Postponement and Information in a Supply Chain

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Abstract

We model a supply chain consisting of a production facility, a distribution center and two differentiated markets. Demand information is used to mitigate the effects of uncertainty in the output markets. We study the firm's operational performance under alternative business processes, comparing *early* and *delayed* product differentiation. Whereas most studies in this area require either approximation or numerical experimentation, we are able to derive the exact solutions (the firm's optimal production, distribution and inventory policies) under each process, and compare them, in a dynamic (multiperiod) setting. The comparison yields the *value of postponement*. Our results show that informational considerations have a paramount effect on the effectiveness of postponement strategies. Specifically, the value of postponement is determined in combination by three important attributes of information– *(i)* its *timing* within the supply chain (relative to material flows), *(ii)* its *level of aggregation*, and *(iii)* its *precision*. We also quantify the effects of demand variability, end-product demand correlation and inventory holding costs on the value of postponement.

1 Introduction

Demand uncertainty is an important feature of contemporary business environments. Proliferation in product varieties [Fisher *et al* (1994)], accelerated clockspeed [Mendelson and Pillai (1999)] and the volatility of the global marketplace increase demand uncertainty, amplifying the tradeoff between inventory-levels and service-levels: inventories can improve firms' responsiveness to customer demand, but this is accomplished at a cost. Some firms have successfully employed "accurate response" strategies to reduce demand uncertainty, usually involving the extensive use of Information Systems for data collection and demand forecasting [Anand and Mendelson (1997)]. Nonetheless, uncertainty remains an important factor, especially in fast-clockspeed industries like high-tech or industries with long production and distribution lead-times like the apparel industry. Firms are thus restructuring the business processes that underlie their supply chains to better cope with demand uncertainty. One strategy that has received considerable attention is design for postponement, or delayed product differentiation.

Consider the process used to make multiple related products, say widgets that are differentiated by color. At the *point of differentiation*, the widgets pass through a painting station and become differentiated (in this case by color). All processes upstream to the point of differentiation are identical for all widgets; a widget gains its particular identity only at that point. Beyond the point of differentiation, the entire order fulfillment process (i.e., both the firm's operations and its information systems) must distinguish between the widgets based on their different colors.

Differentiation can apply in a similar fashion to different product attributes, target market segments and sales regions. *Postponement* is a strategy that delays the point of differentiation, moving it further downstream, often by restructuring the supply chain. The result is increased flexibility in meeting uncertain and changing customer demands. While this concept has its roots in the manufacturing sector, services (for example, fast food) are beginning to apply postponement to their processes as well.

Postponement enables the firm to get the best (or, rather, avoid the worst) of make-to-stock and make-to-order policies. Under a make-to-stock policy, demand needs to be estimated accurately, since forecasting errors are severely punished: in industries like high-tech or apparel, a product can lose 1% of its value per week, and the loss rate is even higher for perishable-goods. Under make-to-order, long lead times make it difficult for the firm to compete. Postponement is a *via media*: lead times are reduced by making an intermediate good to stock, while inventory markdowns as well as lost sales are also reduced by customizing the intermediate good based on observed demand or order patterns. These improvements can be achieved simultaneously due to better use of information.

Benetton provides a celebrated example of using postponement to cope with long production lead-times and fickle fashion trends, using undyed yarn to knit about half of its clothing [Benetton (1984)]. Dyeing is thus postponed to a later stage, when Benetton has a better idea of the popular colors for the season. Another well-known example of postponement, due to Hau Lee and collaborators, is Hewlett-Packard (HP) [*cf* Lee *et al* (1993); Feitzinger and Lee (1997)]. HP manufactured its Deskjet-Plus printers in its Vancouver, Washington Division, and shipped the printers to three Distribution Centers (DCs) in North America, Europe and Asia. The transit time by sea, to the two non-U.S. DCs, was about a month. Depending on the eventual destination country, different power supply modules had to be installed in the printers to accommodate local voltage, frequency and plug conventions. The manuals and labels also had to be localized due to language differences. HP redesigned the printer so that the power module could be added as a simple plug-in, manufactured a generic Deskjet-Plus printer in the U.S. (sans power supply module, manual and labels) and later localized the generic product in Europe, based on observed demand conditions. Restructuring its printer production process in this fashion enabled HP to maintain the same service-levels with an 18% reduction in inventory, saving millions of dollars [Lee *et al* (1993)].

1.1 Summary of Analysis and Findings

This paper models the firm's internal (production and distribution) activities, its Information Structure, its output markets characterized by demand uncertainty, and the interaction among the three. The firm responds to demand uncertainty by using forecasts to make informed decisions. The effectiveness of forecasts is determined by a number of attributes of demand information. The important, relevant attributes include information precision, information timing and level of aggregation of information (across multiple products). In our modeling context, we use the term 'Information Structure' (IS) to denote the attributes of demand information relevant to forecasting. We explicitly model both the precision and timing of the firm's IS.

The operational benefits of postponement are related to the firm's ability to use its IS to capture relevant market information for demand forecasting. In general, two key principles govern effective forecasting: *(i)* The shorter the time horizon over which prediction is made, the more accurate is the forecast (*cf* Lee and Whang (1998)); and *(ii)* Aggregate forecasts are more accurate than disaggregated ones. The first principle reflects the build-up of uncertainty over time; the second is a consequence of the law of large numbers. Postponement exploits *both* of these principles. First, delayed differentiation implies that the demand forecast needed initially is at a more aggregate level. For example, at the time of manufacture, HP needs to predict the demand for its printers over all of Europe, and Benetton needs to predict the demand for all colors of a particular style. Second, disaggregated forecasts are needed only following the point of product differentiation, when the forecast horizon is shorter.

To study the value of postponement, it is necessary to open the "black box" known as the supply chain. We model a stylized supply chain consisting of a production facility, a distribution center (DC) and two markets where the ultimate product is sold. The demand curves in the end-markets are stochastic, i.e., the quantity demanded is a stochastic function of the corresponding price; thus, both prices and quantities (and hence, revenues) are functions of the firm's cost parameters. We consider two alternative business processes for managing this supply chain. Under the *Early Differentiation* (ED) process, the firm differentiates its products at the production stage. As a result, the two products are already differentiated through all the subsequent stages of the supply chain. Under the *Delayed Differentiation* (DD) process, the firm delays product differentiation until further information about the demand for each product is received. We apply these alternative business processes to the same demand and cost environment. In a dynamic (multiperiod) setting, we derive the optimal production, shipping and inventory policies for each process in closed-form, and compare the resulting production, inventory and sales. We analyze the factors that determine profits under each process and study the *value of postponement* (VOP), defined as the difference between the two expected profits. We also study the effect of changes in the demand variability, demand correlation, information precision and inventory holding costs on the VOP.

We prove that the optimal production, sales and inventory policies under both ED and DD are myopic, and derive the optimal solutions in closed form. While the sales strategies are not identical under ED and DD, expected sales are the same. We also prove that the average inventory is lower under DD, i.e., postponement reduces average inventory. In fact, although both ED and DD profits clearly fall as the holding cost increases, we find that the VOP *increases* in the holding cost.

We find that demand variability is an important factor in determining the VOP. For any IS, the VOP is *increasing* in demand variability, even though both ED and DD profits individually fall with variability. Thus, postponement is particularly effective in highly dynamic environments.

Postponement is further affected by the demand correlations. While greater correlation increases ED profits, its effect on DD profits is ambiguous. However, the net effect on VOP is unambiguous: it falls with the demand correlation.

Our results demonstrate that the effectiveness of postponement strategies is closely linked to the quality of demand information provided by the firm's IS. Firstly, as our model formulation makes clear, the *timing* of demand information (relative to the material flows) is the fundamental difference between the ED and DD processes. Two additional key attributes of information quality are *precision* and *detail* (level of *disaggregation*). Our results show that more precise demand information increases profits under both ED and DD, as might be expected. Under most settings, the VOP also increases with IS precision– this is related to the ability of delayed product differentiation to better exploit the available information about demand. In addition, we show that disaggregation is an important precondition for a positive VOP.

The rest of this paper is organized as follows. In Section *1.2*, we review the extant academic literature on postponement. In Section *2*, we present our two models of the alternative business processes. We derive the optimal solutions for each in Section *3*. In Section *4*, we compare the optimal policies, sales, inventories and profits under each model, and discuss the effect of information (and hence, forecast) aggregation on the performance of each. We then develop a numerical example to shed further insights. Section *5* concludes with a discussion of possible extensions and future research.

1.2 Literature Review

This paper develops an analytical model of postponement under demand uncertainty, extending the academic literature on postponement. With the notable exception of Anand and Girotra (2007), all these papers largely focus on monopoly settings, as we do. Most of the extant academic literature focuses on cost-minimization strategies (*cf* Aviv and Federgruen (2001a; 2001b), Eppen and Schrage (1981), Federgruen and Zipkin (1984), Gavirneni and Tayur (1998), Lee (1996), Lee and

Tang (1999), Lee and Whang (1998), Schwarz (1989) and Swaminathan and Tayur (1998)); notable exceptions (discussed below) are Van Mieghem and Dada (1999), Chod and Rudi (2005) and Anand and Girotra (2007). Lee and Tang (1999) minimize the sum of investment, processing and inventory costs subject to service-level constraints. In other models, the total cost to be minimized is the sum of production or ordering costs, holding costs, and shortage costs due to unmet demand [Gavirneni and Tayur (1998), Swaminathan and Tayur (1998)] or backorder costs due to backlogging [Aviv and Federgruen (2001a; 2001b), Lee (1996), Lee and Whang (1998)]. Earlier models [Eppen and Schrage (1981), Federgruen and Zipkin (1984), Schwarz (1989)] show how delayed allocation decisions create "statistical economies of scale" and analyze the associated tradeoffs. In our model, both prices and sales revenues are endogenously determined, as a function of quantities. Hence retail prices are not exogenously specified, and the resultant profit-maximization problem is not reducible to a equivalent cost-minimization problem.

The previous literature may also be distinguished based on their models of the supply chain structure: models with virtual depots (points of product differentiation/distribution where no inventories are held) and those with physical depots (distribution points where inventories can be held).[1] These are discussed below.

