximations* (Bühlmann, 1970). But in dealing with CAT risk, this kind of learning or approximating model may not work given that CAT events are very infrequent and learning seems impossible in this setting.

In our current robust control settings, the perspective of a decision maker differs substantially from that of one who learns. The agent accepts model misspecification as a permanent state rather than using data to improve his model specification over time. The agent is assumed to deal with model uncertainty as follows: First, having noticed the unreliable aspects of reference model based on existing information, he evaluates the future prospects under alternative models. Second, acknowledging that the reference model is indeed the best statistical characterization of the available information, he penalizes the choice of alternative models by a distance function measuring how far it deviates from the reference model. On the other hand, an agent in the framework of credibility theory balances between the past claim experience and an estimator from other information, but does not show ambiguity aversion to the inaccuracy of the experience information. Our previous discussions illustrate that the ambiguity aversion assumption does help explain the stylized puzzles about catastrophe risk pricing.

As mentioned in the introduction, Froot (1999) has argued that the puzzles concerning CAT reinsurance are instead due to some form of capital market imperfection. He lists several underlying causes driving to the imperfection and concludes that the most compelling reason is that the information asymmetry between shareholders and mangers of reinsurance companies drives up capital cost and prevents reinsurers to raise external capital to do catastrophe reinsurance business. Since by catastrophe risk securitization, we may effectively take the catastrophe reinsurance out of corporate form and thus completely eliminate managerial discretion and misbehavior, this inefficient corporate form reason may not be used to explain empirical puzzles for catastrophe-linked securities.

Another alternative approach that may help explain the high level of CAT bond spread and several other empirical facts is Dieckmann (2008), which adapts Campbell and Cochrane (1999)'s external habit model to include the effect of catastrophe risk. It is therefore important for us to understand what new insight our robust control framework model may add to the external habit model. Liu et al. (2005) has shown that Campbell and Cochrane's habit model may not generate the option smile pattern, therefore we may reasonably conjecture that it is also not easy for the external habit model to explain the second stylized fact mentioned in the introduction though the rich dynamic properties of the habit model may help explain many other behavior of catastrophe bond spread. Another possible advantage of our approach is that we apply a partial equilibrium model and so avoid the worry about the consistence problem a general equilibrium approach may incur to calibrate all economic phenomena in a single model. Cummins and Weiss (2009) shows that during the normal period prior to the subprime financial crisis, CAT bond returns have almost no correlation with returns of alternative financial investments. The data set in the period of course already includes the event of Hurricane Katrina, so it seems reasonable to deal with pure catastrophe risk in a partial equilibrium framework. The significant increase of the correlation of the CAT bonds with other financial instruments during the subprime crisis period is not due to a natural disaster and as mentioned in Remark 2.3, our ambiguity aversion model may give some insight of explaining this interesting phenomena.

The difference of our approach from Liu et al. (2005) is that we have injected the aggregate catastrophe loss as an exogenous factor to affect the ambiguity aversion magnitude of the representative agent. In Liu et al. (2005), the ambiguity aversion magnitude is a constant so their model is in essence still a static one. In our model, the shock of the aggregate catastrophe loss is assumed to be correlated with the ambiguity aversion magnitude and this assumption can be used to explain the cyclical behavior of CAT bonds spread. Our dynamic specification is in conjunction with the discussion in Cummins and Weiss (2009), which ascribes the CAT bond cycles to two primary factors: (1) Increasing uncertainty about the accuracy of loss models following a large event, and (2) time lags in the development of expertise required to participate in the CAT bond market following a surge in demand. Both factors may be seen as equivalent to more ambiguity aversion of the representative agent after a large event occurs. Another difference of our approach from Liu et al. (2005) is that we have introduced a more general penalty function form which may be chosen to fit empirical test. We find that ambiguity aversion premium emerging from Liu et al. 's penalty function choice seems too high for empirical test of catastrophe bonds data. Moreover, the power utility function chosen by Liu et al. (2005) seems not so suitable for the discussion of natural catastrophe shock in our model.

4. Conclusion

Motivated by several stylized facts concerning catastrophe-linked securities, we propose an intertemporal equilibrium model by allowing agents to act in a robust control framework against model misspecification with respect to rare events. There are several extensions to our current investigation.