1.2.1 Models with a Virtual Depot

Eppen and Schrage (1981) establish the foundations of a basic model for studying the value of postponement. They consider a distribution system with a supplier, a "virtual" depot that holds no inventory and N symmetric warehouses. It takes L periods to replenish the depot. Once product is received at the depot, it is allocated to the warehouses; actual delivery is subject to a delay of l periods. Demands at the warehouses are stationary and Normally distributed, and unsatisfied demands are backordered. The objective is to minimize expected long-run average cost, consisting of the costs of ordering, backlogs and inventory holding. Eppen and Schrage (1981) compare a *decentralized* system that uses no depot to the above model, which we call the *postponement*

[1]The reader is referred to Swaminathan and Lee (2003) for an alternative categorization of the literature based on three postponement "enablers": process standardization, process re-sequencing and component standardization.

model, subject to strong assumptions that include the use of a base-stock policy.[2] They derive approximately-optimal policies, showing how postponement reduces inventories, and hence costs, due to *statistical scale economies*, namely the pooling of demands at the "virtual" depot.

Federgruen and Zipkin (1984) study similar systems, extending the Eppen-Schrage (1981) model in many respects. They allow more general demand processes and cost structures, study the finite horizon case and most importantly derive (rather than assume the structure of) the order policies. They show how the model can be approximated by a single-location inventory problem, and their results like those of Eppen and Schrage (1981) can be used to determine the value of statistical scale economies.

Lee (1996)'s Build-to-Stock model follows the Eppen-Schrage (1981) structure, showing that postponement always lowers the expected level of finished good inventory. Lee (1996) also shows how these results translate into meaningful postponement strategies in actual operational practice.

Aviv and Federgruen (2001a) study a dynamic, multi-item inventory system with random demands and periodic review. They find near-optimal heuristic strategies and derive an approximate lower-bound for the benefits of postponement. Aviv and Federgruen (2001b) and Lee and Whang (1998) introduce an additional factor that contributes to the value of postponement: the value of learning from past realizations of demand. Such learning effects can occur when consecutive demands are correlated or when the parameters of the demand distribution are unknown. Aviv and Federgruen (2001b) extend the Federgruen-Zipkin (1984) optimization model to a Bayesian framework, derive approximately-optimal policies and show that learning increases the value of postponement. Lee and Whang (1998) extend the Eppen-Schrage (1981) results to the case where demand follows a random walk. They derive the resulting savings in safety stock and show that *(i)* learning increases the value of postponement, and *(ii)* as product variety increases, the percentage reduction of safety stock through postponement also increases.

The above models assumed that statistical scale economies can be achieved at no cost. Schwarz (1989) allows for a depot that holds no inventory to achieve statistical scale economies at a cost. Specifically, Schwarz (1989) assumes that postponement results in additional lead time due to

[2] Another assumption is the "allocation assumption" whereby quantities are always sufficient to allocate the same fractile point to each warehouse. They also consider a "centralized" model with a single retailer, which is not of interest here.

warehouse allocation and shipment as well as an additional "pipeline inventory" holding cost. This leads to a tradeoff analysis, seeking the breakeven additional lead-time in the postponed system for which the level of service (expected backorders per order cycle) and expected inventory holding cost would be the same under both systems. Schwarz (1989) finds that the value of postponement depends heavily on the pipeline inventory holding cost; if this cost can be avoided, statistical scale economies have a significant impact.

Van Mieghem and Dada (1999) and Chod and Rudi (2005) are two other studies in this genre, that use a demand model similar to ours (with downward-sloping demand curves, implying that firms choose prices in addition to production quantities). Van Mieghem and Dada (1999) analyze different combinations of strategies based on which of these decisions (setting prices or production quantities) are made before or after uncertainty resolution. The analysis in Van Mieghem and Dada (1999) is a static (single-period) setting for a single product; Chod and Rudi (2005) extend the analysis to two correlated markets, albeit in a single-period setting. In contrast, ours is a dynamic model with two differentiated products. A further important distinction is that we model market information which the firm can use to update its demand curve estimates and make more informed decisions. While both Van Mieghem and Dada (1999) and Chod and Rudi (2005) implicitly incorporate the role of information timing (which, by interaction with material flows, determines early or delayed differentiation), the information precision in their settings is 'perfect', i.e., demand curves are perfectly predictable once the market signals are received. In our model, we parameterize the level of information precision, which leads to demand updation in a Bayesian fashion. In addition to information precision, our model studies the role of information aggregation. We show that the value of postponement is determined in combination by all three important attributes of information– (i) its timing within the supply chain (relative to material flows), (ii) its level of aggregation, and (iii) its precision.

1.2.2 Models with a Physical Depot

In the models considered above, the "virtual" depot did not hold physical inventory. Lee (1996)'s Build-to-Order model extends the analysis to the case where the depot holds physical inventory. Lee considers the inventory position needed to satisfy a customer response-time target (specified as an expected-value or probability target), where the unit cost rate is an increasing function of the degree of postponement (reflecting the increase in holding cost along the value chain.) He finds that the effect of postponement on expected holding cost is ambiguous. However, if the marginal increase in unit holding cost at the point of differentiation is small, postponement reduces the expected holding cost. Lee (1996) shows how this result translates into practical considerations for supply chain design.

Lee and Tang (1999) extend the model by allowing inventories to be held at different points along the supply chain. They consider and compare three types of cost: *(i)* inventory holding costs, *(ii)* processing costs, and *(iii)* the investment needed to change a custom operation to a common one. Lee and Tang (1999) study three methods of delaying product differentiation: standardization of components, modular design and process restructuring. Their analysis focuses on the insights that can be obtained by comparing the cost expressions. Lee (1996)'s Build-to-Order model and Lee and Tang (1999)'s extension introduce two additional factors that contribute to the value of postponement: *(i)* economies of scale in holding physical inventory, and *(ii)* postponing the increase in value-added per unit, which reduces the inventory holding cost per unit.

Anand and Girotra (2007) model two competing firms– each with the option of employing early or delayed differentiation in their supply chains, and study the resulting equilibria. Unlike the current study, and as in Van Mieghem and Dada (1999) and Chod and Rudi (2005) (discussed above), their model is static, and firms' IS are always "perfect", i.e., the precision of demand information is not parameterized.

Swaminathan and Tayur (1998) study the component commonality problem in detail. They determine the optimal configuration and inventory levels for intermediate products (called "vanilla boxes") given a specification of the bill of materials and demand subject to capacity constraints. They derive a solution procedure for minimizing the sum of the expected inventory holding and stockout costs. Comparing their preconfigured "vanilla box" process to make-to-stock and assemble-to-order processes, they find that the value of using "vanilla boxes" crucially depends on the degree of demand variability and the sign and strength of the demand correlations.

Gavirneni and Tayur (1998) study a model that takes postponement to the limit: two customers, each following a known *(s,S)* ordering policy, use a common capacitated supplier. The supplier incurs inventory and shortage penalty costs. Gavirneni and Tayur (1998) compare the case where the products are differentiated (i.e., the supplier pre-allocates capacity to each product) to the case where the products are identical from the supplier's perspective (corresponding to a limiting case of postponement). Another important dimension of their model is the effect of information sharing regarding actual (customer) inventory levels between the customers and the supplier.

From among the above studies, the business process we consider is closest to Lee (1996)'s physical depot model. However, our model explicitly considers the interaction between the price-quantity relationship in the ultimate markets and the production-inventory decisions in the supply chain. In addition, our model is designed to explicitly consider the firm's ability to exploit market information, modeled as a vector of binary signals. Indeed, our model enables us to explicitly study the interaction between the firm's postponement strategies and the accuracy of its market information.[3] Whereas most studies in this area require either approximation or numerical experimentation, we are able to derive the exact solution (the firm's optimal production, distribution and inventory policies) to our dynamic optimization problem. Further, our analysis enables us to study both revenue (sales) and cost (production/inventory) components of the value of postponement. We also quantify the effects of demand variability, information (forecast) precision, information aggregation across products, end-product demand correlation and inventory holding costs, on the value of postponement.

2 Models of Alternative Business Processes

Consider a firm that produces and sells two related products. For concreteness, we assume that the products are sold in two distinct product markets (our model is equally applicable when a single product is sold in geographically separated markets). The firm's supply chain consists of a production facility, a Distribution Center (DC) and two retail outlets, one for each market. Figure 1 depicts the firm's supply chain. The firm's objective is to maximize its discounted expected profits over the long-term (infinite) horizon, where the one-period discount factor is β.

INSERT FIGURE 1

[3]None of the "physical-depot" models discussed in this Section, including ours, model "learning" across periods. In our model, the demand curves are correlated across products but uncorrelated across periods; the latter assumption enables closed-form derivation of optimal policies. See Lee and Whang (1998) and Aviv and Federgruen (2001b) for "virtual-depot" models that study the effect of learning on postponement.

Our specification of demand reflects a tradeoff between the quantity sold in each market and the market price. Unlike traditional models of manufacturing operations, supply chain management models call for an analysis of the impact of the firm's operational decisions on the *entire* supply chain. Thus, "demand" does not represent an exogenous quantity; rather, the firm optimizes the entire supply chain, tying together production, logistics and pricing decisions.

The supply chain is held together by two types of "glue": physical (inventory) and virtual (information). With respect to the former, we allow the firm to hold inventory at the DC. We assume that any quantity of intermediate goods may be held at the DC, incurring an inventory holding cost of h per unit per period.[4]

The informational building blocks of the model focus on demand information. We assume that the state of demand in each market i is a binary random variable S_i, with $S_i = 1$ corresponding to a "high" demand state (occurring with probability p) and $S_i = 0$ corresponding to a "low" demand state (occurring with probability $(1 - p)$). The demand curves are linear with slope parameter b and a random intercept. The demand intercept in a market with the "high" demand state is a_H, and in the "low" demand state it is a_L, where $a_H > a_L$. Thus the prices (for any sales quantity q) are $a_H - bq$ in the "high" demand state and $a_L - bq$ in the "low" demand state. The corresponding revenue functions are $R_H(\cdot)$and $R_L(\cdot)$, with

$$R_H(q) = (a_H - bq)q \quad \text{and} \quad R_L(q) = (a_L - bq)q.$$

The demand states in the two markets may be correlated, with correlation coefficient ρ. When the two markets represent geographically-separated retail locations for the same physical product (e.g., laser printers sold in Europe vs. the US), this correlation reflects the effects of common demand patterns on the localized markets. When the markets are for related products (that is, the separation is in product space, e.g., black-and white vs. color laser printers; or apparel with different colors), the correlation coefficient quantifies the degree to which information about the state of demand for one product can help predict the demand for the other. In both situations, the correlation is typically non-negative, and we assume this is the case through the rest of this paper.[5]

[4]The model is altered in a straightforward fashion if the DC has only a finite storage capacity.

[5]Extending the analysis to negatively correlated demands is straightforward. We focus on the more common case of $\rho \geq 0$ to avoid clutter.

The joint distribution of the vector of demand states (S_1, S_2) is given by

$$P[S_1 = S_2 = 1] = p^2 + p(1-p)\rho,$$

$$P[S_1 = S_2 = 0] = (1-p)^2 + p(1-p)\rho,$$

and

$$P[S_1 = 1,\ S_2 = 0] = P[S_1 = 0,\ S_2 = 1] = p(1-p)(1-\rho).$$

The firm has a binary *Information Structure* (IS) $\underline{X} = (X_1, X_2)$ that helps it infer the states of demand in the two markets. The IS is modeled as a two-dimensional vector of binary signals, one from each market, taking on the values 0 or 1.[6] These signals are received at the DC after the production decision has been implemented, but prior to the decision on the quantities to be refined and shipped to each market. The DC uses this information in implementing its refinement, distribution and inventory strategies, updating the probability distribution of the states of the two markets in a Bayesian fashion.

Each signal X_i represents the state of demand in market i with noise: $X_i = S_i$ with probability $(1-\alpha)$ and $(1-S_i)$ with probability α $(i = 1, 2)$. The parameter α is a measure of the imprecision of the firm's IS, with a smaller α corresponding to more informative signals. We assume, without loss of generality, that $0 \leq \alpha \leq \frac{1}{2}$. This information structure is similar to the one modeled in Anand and Mendelson (1997), where the supply chain was a "black box", the model had a single period and demands were assumed independent across markets.

The sequence of events is thus as follows (see Figure 2). At the beginning of each period, intermediate goods are delivered to the DC. Then, the firm obtains the signal $\underline{X}$ and uses this information, and the available quantities, to decide how much of the intermediate goods are to be refined to end-products for each market, and how much inventory to carry at the DC. The refinement cost for either product is θ per unit; θ also subsumes any transportation costs from the DC to the respective markets. After shipment, the demand curves for each market are realized and sales are made. The firm then makes its production decisions for delivery to the DC next period. The production cost is k per unit of intermediate good. This sequence is repeated period after period, with the (two-dimensional) binary random vectors representing the market states and corresponding signals for the two markets being i.i.d.

[6]The binary nature of the forecasts is an artifact of the binary state variables.

INSERT FIGURE 2

We denote by $P(x_1, x_2)$ the probability that $X_1 = x_1$ *and* $X_2 = x_2$, and by $P_{x_1x_2} = Pr[S_1 = 1 \mid X_1 = x_1,\ X_2 = x_2]$ the probability that a market is in the "high-demand" state, when x_1 is the signal from that same market and x_2 is the signal from the *other* market.[7]

We make the technical assumption that $a_L > \frac{k}{\beta} + \theta$, which guarantees that it is optimal for the firm to produce positive quantities of the products. We also assume that the production cost k is greater than the inventory holding cost h; otherwise, it is never optimal to hold any inventory of either product.

Suppose that the signals received for the two products are $X_1 = x_1$ and $X_2 = x_2$. The expected revenues from shipping quantity q_1 of product 1 are $P_{x_1x_2} \cdot R_H(q_1) + (1 - P_{x_1x_2}) \cdot R_L(q_1)$, and the expected revenues from shipping quantity q_2 of product 2 are $P_{x_2x_1} \cdot R_H(q_2) + (1 - P_{x_2x_1}) \cdot R_L(q_2)$. The shipment quantities q_1 and q_2 are constrained by the quantities of the intermediate good(s) available at the warehouse. For convenience, define

$$f(P_{x_1x_2}) = P_{x_1x_2} \cdot a_H + (1 - P_{x_1x_2})a_L. \tag{1}$$

Thus, for example, $f(P_{10})$ is the expected value of the demand intercept for product 1 when $X_1 = 1$ and $X_2 = 0$.

We can now formulate the firm's decision problem as follows (This general formulation applies to both the Early Differentiation (ED) and Delayed Differentiation (DD) processes, and will be subsequently specialized for each process.). Let $Q^{1,t}$ and $Q^{2,t}$ denote the production quantities of products 1 and 2 for period t, and let $q^{1,t}$ and $q^{2,t}$ denote the corresponding shipment quantities to the respective markets. Also let $\xi^{1,t}$ and $\xi^{2,t}$ denote the intermediate good inventories carried over from period t to the next period. Since the market states are independent over time, inventories completely specify the firm's state at the start of each period. Let $\underline{X}_t = (X_{1,t}, X_{2,t})$ be the vector of information signals received from the two markets in period t. Correspondingly, let $R^{1,t}(\cdot|\underline{X}_t)$

[7] The derivations of the conditional state distributions $P_{x_1x_2}$ and the proofs of all Theorems are provided in the Appendix.

and $R^{2,t}(\cdot|\underline{X}_t)$ denote the conditional expected revenue functions from sales in markets 1 and 2 in period t. Suppose that the supply chain starts operating at $t = 0$ with zero inventories (i.e., $\xi^{1,0} = \xi^{2,0} = 0$). The firm's problem is to maximize its expected discounted profits, namely

$$\max_{\substack{Q^{1,0}, Q^{2,0}, \{Q^{1,t}, Q^{2,t}\}_{t=1}^{\infty} \\ \{q^{1,t}, q^{2,t}, \xi^{1,t}, \xi^{2,t}\}_{t=1}^{\infty}}} \left\{ -k\left(Q^{1,0} + Q^{2,0}\right) + E\left[\sum_{t=1}^{\infty} \beta^t \cdot \left(R^{1,t}(q^{1,t}|\underline{X}_t) + R^{2,t}(q^{2,t}|\underline{X}_t)\right)\right.\right.$$
$$\left.\left. - \left[\sum_{t=1}^{\infty} \beta^t \left(\theta(q^{1,t} + q^{2,t}) + h \cdot (\xi^{1,t} + \xi^{2,t}) + k \cdot \left(Q^{1,t} + Q^{2,t}\right)\right)\right]\right]\right\} \quad (2)$$

subject to (i) non-negativity constraints on all the decision variables, and (ii) the inventory transition law, which states that for any item, the inventory in each period is equal to the sum of the inventory and production from the previous period less the shipment quantity in the current period.

The above formulation is common to both the Early Differentiation (ED) and Delayed Differentiation (DD) processes, and is specialized for each process in the following Sections.

2.1 Early Differentiation (ED) Process

Under ED, the products are already segregated in the production stage, resulting in non-substitutable intermediate goods. Thus, the production quantities of the intermediate goods are separately determined for *each* product. In the second (refinement) stage, the decisions to be made involve the refinement, shipment and inventory quantities for each product. The firm's problem under ED is thus to solve the maximization problem (2) subject to non-negativity constraints on the decision variables $Q^{1,0}, Q^{2,0}$ and $\{Q^{1,t}, Q^{2,t}, q^{1,t}, q^{2,t}, \xi^{1,t}, \xi^{2,t}\}_{t=1}^{\infty}$, and the inventory transition laws given by

$$\xi^{1,t} = \xi^{1,t-1} + Q^{1,t-1} - q^{1,t}, \text{ and} \quad (3)$$

$$\xi^{2,t} = \xi^{2,t-1} + Q^{2,t-1} - q^{2,t}, \quad (4)$$

where $\xi^{1,0} = \xi^{2,0} = 0$. Clearly, this problem is separable across the two products – a direct result of early differentiation.

2.2 Delayed Differentiation (DD), or Postponement, Process

Under the DD process, manufacturing is common to the two products. Thus, at the production stage the firm has to decide only on the production quantity of the common intermediate good. Later, in the refinement stage, each unit of the common intermediate good can be customized at the DC to produce a unit of either good.

The objective function under DD is given by expression (2). However, the additional flexibility offered by the common intermediate good is reflected in the more flexible transition law,

$$\left(\xi^{1,t}+\xi^{2,t}\right)=\left(\xi^{1,t-1}+\xi^{2,t-1}\right)+\left(Q^{1,t-1}+Q^{2,t-1}\right)-\left(q^{1,t}+q^{2,t}\right). \tag{5}$$

Condition (5) replaces conditions (3) and (4) of the ED model; the other (non-negativity) constraints are the same for both ED and DD. Transition law (5) means that the problem under DD is no longer separable across the two products.

To simplify the problem representation under DD, let $\xi^t=\xi^{1,t}+\xi^{2,t}$ and $Q^t=Q^{1,t}+Q^{2,t}$. The firm's problem under DD now simplifies to:

$$\max_{Q^0,\{Q^t,q^{1,t},q^{2,t},\xi^t\}_{t=1}^{\infty}} E\left[-kQ^0+\sum_{t=1}^{\infty}\beta^t\cdot\left(R^{1,t}(q^{1,t}|\underline{X}_t)+R^{2,t}(q^{2,t}|\underline{X}_t)-\theta(q^{1,t}+q^{2,t})-h\cdot\xi^t-k\cdot Q^t\right)\right],$$

subject to the non-negativity constraints on the decision variables $Q^t,\xi^t,q^{1,t}$ and $q^{2,t}$, the inventory transition law $\xi^t=\xi^{t-1}+Q^{t-1}-q^{1,t}-q^{2,t}$, and the initial condition $\xi^0=0$.