First, we have considered the issue of uncertainty aversion in a robust control framework and thus envision the model misspecification as a permanent psychological characteristic of the decision maker. On the other hand, an agent may learn the model through successive approximations. Cummins and Weiss (2009) shows that the empirical CAT bond spread has declined over time and this phenomenon can also be noticed in Fig. 3 generated from exogenously given dynamic assumption in our model. It would be an important extension to incorporate forms of learning, for example, the credibility theory, into our framework to give an endogenous explanation of the assumption.

Second, we have used issue prices during the period 1997–2000 as the empirical data for fitting, which is at the risk of losing intra-year shifts in pricing. But using secondary market data instead to get a contemporaneous view will be at the risk that estimated loss data has shifted. It will be a challenging problem to get the updating of the probability of first loss as well as price data to test the dynamic behavior of our model.

Thirdly, we have introduced a more general penalty function form than that in Hansen and Sargent (2001) which can be chosen to fit empirical test. But the general penalty term introduced here is still a function of the relative entropy which measure the distance between the reference and alternative model. It will be very interesting to generalize the penalty function to a form of more practical interest. A potential choice may be related to the expected discounted penalty function introduced in the seminal paper of Gerber and Shiu (1998), which has many tractable analytical properties. In Appendix D we give a possible connection of the Gerber–Shiu penalty function with the penalty term (2.2). See for example Gerber and Landry (1998) and Lin and Wang (2009) for application of the Gerber–Shiu penalty function to the pricing of perpetual put options.

Finally, our equilibrium model exploits a representative paradigm. The representative agent is assumed to underwrite the total catastrophe risk and thus there is no insurability problem in our model, but just as Gollier (2005) pointed out, an insurability problem may occur if agents in the economy are different. An introduction of heterogeneous agents might result in new insights about the role of robust control theory in catastrophe insurance research. It would be a very interesting topic to study the idiosyncratic degrees of agents' ambiguity aversion and its effect on insurance coverage window and the window shift after a large catastrophe event occurs.

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Appendix A. Proof of Proposition 1

With assumption that $g_s = g$, the HJB equation (3.5) is reduced to:

$$\sup_{\pi_s}\left\{J_s - \rho J + (\mu - r)\pi_s J_W + \frac{1}{2}\sigma^2\pi_s^2 J_{WW} + rWJ_W \right.$$
$$+ \inf_{h_s}\left\{\tilde{E}_s J\left(W, s, y + \frac{\mathrm{d}Y_s}{\mathrm{d}s}, m, g\right) - J(W, s, y, m, g)\right.$$
$$\left.\left. - \lambda_s\gamma g(1 + \mathrm{e}^{h_s}(h_s - 1))J\right\}\right\} = 0.$$

The maximum of the equation is achieved at $\pi_s^* = -\frac{(\mu-\delta)}{\sigma^2}\frac{J_W}{J_{WW}}$. We conjecture that the indirect function J is of the form $J(W, s, y, m, g) = V(W, s)\mathrm{e}^{\gamma m y}f(s, m, g)$ where $V(W, s)$ is the usual *value function* defined as

$$V(W, s) = -\frac{1}{\gamma}\exp\left(-\gamma W\mathrm{e}^{r(T-s)} - \frac{(\mu - r)^2}{2\sigma^2}(T - s) - \rho(T - s)\right),$$

which satisfies the traditional HJB equation

$$V_s - \rho V + rWV_W - \frac{(\mu - r)^2}{2\sigma^2}\frac{V_W^2}{V_{WW}} = 0, \qquad \text{(A.1)}$$

and $f = f(s, m, g)$ is a time-dependent function.

Inserting the expression for J and π_s^* into the HJB equation, we can rewrite the equation as

$$V_s f + Vf_s - \rho Vf - \frac{(\mu - r)^2}{2\sigma^2}\frac{V_W^2}{V_{WW}}f + rWV_W f$$
$$+ \inf_{h_s}\{\lambda_s\mathrm{e}^{h_s}V(M_{L_s}(m\gamma) - 1)f(s, m, g)$$
$$- \lambda_s\gamma g(1 + \mathrm{e}^{h_s}(h_s - 1))Vf\} = 0.$$

Since the value function V satisfies Eq. (A.1), the first, third, fourth and fifth terms cancel and canceling $V < 0$ from the remaining terms yields the following partial differential equation for f

$$f_s + \sup_{h_s}\{\lambda_s\mathrm{e}^{h_s}(M_{L_s}(\gamma m) - 1)f$$
$$- \lambda_s\gamma g(1 + \mathrm{e}^{h_s}(h_s - 1))f\} = 0, \qquad \text{(A.2)}$$

with boundary condition $f(T, m, g) = 1$.