We define the value of postponement (VOP) as the difference in discounted expected profits between the DD and ED processes. Comparing conditions (3) and (4) with (5), it is easy to see that the DD problem is a relaxation of ED, hence the VOP is non-negative. The more flexible inventory transition law reflects the DC's greater flexibility under DD. Under ED, the firm commits early to the production quantities of each product, and the DC cannot alter the sales mix in response to demand information as much as it can under DD. However, the firm might prefer the ED structure due to cost considerations not directly addressed in our model: for example, higher production costs under DD, or the cost of redesigning the products and restructuring the supply chain for postponement. As we demonstrate below, there are circumstances under which postponement does not add enough value to justify its costs.[8]

3 Optimal Solutions

We now derive the firm's optimal policies under both ED and DD. We show that in both cases, the optimal policy is *myopic*, i.e., the decisions taken in each period can disregard the consequences for future periods. Sobel (1981) derives sufficiency conditions for the optimality of myopic policies. Myopic optimal policies when Sobel (1981)'s conditions are not met are a rare occurrence in such dynamic models. We prove that the optimal policies for our problem are myopic even though our model does not satisfy Sobel (1981)'s conditions, and derive them in closed form.

3.1 Early Differentiation

Theorem 1 (Optimal Policies under ED) The following myopic production, shipment and inventory policies (for each product) are optimal:

(A) Shipping/Inventory:

In each period, conditional on the information signals $X_1 = x_1$ and $X_2 = x_2$, the DC refines and ships its entire on-hand quantity of product 1 up to the threshold $q^{ED}_{x_1x_2} = \frac{f(P_{x_1x_2})-(k+\theta-h)}{2\cdot b}$, where $f(\cdot)$ was defined in (1). The threshold for product 2 is $q^{ED}_{x_2x_1} = \frac{f(P_{x_2x_1})-(k+\theta-h)}{2\cdot b}$. Any surplus above these thresholds is held as inventory.

(B) Production:

Each period, the firm produces to bring the DC's intermediate good inventories for the next

[8]See Section 4.5; in particular, the discussion around figure 8.

period (for each product) to $Q^*{}_{ED}$, where $Q^*{}_{ED} = \max\left\{Q^*_{1,ED}, Q^*_{2,ED}, Q^*_{3,ED}, Q^*_{4,ED}\right\}$, and

$$
\begin{aligned}
Q^*_{1,ED} &= \frac{f(P_{11}) - (k+\theta-h) - \frac{\left(\frac{k}{\beta}-k+h\right)}{P(1,1)}}{2\cdot b},\\
Q^*_{2,ED} &= \frac{\left(\frac{P(1,1)f(P_{11})+P(1,0)f(P_{10})-\left(\frac{k}{\beta}-k+h\right)}{P(1,1)+P(1,0)}\right) - (k+\theta-h)}{2\cdot b},\\
Q^*_{3,ED} &= \frac{\left(\frac{P(1,1)f(P_{11})+P(1,0)(f(P_{10})+f(P_{01}))-\left(\frac{k}{\beta}-k+h\right)}{P(1,1)+2\cdot P(1,0)}\right) - (k+\theta-h)}{2\cdot b}, \text{ and}\\
Q^*_{4,ED} &= \frac{P(1,1)f(P_{11}) + P(1,0)\left(f(P_{10})+f(P_{01})\right) + P(0,0)f(P_{00}) - \left(\frac{k}{\beta}+\theta\right)}{2\cdot b}. \quad \blacklozenge
\end{aligned}
$$

To interpret the solution, first consider the firm's optimal shipment/inventory policy for product 1. The expected revenue from shipping the quantity q is $(f(P_{x_1x_2}) - b\cdot q)\cdot q - \theta\cdot q$, hence the expected marginal revenue is $f(P_{x_1x_2}) - 2b\cdot q - \theta$. Moreover, the marginal cost of replenishing the warehouse for the next period is k per unit, and the savings from holding a unit less at the warehouse is h. Thus, given the choice between shipping an additional unit of product 1 versus holding it in inventory, the shipping option is preferable as long as the expected marginal revenue from that unit is greater than the additional cost; i.e., as long as $f(P_{x_1x_2}) - 2b\cdot q - \theta \geq k - h$, or equivalently, $q \leq \frac{f(P_{x_1x_2})-(k+\theta-h)}{2b} = q^{ED}_{x_1x_2}$. Thus, $q^{ED}_{x_1x_2}$ is the maximum quantity of product 1 that the DC would ever ship when the signals are x_1 and x_2; above this threshold, the firm is better off keeping inventory.

The production policy is a stationary threshold policy, independent of demand information by necessity. The threshold value $Q^*{}_{ED}$ is shaped by the shipment policy. For each product, there are four possible ship-up-to levels based on the four possible combinations of the binary market signals X_1 and X_2. The ship-up-to levels are related as $q^{ED}_{11} \geq q^{ED}_{10} \geq q^{ED}_{01} \geq q^{ED}_{00}$; hence, q^{ED}_{11} is the maximum quantity ever shipped in any period (of either product). It follows that the optimal build-up-to level $Q^*{}_{ED}$ for either product never exceeds q^{ED}_{11}. Hence, the four possible ranges for $Q^*{}_{ED}$ are: *(i)* $q^{ED}_{11} > Q^*_{ED} \geq q^{ED}_{10}$; *(ii)* $q^{ED}_{10} > Q^*_{ED} \geq q^{ED}_{01}$; *(iii)* $q^{ED}_{01} > Q^*_{ED} \geq q^{ED}_{00}$; and *(iv)* $q^{ED}_{00} > Q^*_{ED}$. The four possible values of Q^*_{ED} posited in the Theorem correspond to these four cases; which of these cases actually arises depends on the cost-benefit tradeoff between lost sales-revenue and excessive inventory build-up. For example, suppose $q^{ED}_{01} > Q^*{}_{ED} > q^{ED}_{00}$. When $(X_1, X_2) = (0,0)$ (an event which has probability $P(0,0)$), the surplus above q^{ED}_{00} is stored as inventory; hence, lowering the production build-up saves holding costs and defers production costs. On the other hand, for all other realizations of (X_1, X_2), a lower build-up would have resulted in more lost sales and lower profits.

3.2 Delayed Differentiation

We next derive the optimal solution for the DD process. For notational convenience, define $q^{DD}_{x_1x_2} = \frac{\frac{f(P_{x_1x_2})+f(P_{x_2x_1})}{2}-(k+\theta-h)}{b}$. Thus, $q^{DD}_{11} = \frac{f(P_{11})-(k+\theta-h)}{b}$; $q^{DD}_{00} = \frac{f(P_{00})-(k+\theta-h)}{b}$, and $q^{DD}_{01} = q^{DD}_{10} = \frac{\frac{f(P_{01})+f(P_{10})}{2}-(k+\theta-h)}{b}$. Further, define $d(x_1,x_2) = \frac{f(P_{x_1x_2})-f(P_{x_2x_1})}{2\cdot b}$. Clearly, $d(x_1,x_2) = -d(x_2,x_1)$, and $d(x,x) = 0$ for all x.

Theorem 2 (Optimal Policies under DD) The following myopic production, shipment and inventory policies are optimal:

(A) Shipping/Inventory:

In each period, conditional on the information signals $X_1 = x_1$ and $X_2 = x_2$ and on-hand quantity Q^* of the intermediate good, the DC refines and ships the entire on-hand quantity up to the threshold $q^{DD}_{x_1x_2}$. Any surplus above this threshold is held as inventory. Of the total quantity q^S $\left(= \min\left\{Q^*, q^{DD}_{x_1x_2}\right\}\right)$ refined, the DC ships $\frac{q^S+d(x_1,x_2)}{2}$ units of product 1 and $\frac{q^S-d(x_1,x_2)}{2}$ units of product 2.

(B) Production:

Each period, the firm produces to bring the DC's intermediate good inventories for the next period to ${Q^*}_{DD}$, where ${Q^*}_{DD} = \max\left\{Q^*_{1,DD}, Q^*_{2,DD}, Q^*_{3,DD}\right\}$, and

$$
\begin{aligned}
Q^*_{1,DD} &= \frac{f(P_{11}) - (k+\theta-h) - \frac{\left(\frac{k}{\beta}-k+h\right)}{P(1,1)}}{b}, \\
Q^*_{2,DD} &= \frac{\left(\frac{P(1,1)f(P_{11})+P(1,0)(f(P_{10})+f(P_{01}))-\left(\frac{k}{\beta}-k+h\right)}{P(1,1)+2\cdot P(1,0)}\right) - (k+\theta-h)}{b}, \text{ and} \\
Q^*_{3,DD} &= \frac{P(1,1)f(P_{11}) + P(1,0)\left(f(P_{10}) + f(P_{01})\right) + P(0,0)f(P_{00}) - \left(\frac{k}{\beta}+\theta\right)}{b}. \ \blacklozenge
\end{aligned}
$$

The optimal policies under DD have an intuitive interpretation similiar to the case of ED, but with an additional level of complexity. Since the intermediate product can be refined into *either* final product, the DC must choose among *three* options: refining (and shipping) more of product 1, of product 2, or holding the intermediate-good inventory. Consider first the optimal allocation problem between products 1 and 2. For a total shipment quantity q of both products, expected revenues are maximized when the allocation equalizes the expected marginal revenues across the two products. When the market signals are $X_1 = x_1$ and $X_2 = x_2$, this results in an allocation of $\frac{q+d(x_1,x_2)}{2}$ and $\frac{q-d(x_1,x_2)}{2}$ to products 1 and 2 respectively – the allocation difference across the two products is exactly $d(x_1, x_2)$, for all $q \geq d(x_1, x_2)$. By arguments similiar to those for ED, $q^{DD}_{x_1x_2}$ is the ship-up-to level; thus, $\min\{Q^*_{DD}, q^{DD}_{x_1x_2}\}$ is the total shipment quantity, and $\left(Q^*_{DD} - q^{DD}_{x_1x_2}\right)^+$ is held as inventory.

The production policy is a stationary threshold policy, with the threshold given by Q^*_{DD}. The three possible ranges for Q^*_{DD} are: *(i)* $q^{DD}_{11} > Q^*_{DD} \geq q^{DD}_{10} (= q^{DD}_{01})$; *(ii)* $q^{DD}_{10} > Q^*_{DD} \geq q^{DD}_{00}$; and *(iii)* $q^{DD}_{00} > Q^*_{DD}$, and the three possible values of Q^*_{DD} posited in the Theorem correspond to these ranges, and reflect a tradeoff between lost sales-revenue and excessive inventory build-up.

4 Alternative Business Processes: Operational Implications

In this Section, we compare the optimal policies and operational performance of the DD and ED processes. In particular, we compare their ship-up-to and build-up-to levels, inventories and sales. We then derive and study the expected profits under each process.

4.1 Optimal Policies

Theorem 3 (Comparison of Optimal Policies)

(i) The ship-up-to levels under the DD and ED structures are related by $q^{DD}_{x_1x_2} = q^{ED}_{x_1x_2} + q^{ED}_{x_2x_1}$. Thus, $q^{DD}_{00} = 2 \cdot q^{ED}_{00}$, $q^{DD}_{11} = 2 \cdot q^{ED}_{11}$ and $q^{DD}_{10} = q^{DD}_{01} = q^{ED}_{10} + q^{ED}_{01}$.

(ii) The build-up-to levels under the two processes satisfy the relationship $2 \cdot Q^*_{ED} \geq Q^*_{DD}$.

$$
\text{Specifically, } 2\cdot Q^*_{ED} - Q^*_{DD} = \begin{cases} \left(\frac{P(1,0)}{b\cdot P(1,1)(P(1,1)+P(1,0))}\right)\left[\left(\frac{k}{\beta}-k+h\right) - P(1,1)\cdot\left(f(P_{11})-f(P_{10})\right)\right]^+, \\ \qquad \text{when } P(1,1)\left(f(P_{11}) - \frac{f(P_{10})+f(P_{01})}{2}\right) \geq \frac{k}{\beta}-k+h; \\ \left(\frac{P(1,0)}{b\cdot\left(P(1,1)+P(1,0)\right)\left(P(1,1)+2\cdot P(1,0)\right)}\right)\Big[P(1,1)\cdot f(P_{11}) + P(1,0)\cdot f(P_{10}) \\ -\left(P(1,1)+P(1,0)\right)f(P_{01}) - \left(\frac{k}{\beta}-k+h\right)\Big]^+, \text{ otherwise.} \end{cases} \blacklozenge
$$

To understand the intuition behind Theorem 3, recall that the ship-up-to levels reflect a tradeoff between the expected revenues from selling an additional unit and the value of holding that unit in inventory. Under DD, although the DC can customize its inventories for either market, the thresholds for each market at which additional sales are less profitable than keeping inventory are still operative; these are $q^{ED}_{x_1x_2}$ and $q^{ED}_{x_2x_1}$ respectively. Below the total shipment level $\left(q^{ED}_{x_1x_2} + q^{ED}_{x_2x_1}\right)$, the DC will find it profitable to ship to one or both markets. Above this threshold, both markets are saturated, and inventory is the dominant option. By definition, the ship-up-to level under DD is $q^{DD}_{x_1x_2} = \left(q^{ED}_{x_1x_2} + q^{ED}_{x_2x_1}\right)$.

Since delayed product differentiation enables pooling of the intermediate good, the build-up-to level under DD (Q^*_{DD}) is no more than the total ($2\cdot Q^*_{ED}$) of the build-up-to levels of the two products under ED, in accord with our intuition. To understand when $2\cdot Q^*_{ED}$ is *strictly* greater than Q^*_{DD}, consider when substitutability across the two products (i.e., commonality of the intermediate good) would increase total sales and lower inventories compared to the ED optimal policy. When the information signals received are $(0,0)$ or $(1,1)$, substitutability across the two products (and hence pooling) would have no effect on ED sales under the optimal policy. Now suppose the signals are $(1,0)$ or $(0,1)$. When $Q^*_{ED} \leq q^{ED}_{01}$, the entire available quantity is shipped to the markets, and when $Q^*_{ED} > q^{ED}_{10}$, inventories of both products are carried over to the next period: substitutability would not have increased total sales in either case. However, when $Q^*_{ED} \in \left(q^{ED}_{01}, q^{ED}_{10}\right)$, there is a shortage of the higher-demand product and an excess of the other; intermediate good commonality would have boosted sales of the higher-demand product and lowered total inventory. Conversely, a lower build-up-to level is adequate under DD. In fact, it is easily demonstrated that the conditions (given by Theorem 3) for $2\cdot Q^*_{ED} > Q^*_{DD}$ are identical to the conditions for $Q^*_{ED} \in \left(q^{ED}_{01}, q^{ED}_{10}\right)$.

4.2 Sales and Inventories

We now compare the sales and inventories under the two business processes. Clearly, the average total (2-product) inventory under ED is given by

$$\begin{aligned} I_{ED} &= 2 \cdot \Big[P(1,1) \cdot \big(Q^*_{ED} - q^{ED}_{11}\big)^+ + P(1,0) \cdot \big(Q^*_{ED} - q^{ED}_{10}\big)^+ + P(0,1) \cdot \big(Q^*_{ED} - q^{ED}_{01}\big)^+ \\ &\quad + P(0,0) \cdot \big(Q^*_{ED} - q^{ED}_{00}\big)^+\Big]. \end{aligned} \tag{6}$$

The average inventories under DD are

$$\begin{aligned} I_{DD} &= P(1,1) \cdot \big(Q^*_{DD} - q^{DD}_{11}\big)^+ + P(1,0) \cdot \big(Q^*_{DD} - q^{DD}_{10}\big)^+ + P(0,1) \cdot \big(Q^*_{DD} - q^{DD}_{01}\big)^+ \\ &\quad + P(0,0) \cdot \big(Q^*_{DD} - q^{DD}_{00}\big)^+. \end{aligned} \tag{7}$$

Since the build-up-to levels in each period are $2 \cdot Q^*_{ED}$ and Q^*_{DD} under ED and DD respectively, the expected (average) sales under each process are $S_{ED} = 2 \cdot Q^*_{ED} - I_{ED}$ and $S_{DD} = Q^*_{DD} - I_{DD}$. Thus, the difference in expected sales is

$$S_{ED} - S_{DD} = (2 \cdot Q^*_{ED} - Q^*_{DD}) - (I_{ED} - I_{DD}). \tag{8}$$

The next Theorem compares sales and inventories under the two business processes.

Theorem 4 *(i)* The average inventory under ED, I_{ED}, is not less than the average inventory under DD, I_{DD}. Specifically, $I_{ED} - I_{DD} = 2 \cdot Q^*_{ED} - Q^*_{DD}$, i.e., the difference in inventories is equal to the difference in build-up-to levels given by Theorem 3.

(ii) The expected sales under ED and DD are identical. ⧫

Theorem 4 implies that postponement cuts down inventory costs; these savings are entirely attributable to DD's lower build-up-to levels. To understand what drives the second part of the Theorem, observe that, since $Q^*_{DD} \leq 2 \cdot Q^*_{ED}$, the DC has a larger quantity available for sales under ED than under DD, in each period. We also know that $Q^*_{DD} \leq q^{DD}_{11}$ and $2 \cdot Q^*_{ED} \leq 2 \cdot q^{ED}_{11}$. Thus, when the signals received are $X_1 = X_2 = 1$, the total sales under ED are always at least as large as under DD. When $X_1 = X_2 = 0$, the ED and DD shipments are identical. Yet, the expected sales are the same due to the greater flexibility under DD when the signals are mixed ($(1,0)$ or $(0,1)$): the DC can choose among shipping *either* product or holding inventory, constrained only by the *total* quantity of the intermediate good available. In particular, the firm can ship a large quantity of the product facing higher expected demand at the expense of the other product. In contrast, under ED there is less opportunity for exploitation of differential shipments across the two markets, hence inventory becomes a more attractive option. It follows that sales are greater under DD when the signals are mixed, compensating for the reverse phenomenon when the signals are $(1,1)$. While the expected sales are the same for ED and DD, the sales strategies are not, reflecting DD's greater responsiveness to demand information, which is enabled by the greater flexibility in distribution under DD.

To sharpen our intuition on the drivers of profit and the role of information, we next compare the optimal production, sales and inventory policies under ED and DD when the demand forecasts aggregate the two products.

4.3 Aggregate vs. Disaggregate Forecasts

We distinguish between two types of forecast: *aggregate* and *disaggregate*. In our model, the information generated by the IS (hence, the demand forecast) are disaggregate, i.e., they are specific to each product.

Now suppose that the firm's IS provides only an estimate of the *aggregate* demand across the two markets. Specifically, we assume that the signals $(1,0)$ and $(0,1)$ are garbled so the firm does not know which market is in the low state and which in the high state. In this case, the demand forecast can be *high* (when $(X_1, X_2) = (1,1)$, which occurs with probability p^2), *medium* (when the signals are 1 and 0, which has probability $2 \cdot p(1-p)$) or *low* (when the signals are both 0, which has probability $(1-p)^2$). Thus, disaggregated information can be inferred perfectly from the aggregate when the forecast is high or low, but not when the forecast is medium. Theorem 5 provides the optimal production, shipment and inventory policies for both DD and ED, under aggregate forecasts.

Theorem 5 (Optimal Policies for Aggregate Forecasts)

(i) The optimal production, shipment and inventory policies under DD are identical to those for disaggregated forecasts, and given by Theorem 2. Only the allocation rule across products is different; in this case, the DC allocates exactly *half* of the total shipment to each product.

(ii) The optimal policies for ED are, *in effect*, identical to those for DD under aggregate forecasts, and obtained by setting $Q^*_{ED} = \frac{Q^*_{DD}}{2}$. ♦

An important implication of Theorem 5 is that when forecasts are made only in the aggregate, the actions taken each period under ED and DD, and hence sales, inventories, revenues and profits, are *identical.* Thus, the availability of disaggregated forecasts (and implicitly, an IS that provides information disaggregated on a product/market basis) is a *necessary* condition for a positive VOP. As will be shown later, however, the availability of disaggregated information is not sufficient for a positive VOP.

A second implication of Theorem 5 is that the total costs under DD are identical for both aggregate and disaggregate forecasts; however, forecast disaggregation may increase revenues through a better-informed allocation of the intermediate good across the two products. Further, by Theorem 4, the total production and inventory costs for ED are always greater than those for DD. ED may in fact generate greater sales revenues than DD, yet DD outperforms ED by its superior management of the marketing-operations interface. We look more closely at the profits under ED and DD in the next Section.