With the increasing and convexity assumption about the function g, the maximum of Eq. (A.2) is achieved at h_s^* which satisfies:

$$M_{L_s}(\gamma m) - 1 = \gamma h_s^* g'(h_s^*\mathrm{e}^{h_s^*} - \mathrm{e}^{h_s^*} + 1).$$

The solution of (A.2) at time 0 is given by

$$f(0, m, g) = \exp\left(\int_0^T\left[\lambda_s\mathrm{e}^{h_s^*}(M_{L_s}(\gamma m) - 1)\right.\right.$$
$$\left.\left. - \lambda_s\gamma g(1 + \mathrm{e}^{h_s^*}(h_s^* - 1))\right]\mathrm{d}s\right),$$

the indirect utility function J at time 0 with $W = w + mc, y = 0$ is thus given by

$$J(w + mc, 0, 0, m, g)$$
$$= V(w + mc, 0)f(0, m, g)$$
$$= -\frac{1}{\gamma}\exp\left\{-\gamma(w + mc)\mathrm{e}^{rT} - \frac{(\mu - r)^2}{2\sigma^2}T - \rho T\right.$$
$$\left. + \int_0^T[\lambda_s\mathrm{e}^{h_s^*}(M_{L_s}(\gamma m) - 1) - \lambda_s\gamma g(1 + \mathrm{e}^{h_s^*}(h_s^* - 1))]\mathrm{d}s\right\}. \qquad \text{(A.3)}$$

The first order condition for m (the second order condition also holds) in Eq. (A.3) gives the following equation for the optimal insurance proportion m^*:

$$c = \mathrm{e}^{-rT}\int_0^T \mathrm{e}^{h_s^*}\lambda_s E[L_s\mathrm{e}^{\gamma m * L_s}]\mathrm{d}s.$$

Since in equilibrium the market is cleared and the representative agent will accept total catastrophe loss under the equilibrium price, in other words in equilibrium we should have $m^* = 1$, the last equation thus simplifies to (2.6) in equilibrium and the equilibrium price is thus given by Proposition 1.

Appendix B. Proof of Proposition 2

Proof of Formula (3.1) is direct. Formula (3.2) is just the modified expected value of Y_T under the equivalent probability measure Q_T. The corresponding moment generating function of the random variable Y_T under the new measure Q_T can be calculated as

$$E[\exp((\gamma + z)Y_T + hN_T)/E[\exp(\gamma Y_T + hN_T)]]$$
$$= \exp[\lambda(M_L(\gamma + z)\mathrm{e}^h - 1)T]/\exp[\lambda(M_L(\gamma)\mathrm{e}^h - 1)T]$$
$$= \exp\left[\lambda M_L(\alpha)\mathrm{e}^h\left(\frac{M_L(\gamma + z)}{M_L(\gamma)} - 1\right)T\right].$$

Hence Y_T under the new measure is again a compound Poisson process, with modified Poisson parameter $\lambda M_L(\gamma)\mathrm{e}^h$ and loss amount becomes a random variable whose moment generating function is $\frac{M_L(\gamma+z)}{M_L(\gamma)}$. The modified expected loss amount can then be calculated as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{M_L(\gamma+z)}{M_L(\gamma)}\right)\Bigg|_{z=0} = E\left[L\frac{\mathrm{e}^{(\gamma+z)L}}{M_L(\gamma)}\right]\Bigg|_{z=0} = E\left[L\frac{\mathrm{e}^{\gamma L}}{M_L(\gamma)}\right].$$

Therefore formula (3.2) is given by

$$\begin{aligned} c &= E\left[\mathrm{e}^{-rT}Y_T\frac{\exp(\gamma Y_T + hN_T)}{E[\exp(\gamma Y_T + hN_T)]}\right] \\ &= \mathrm{e}^{-rT}\lambda \mathrm{e}^h M_L(\gamma)E\left[L\frac{\mathrm{e}^{\gamma L}}{M_L(\gamma)}\right]T = \mathrm{e}^{-rT}\lambda \mathrm{e}^h E[L\mathrm{e}^{\gamma L}]T, \end{aligned}$$

which is exactly formula (3.1).