4.4 Profits under ED and DD

Under both ED and DD, three factors determine the net profit: sales revenues, production and refinement costs, and inventory holding costs. To interpret the profit expressions, consider first the DD process. Comparing Theorems 2 and 5, it is clear that under DD, production and inventory holding costs are the same for both aggregate and disaggregate forecasts. However, disaggregate forecasts increase revenues (and hence profits, since costs are the same as for aggregate forecasts). In fact, DD revenues may be decomposed into two components, based on the level of forecast aggregation: *(i) Revenues using aggregate forecasts:* When only aggregate forecasts are available, the firm estimates the average demand curve across the two markets, and ships half of the total quantity to each market (recall Theorem 5). The result is equivalent to selling the average of the total shipment quantity to two 'virtual' markets, in each of which the demand curve is the average of the demand curves of the two real markets; and *(ii) Additional Revenues from forecast disaggregation:* In addition to the averaging process, the firm can exploit the commonality of the intermediate good to make differential shipments of the two products based on product-specific demand forecasts. (The optimal shipment strategy attempts to equalize the expected marginal revenues from the two markets.) This component of revenue captures the additional value of disaggregated forecasts, and is the difference in revenues from following the DD policies specified in Theorems 2 and 5.

As shown in Theorem 5, ED and DD perform identically under aggregate forecasts; hence their profits are identical. Under forecast disaggregation, however, ED production, sales and inventory policies are markedly different (as given by Theorem 1). The additional value of forecast disaggregation under ED can be computed from the ED profits given by equation (9) of Theorem 6 below and our foregoing results. The following Theorem derives the expected disaggregated-forecast profits for ED and DD under their respective optimal policies.

Theorem 6 The infinite horizon discounted expected profits for the ED and DD processes are given by:

$$\Pi_{ED} = -2 \cdot k \cdot Q^*_{ED} - \frac{2 \cdot \beta}{1-\beta} \left[\sum_{\{x_1,x_2\}} P(x_1,x_2) \cdot \left(k \cdot \min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\} + h \cdot \left(Q^*_{ED} - \min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}\right)\right) \right]$$
$$+ \frac{2 \cdot \beta}{1-\beta} \left[\sum_{\{x_1,x_2\}} P(x_1,x_2) \cdot \left(f(P_{x_1x_2}) - \theta - b \cdot \min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}\right) \cdot \min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\} \right], \quad (9)$$

and

$$\Pi_{DD} = -k \cdot Q^*_{DD} - \frac{\beta}{1-\beta} \left[\sum_{\{x_1,x_2\}} P(x_1,x_2) \cdot \left\{ k \cdot \min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\} + h \cdot \left(Q^*_{DD} - \min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}\right)\right\} \right]$$
$$+ \frac{\beta}{1-\beta} \left[\sum_{\{x_1,x_2\}} \left\{ 2 \cdot P(x_1,x_2) \cdot \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2} - \theta - b \cdot \frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2} \right) \cdot \frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2} \right\} \right]$$
$$+ \frac{\beta}{1-\beta} \left[\sum_{\{x_1,x_2\}} P(x_1,x_2) \cdot \left\{ b \cdot \frac{[d(x_1,x_2)]^2}{2} \right\} \right]. \quad (10)$$

The profit (net present value) under ED, Π_{ED}, can be interpreted as follows. Since the profit contribution under ED is separable across the two products, Π_{ED} is twice the single-product profit.

Since the supply chain starts period 0 with zero inventory, the production cost in period 0 is the cost of producing the build-up-to level Q^*_{ED} for each product, hence the first term in (9). The factor $\frac{\beta}{1-\beta}$ is the familiar infinite-horizon net present value multiplier. Next, for a given signal pair (x_1, x_2), the shipment quantity of product 1 (say) under the optimal policy is $\min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}$, and the quantity held as inventory is $\left(Q^*_{ED} - \min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}\right)$. Under the optimal production policy, the production quantity of product 1 for the next period is $\min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}$. Thus, the next term in (9) captures the production costs, $k \cdot \min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}$, and the inventory holding cost, $h \cdot \left(Q^*_{ED} - \min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}\right)$, weighted by the probability $P(x_1, x_2)$ of observing the signal pair(x_1, x_2). Finally, the revenues from shipping the quantity $\min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}$when the demand signals are (x_1, x_2) are $\left(f(P_{x_1x_2}) - \theta - b \cdot \min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}\right) \cdot \min\left\{Q^*_{ED}, q^{ED}_{x_1x_2}\right\}$; the last term in (9) thus reflects the expected revenues.

Unlike ED, the net profit under DD (expression (10)) is not separable across the products. The first two terms of Π_{DD} capture the production and holding costs, which are lower than ED's, since both the build-up-to level and the average inventory are lower. The third term in Π_{DD} is the revenue from aggregate forecasts. To interpret this term, observe that when the signals received are (x_1, x_2), the total shipment quantity to the two markets is $\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}$; on average, each market receives $\frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2}$. If just this average quantity were shipped to each market, the revenues from the two markets would be $\left(f(P_{x_1x_2}) - \theta - b \cdot \frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2}\right) \cdot \frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2}$ and $\left(f(P_{x_2x_1}) - \theta - b . \frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2}\right) . \frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2}$. The total revenue for the firm would then be $2 \cdot \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2} - \theta - b \cdot \frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2}\right) . \frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2}$, which is twice the revenue obtained by shipping the "average" quantity $\frac{\min\left\{Q^*_{DD}, q^{DD}_{x_1x_2}\right\}}{2}$ to a market with "average" demand, given by the revenue function $R_{av}(q) = \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2} - \theta - b \cdot q\right) q$. The final term in $\Pi_{DD}, \sum_{\{(x_1,x_2)\}} P(x_1, x_2) \cdot \frac{b}{2} \cdot \ [d(x_1, x_2)]^2$, reflects the additional expected revenues from exploiting disaggregated forecasts by making differential shipments. To illustrate, suppose the market signals are x_1 and x_2, and the total quantity shipped is q. If the average quantity $\frac{q}{2}$ were shipped to each market, the total expected revenues would be $\left(f(P_{x_1x_2}) - \theta - b \cdot \frac{q}{2}\right) \cdot \frac{q}{2} + \left(f(P_{x_2x_1}) - \theta - b \cdot \frac{q}{2}\right) \cdot \frac{q}{2}$, which simplifies to $\left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2} - \theta\right) q - b\frac{q^2}{2}$. Now suppose the firm allocates the quantity q optimally, namely $\frac{q + d(x_1,x_2)}{2}$ to market 1 and $\frac{q - d(x_1,x_2)}{2}$ to market 2, thereby equalizing the respective marginal revenues. Total revenues are then $\left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2} - \theta\right) q - b\frac{q^2}{2} + b\frac{(d(x_1,x_2))^2}{2}$. Thus, the difference in revenues between the optimal allocation and an allocation of the same quantity to each market is exactly $b \cdot \frac{(d(x_1,x_2))^2}{2}$.

To summarize, the additional value of forecast disaggregation under DD is $\sum_{\{(x_1,x_2)\}} P(x_1,x_2) \cdot \frac{b}{2} \cdot [d(x_1,x_2)]^2$. Since VOP$= \Pi_{DD} - \Pi_{ED}$, the additional value of disaggregate forecasts under ED is $\sum_{\{(x_1,x_2)\}} P(x_1,x_2) \cdot \frac{b}{2} \cdot [d(x_1,x_2)]^2 - VOP$. Thus, DD benefits more from forecast disaggregation than ED; the difference in benefits is equal to the value of postponement.

It should also be clear from the preceding discussion that information disaggregation does not guarantee a positive VOP unless it translates operationally into differential shipments to the two markets. For example, in the extreme case where the two markets are perfectly positively correlated ($\rho = +1$), the posterior expected demand curves (given the signals) are always identical for the two markets — in the absence of differential shipments, the VOP is identically zero. Mathematically, when $\rho = +1$, $P_{10} = P_{01}$, and so $d(x_1,x_2) = 0$ for all x_1 and x_2. Hence, forecast disaggregation does not improve ED or DD profits, and the VOP is zero. More generally, postponement adds little value when the demands for the end-products are highly positively correlated ($P_{10} \approx P_{01}$), since (operationally) shipment quantities are not highly differentiated across products.

We develop an illustrative numerical example in the next Section.

4.5 Illustrative Example

We consider the demand intercepts $a_H = \$70$ (high) and $a_L = \$40$ (low) with a slope of $b = 1$ for the demand curve. Also, the unit production cost is $k = \$18$, the refinement (or customization and shipping) cost is $\theta = \$10$ and the discount factor is $\beta = 0.9$. We initially set the *á priori* probability of high demand, p, to $\frac{1}{2}$, and the inventory holding cost h to \$6. We varied the information precision parameter α from 0 to 0.5 in increments of 0.1. The coefficient of correlation ρ between the two markets was varied from 0 to 1, also in increments of 0.1.

Figure 3 shows the expected profits[9] under ED as α and ρ are varied. Predictably, for a fixed ρ, the profits increase as α falls, i.e., as information precision increases. However, for a fixed α, since the optimal solution and profits for the two products are separable, one might naively expect that changing the demand correlation ρ will not affect profits. In fact, we find that the profits

[9]Of course, the dollar profits under ED and DD and the dollar value of the VOP depend on the quantity units (e.g. thousands or millions).

increase in ρ, particularly for intermediate levels of information precision ($\alpha = 0.1$ or 0.2) and moderate to high levels of correlation ($\rho \geq 0.4$). The reason is that the signals from both markets are used in the Bayesian demand prediction for each, so the forecast accuracy for either market improves with the demand correlation. The incremental value of the correlation in making better demand predictions is higher for moderate values of the information precision parameter α: when $\alpha = 0$, the demand curves for each product are perfectly predictable using the signals from their respective markets, and the correlation does not add any value; on the other hand, when $\alpha = 0.5$ (pure noise), the information signal from the *other* market is also useless, hence the correlation does not improve prediction. In both of these cases, the profits plotted against ρ are flat lines (see the top and bottom curves in Figure 3).

INSERT FIGURE 3

Figure 4 shows that the profits under DD also increase, *ceteris paribus*, in information precision. The effect of the demand correlation on profits under DD is less obvious, as two counteracting effects come into play. The first is the Bayesian effect observed under ED, which tends to increase profits with correlation. On the other hand, a *pooling* effect operates in the opposite direction: the value of pooling enabled by delayed differentiation is a decreasing function of the correlation between the two markets. For most values of α and ρ, the pooling effect dominates, and so the profits under DD fall with increasing ρ. However, for moderate levels of information precision ($\alpha = 0.1, 0.2$ and 0.3) and high levels of correlation ($\rho \geq 0.8$), the Bayesian effect prevails, and profits increase with ρ in this region. When $\alpha = 0.4$, the pooling effect is stronger for all ρ. When information is completely precise ($\alpha = 0$), Bayesian analysis provides no additional information, and only the pooling effect is observed. Hence, unlike under ED, the profits fall with ρ even for $\alpha = 0$. When $\alpha = 0.5$, the signals are too noisy to provide any useful information, both effects disappear and profits are flat in ρ.

INSERT FIGURE 4

Figure 5 plots the Value of Postponement (VOP) for the same parameter values. The VOP is non-negative in our setting, because delayed product differentiation always outperforms early differentiation due to the DC's greater flexibility under DD in its customization and shipment choices.[10] Clearly, DD is better suited to exploit information than ED, and Figure 5 demonstrates

[10]In practice, this additional flexibility usually comes at a price: an additional fixed cost of

that for the chosen parameter values, the VOP increases in information precision; in the extreme case when information is so noisy ($\alpha = 0.5$) that it precludes a useful revision of forecasts, the VOP is zero. It should also be clear from the preceding discussion (Figures 3 and 4) that the VOP falls as the market correlation increases, and this is also confirmed by Figure 5. Thus, design for postponement is generally not cost-effective when the demands for the products sold are highly correlated or demand information is very noisy.

INSERT FIGURE 5

Figure 6 shows the effect of demand variability, measured by the intercept spread $a_H - a_L$, on the VOP. The average demand intercept, $\frac{a_H + a_L}{2}$, was kept constant at \$55, and the spread was varied from \$24 to \$34 (ρ was set at 0). For all levels of α, the VOP is *increasing* in demand variability *in absolute terms*, even though both ED and DD profits individually fall with variability (except for $\alpha = 0.5$, corresponding to useless information, where the VOP is identically zero). Thus, demand variability is clearly a key factor in determining the VOP: the more dynamic the demand environment, the higher the VOP (and the higher the payoff from redesigning the products and processes for postponement).

INSERT FIGURE 6

Figure 7 studies the effect of the *á priori* demand distribution on the VOP, by varying the value of p, the prior probability of high demand. Clearly, for $p = 0$ and $p = 1$, there is no uncertainty in demand– hence IS is rendered moot, and the VOP is zero for all values of the precision parameter α. Also, for $\alpha = 0.5$ (information is pure noise), the VOP is zero irrespective of the *á priori* demand distribution. For all other values of α, the VOP plotted against p is roughly umbrella-shaped, reaching a peak at around $p = 0.4$. Further, for most ranges of the parameter values, the VOP is increasing in IS precision; this trend is reversed only for very low α (0 and 0.1) and high values of p (> 0.7). For these values, the scale effect (both ED and DD profits increase in p, the probability of high demand) confounds the pooling effect (which enables better exploitation of demand information under DD).

INSERT FIGURE 7

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redesigning the supply chain for postponement, or higher unit costs for the DD process. (Figure 9 illustrates these tradeoffs.)

Figure 8 illustrates the effect of the inventory holding cost h on the VOP.[11] We find that although both ED and DD profits fall with h, the VOP *increases* in h. For sufficiently high levels of the holding cost, holding any inventory is too costly to be viable under either process, and the VOP flattens to a constant.

INSERT FIGURE 8

Finally, Figure 9 illustrates the tradeoff between the benefits of building postponement into the firm's production process and its costs. As discussed previously, restructuring the production process of the supply chain for postponement would incur additional fixed costs. Further, the DD production process might incur higher unit production costs. In the example, we set $\rho = \alpha = 0$, fix the unit production cost k at \$18 for ED, and let k vary for DD. Figure 9 is the indifference curve (which is nearly linear) showing the combinations of additional fixed costs and incremental unit production costs for which the firm would do equally well under ED or DD. Firms whose added DD costs are below this curve are better off implementing postponement; firms above this curve should opt for early differentiation. Thus, when the incremental marginal cost is 0 (i.e., $k = \$18$ under DD as well), the incremental fixed cost up to which the firm will prefer DD is 359.25, which is the value of postponement at these parameter values. As k under DD increases, the break-even incremental fixed costs between ED and DD falls. We find that when $k = \$19.47$, i.e., the marginal production cost under DD is 8.18% higher than under ED, DD and ED do identically (assuming no additional fixed costs under DD). In our example, DD is never preferred when the unit production cost increases to more than 8.18% of ED's costs.

INSERT FIGURE 9

Thus the choice of business process should be informed by demand conditions (including demand information) as well as cost data. As illustrated by the example, the VOP strongly depends on the demand variability, information precision, end-product demand correlation and inventory holding costs.

[11] We plot the inventory holding cost as a percentage of the costs of production and distribution, $\frac{h}{k+\theta}$.

5 Concluding Remarks

A firm's ability to forecast demand is intimately connected to the data gathered by its IS; its ability to respond depends on the nature and timing of the activities in its supply chain. Therefore, our model incorporates both material and information flows, including IS precision (accuracy), IS timing, and the activities (production and distribution) internal to the supply chain. We show that the value of postponement largely derives from the ability to make better use of information about demand. Three important attributes of information are *(i)* its timing within the supply chain (relative to the material flows), *(ii)* its level of aggregation, and *(iii)* its precision. All three attributes play an important role in determining the value of postponement.

The role of information timing, which is implicit in our model structure, illustrates the close link between operations and information systems in determining a firm's optimal choice of business process and profits. If demand information is received early enough to affect the firm's production decisions, delayed product differentiation would not substantively alter the firm's operational decisions, and would probably increase the firm's costs. If demand information is received very late (after the products are shipped to their respective markets), postponement again becomes irrelevant. Postponement is only meaningful when information is received after the initial production decision and prior to customization/distribution, as in our model. The synchronization of material and information flows is thus an important enabler of postponement as a strategic lever. Our results showed that the second attribute of information, viz., level of aggregation, is also an important factor influencing the choice of business process. Under aggregate forecasts, both early and delayed differentiation perform identically. In our context, disaggregate forecasts, stemming from product- (or market-) specific information and leading operationally to differential product shipments, are necessary for postponement to add value to the firm. Finally, our analysis showed that the value of postponement was strongly dependent on the precision of the firm's IS.

Our model suggests a number of open avenues for future research. While our dynamic model assumes that only the DC can hold inventory, the model could be extended to the case where retail outlets also hold inventory. The optimal solution to this problem may not possess a simple structure, and may not be derivable in closed form. Nevertheless, numerical analysis of this solution may yield further insights into the costs and benefits of postponement.

A recent stream of literature incorporates Bayesian "learning" in dynamic models with inventories (See Chen and Plambeck (2008) and references therein). A potentially interesting extension of our "physical-depot" model (see Section 1.2.2) is to study the effect of "learning" on the value of postponement strategies, perhaps by introducing serial correlation of demand across periods. Such non-stationary dynamic models with learning are probably analytically intractable, and hence will require heuristic approaches or numerical approximations.

Our analysis assumed that the incentives of the entire supply chain were aligned. Hence, two possible extensions to this research– incorporating incentive conflicts in a horizontal and/or a vertical context– arise naturally. Anand and Girotra (2007) modeled horizontal competition in which early/delayed differentiation was a decision variable for firms. In their static model, firms' IS are always "perfect". It would be interesting to extend this study to a *dynamic* setting with a richer model allowing for *imperfect* information (for example, by parameterizing the precision of demand information). Further, modeling costly investments in IS (so that IS precision is a decision variable) and linking this to the choice of business process would be a worthwhile avenue for further research.

Another important area relates to the interaction of multiple players within the supply chain vertical, linking inventory, information and incentives issues (*cf* Anand *et al* (2008), Anupindi *et al* (2001), Cachon (2001), Lee and Whang (1999) and Netessine and Zhang (2005)). When the different parts of a complex supply chain are controlled by different firms, as is often the case, coordinating their actions through proper contracting mechanisms becomes an important issue (Cachon (2003)). The study of postponement in such a setting, relating the effects of contract structure to the value and impact of postponement, is another promising area for future research.

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Appendix: Proofs and Detailed Derivations

Derivation of Conditional State Probabilities given Information Signals:

The conditional state distributions are given below, and derived using Bayes' rule.