Appendix C. Proof of Proposition 4

The loss fraction f_{12} of the principal is given by

$$\begin{aligned} f_{12} = f_{12}(L) &= \mathrm{Max}[0, \mathrm{Min}(L - B_1, B_2 - B_1)]/(B_2 - B_1) \\ &= (\mathrm{Max}[0, L - B_1] - \mathrm{Max}[0, L - B_2])/(B_2 - B_1) \\ &= ((L - B_1)_+ - (L - B_2)_+)/(B_2 - B_1). \end{aligned}$$

Thus the cash flow the agent gets back at time $t + 1$ can be described as

$$\varphi(Y_1) = p \cdot (\mathrm{e}^r + \tilde{l} - I(N_1 > 0)f_{12}(L_1)),$$

where I is an event indicator function, L_1 denotes the first conditional catastrophe loss. The probability that at least one catastrophe loss occurs in the period is given by $E[I(N_1 > 0)] = (1 - \mathrm{e}^{-\lambda})$. The mean of the bond principal loss proportion is thus given by $l = E[I(N_1 > 0)f_{12}(L_1)] = (1 - \mathrm{e}^{-\lambda})E[f_{12}]$.

Now we apply the equivalent martingale measure $Q = \{Q_s\}$ in Proposition 3 instead to price the above CAT bond. The basic equation can be written as:

$$p = E^Q[p \cdot \mathrm{e}^{-r}(\mathrm{e}^r + \tilde{l} - I(N_1 > 0)f_{12}(L_1))],$$

from which we can get the formula to calculate the spread premium as below,

$$\begin{aligned} \tilde{l} &= E[\exp(\gamma Y_1 + hN_1)I(N_1 > 0)f_{12}(L_1)]/E \\ &\quad \times [\exp(\gamma Y_1 + hN_1)], \end{aligned} \tag{C.4}$$

which can be calculated as

$$\tilde{l} = \frac{(1 - \mathrm{e}^{-\lambda \mathrm{e}^h M_L(\gamma)})E[f_{12}(L)\mathrm{e}^{\gamma L}]}{M_L(\gamma)},$$

thus the spread premium to the mean principal loss ratio is as given in Proposition 4.

Appendix D. A connection between the Gerber–Shiu penalty function and the penalty term (2.2)

Define a special Gerber–Shiu penalty function as

$$\begin{aligned} &\phi(u; \omega(x), \vartheta) \\ &\quad = E[\omega(V(S))I_{\{S<\infty\}}|V(0) = u,\ \text{the loading factor is } \vartheta], \end{aligned}$$

with S denoting the ruin time and $V(t)$ is interpreted as an insurer's surplus at time t; then we have

$$\phi(0; \mathrm{e}^{-\gamma x}, \vartheta) = \frac{1}{(1 + \vartheta)E[L]}\frac{M_L(\gamma) - 1}{\gamma}. \tag{D.5}$$

See, e.g., Bowers et al. (1997, Chapter 13) for a proof of (D.5). In the robust control framework, people consider distorted probability measure $\tilde{P} = (\tilde{P}_s)$, and under this new measure, the fair premium rate should be $\lambda \mathrm{e}^h E[L]$. But under the old measure, the fair premium rate is $\lambda E[L]$, so we have $(1 + \vartheta) = \mathrm{e}^{-h}$. With h satisfying $\frac{M_L(\gamma)-1}{\gamma} = hg'(h\mathrm{e}^h - \mathrm{e}^h + 1)$ in Proposition 2, we have

$$\begin{aligned} \phi(0; \mathrm{e}^{-\gamma x}, \vartheta) &= \frac{1}{(1 + \vartheta)E[L]}\frac{M_L(\gamma) - 1}{\gamma} \\ &= \frac{1}{E[L]}\mathrm{e}^h hg'(h\mathrm{e}^h - \mathrm{e}^h + 1) \\ &= \frac{1}{E[L]}[g(h\mathrm{e}^h - \mathrm{e}^h + 1)]'. \end{aligned}$$

Since in the robust control framework, the penalty term is defined as

$$\psi(h) = \gamma U\lambda g(h\mathrm{e}^h - \mathrm{e}^h + 1).$$

It seems that the optimal distortion factor h is chosen so that the marginal penalty term is just a multiple of the special Gerber–Shiu penalty function, or more precisely,

$$\psi'(h) = \gamma U\lambda[g(h\mathrm{e}^h - \mathrm{e}^h + 1)]' = \gamma U\lambda E[L]\phi(0; \mathrm{e}^{-\gamma x}, \vartheta).$$

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