$P_{00} = \frac{p\left[\alpha^2(p+(1-p)\cdot\rho)+\alpha(1-\alpha)(1-p)(1-\rho)\right]}{p[\alpha^2(p+(1-p)\cdot\rho)+\alpha(1-\alpha)(1-p)(1-\rho)]+(1-p)\cdot[\alpha(1-\alpha)p\cdot(1-\rho)+(1-\alpha)^2((1-p)+p\cdot\rho)]}$;

$P_{11} = \frac{p\left[(1-\alpha)^2(p+(1-p)\cdot\rho)+\alpha(1-\alpha)(1-p)(1-\rho)\right]}{p[(1-\alpha)^2(p+(1-p)\cdot\rho)+\alpha(1-\alpha)(1-p)(1-\rho)]+(1-p)\cdot[\alpha(1-\alpha)p\cdot(1-\rho)+\alpha^2((1-p)+p\cdot\rho)]}$;

$P_{01} = \frac{p\left[\alpha(1-\alpha)(p+(1-p)\cdot\rho)+\alpha^2(1-p)(1-\rho)\right]}{p[\alpha(1-\alpha)(p+(1-p)\cdot\rho)+\alpha^2(1-p)(1-\rho)]+(1-p)\cdot[(1-\alpha)^2p\cdot(1-\rho)+\alpha(1-\alpha)((1-p)+p\cdot\rho)]}$; and

$P_{10} = \frac{p\left[\alpha(1-\alpha)(p+(1-p)\cdot\rho)+(1-\alpha)^2(1-p)(1-\rho)\right]}{p[\alpha(1-\alpha)(p+(1-p)\cdot\rho)+(1-\alpha)^2(1-p)(1-\rho)]+(1-p)\cdot[\alpha^2p\cdot(1-\rho)+\alpha(1-\alpha)((1-p)+p\cdot\rho)]}$.[12]

By Bayes' rule,

$$P_{x_1x_2} = Pr[S_1 = 1 \mid X_1 = x_1,\ X_2 = x_2] = \frac{Pr[X_1 = x_1,\ X_2 = x_2 \mid S_1 = 1].Pr[S_1 = 1]}{Pr[X_1 = x_1,\ X_2 = x_2]}. \quad (11)$$

Further,

$$\begin{aligned} Pr[X_1 = x_1,\ X_2 = x_2 \mid S_1 = s] &= Pr[X_1 = x_1,\ X_2 = x_2 \mid S_1 = s, S_2 = 1] \cdot \Pr[S_2 = 1|S_1 = s] \\ &+Pr[X_1 = x_1,\ X_2 = x_2 \mid S_1 = s, S_2 = 0] \cdot \Pr[S_2 = 0|S_1 = s] \end{aligned} \quad (12)$$

Also,

$$\begin{aligned} \Pr[S_2 = 1|S_1 = 1] &= \frac{\Pr[S_1 = S_2 = 1]}{\Pr[S_1 = 1]} = \frac{p^2 + p(1-p)\rho}{p} = p + (1-p)\rho; \\ \Pr[S_2 = 0|S_1 = 0] &= \frac{\Pr[S_1 = S_2 = 0]}{\Pr[S_1 = 0]} = \frac{(1-p)^2 + p(1-p)\rho}{1-p} = (1-p) + p \cdot \rho; \\ \Pr[S_2 = 0|S_1 = 1] &= 1 - \Pr[S_2 = 1|S_1 = 1] = (1-p) \cdot (1-\rho)\,;\ \text{and} \\ \Pr[S_2 = 1|S_1 = 0] &= 1 - \Pr[S_2 = 0|S_1 = 0] = p \cdot (1-\rho)\,. \end{aligned} \quad (13)$$

[12]When $\alpha = 0$ and $\rho = 1$, we let $P_{01} = P_{10} = \frac{1}{2}$.

Letting $X_1 = \ X_2 = 0$ and $s = 1$ in equation (12), and using equations (13),

$$\begin{aligned} Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 1] = & \quad Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 1, S_2 = 1]. \Pr[S_2 = 1 | S_1 = 1] \\ + & \quad Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 1, S_2 = 0]. \Pr[S_2 = 0 | S_1 = 1] \\ = & \quad \alpha^2 (p + (1-p)\rho) + \alpha(1-\alpha) \cdot (1-p) \cdot (1-\rho) . \end{aligned} \tag{14}$$

Similiarly, from equations (12) and (13),

$$\begin{aligned} Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 0] = & \quad Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 0, S_2 = 1]. \Pr[S_2 = 1 | S_1 = 0] \\ + & \quad Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 0, S_2 = 0]. \Pr[S_2 = 0 | S_1 = 0] \\ = & \quad \alpha(1-\alpha)(p \cdot (1-\rho)) + (1-\alpha)^2 ((1-p) + p \cdot \rho) . \end{aligned} \tag{15}$$

Finally,

$$\begin{gathered} Pr[X_1 = 0,\ X_2 = 0] = Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 1] \cdot Pr[S_1 = 1] \\ + Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 0] \cdot Pr[S_1 = 0] \\ = p \cdot \Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 1] + (1-p) \Pr[X_1 = 0,\ X_2 = 0 \mid S_1 = 0]. \end{gathered} \tag{16}$$

From equations (11), (14), (15) and (16), we see that

$$P_{00} = \frac{p\left[\alpha^2 (p + (1-p) \cdot \rho) + \alpha(1-\alpha)(1-p)(1-\rho)\right]}{p[\alpha^2 (p + (1-p) \cdot \rho) + \alpha(1-\alpha)(1-p)(1-\rho)] + (1-p) \cdot [\alpha(1-\alpha) p \cdot (1-\rho) + (1-\alpha)^2 ((1-p) + p \cdot \rho)]} .$$

The derivation of the other probabilities is similiar.♦

Proof of Theorem 1:

We prove the optimality of the proposed policy for a single product; optimality for multiple products follows from the separability of the problem. The proof proceeds in three steps:

(i) We expand the space of feasible policies by allowing the firm to also sell its product in a "spot market" for the intermediate good (as described below). We consider a specific policy for this "augmented" problem, and derive the value function for this policy.

(ii) We prove that the policy proposed in *(i)* is optimal in the "augmented" problem. Thus, its value function is an upper bound on the value obtainable in the original problem.

(iii) We derive the value function for the policy proposed in Theorem 1, and show that it is identical to the value function of the optimal policy in the augmented problem. This completes the proof since the augmented problem is a relaxation of the original problem.

Step (i): Let Ψ denote the space of all feasible policies for the firm. Now suppose there is a "spot market" available every period, in which the firm can sell any amount of its available intermediate good at the price $k-h$. Denote the space of feasible policies with the additional spot-price option as $\widetilde{\Psi}$. Clearly, $\widetilde{\Psi} \supseteq \Psi$.

Now consider the following policy in the "augmented" problem:

Augmented policy:

(A) Production: Build up the warehouse inventory of the product each period to Q^*_{ED}. If the quantity available at the warehouse exceeds Q^*_{ED}, produce nothing.

(B) Shipment / Inventories: Conditional on the available quantity Q, and information $X = (x_1, x_2)$, ship up to the quantity $q(X) = q^{ED}_{x_1x_2} = \frac{f(P_{x_1x_2})-(k+\theta-h)}{2\cdot b}$ to the market. If there is any quantity left over, keep up to $[Q^*_{ED} - q(X)]^+ = Q^*_{ED} - \min\{Q^*_{ED}, q(X)\}$ in inventory. Sell the remaining quantity (if any; given by $[Q - \max\{Q^*_{ED}, q(X)\}]^+$) in the spot market.

We now derive the infinite horizon discounted value function $\widetilde{v}(Q)$ for the augmented policy, where the initial "state" (DC inventory) is Q. Under the augmented policy, the production quantity for the next period is $Q^*_{ED} - Q$ when $Q \leq Q^*_{ED}$, and 0 otherwise. The shipment quantity, when Q units are available and the information signal is $X = (x_1, x_2)$, is $\min\{Q, q(X)\}$. The inventory carried forward is the minimum of the remaining quantity and the inventory build-up-to level, which is $\min\{Q - \min\{Q, q(X)\}, Q^*_{ED} - \min\{Q^*_{ED}, q(X)\}\}$. Finally, the quantity sold in the spot market is the leftover after setting aside the shipment and inventory quantities, which is $[Q - \max\{Q^*_{ED}, q(X)\}]^+$. The expected revenue function (from sales of the final product) that period is $R(q|X) = (f(P_{x_1x_2}) - b \cdot q) \cdot q - \theta \cdot q$.

The value function is specified by the following recursive equations.

When $Q \leq Q^*_{ED}$,

$$\begin{aligned}\widetilde{v}(Q) &= -k \cdot (Q^*_{ED} - Q) + \beta \cdot E_X \left[R\left(\min\{Q^*_{ED}, q(X)\}|X\right) - h \cdot (Q^*_{ED} - \min\{Q^*_{ED}, q(X)\})\right] \\ &\quad -\beta \cdot E_X \left[k \cdot \min\{Q^*_{ED}, q(X)\}\right] + \beta \cdot \widetilde{v}(Q^*_{ED}). \end{aligned} \tag{17}$$

When $Q > Q^*_{ED}$,

$$\begin{aligned}\widetilde{v}(Q) &= \beta \cdot E_X \left[R\left(\min\{Q, q(X)\}|X\right) - h \cdot \min\left\{\left(Q - \min\{Q, q(X)\}\right), \left(Q^*_{ED} - \min\{Q^*_{ED}, q(X)\}\right)\right\}\right] \\ &\quad +\beta \cdot (k-h) \cdot E_X \left[\left(Q - \max\left\{Q^*_{ED}, q(X)\right\}\right)^+\right] \qquad (18) \\ &\quad -\beta \cdot k \cdot E_X \left[Q^*_{ED} - \min\left\{\left(Q - \min\{Q, q(X)\}\right), \left(Q^*_{ED} - \min\{Q^*_{ED}, q(X)\}\right)\right\}\right] + \beta \cdot \widetilde{v}(Q^*_{ED}).\end{aligned}$$

Observe that the intermediate good quantity available never exceeds Q^*_{ED} after the first period; thus, under the augmented policy, the spot market is never used after the first period. Also, when $Q^*_{ED} < Q$, $Q^*_{ED} - \min\{Q^*_{ED}, q(X)\} < Q - \min\{Q, q(X)\}$. Simplifying and solving equations (17) and (18) , we get

$$\widetilde{v}(Q) = \begin{cases} k \cdot Q + \widetilde{v}(0), & \text{if } Q \leq Q^*_{ED}; \\ \beta \cdot E_X \left[\Phi(Q; X)\right] + k \cdot Q^*_{ED} + \widetilde{v}(0), & \text{if } Q > Q^*_{ED}; \end{cases} \tag{19}$$

where

$$\widetilde{v}(0) = -k \cdot Q^*_{ED} + \frac{\beta}{1-\beta} \cdot E_X \left[R\left(\min\{Q^*_{ED}, q(X)\}|X\right) - h \cdot (Q^*_{ED} - \min\{Q^*_{ED}, q(X)\}) - k \cdot \min\{Q^*_{ED}, q(X)\}\right], \tag{20}$$

and

$$\Phi(Q; X) = R\left(\min\{Q, q(X)\}|X\right) - R\left(\min\{Q^*_{ED}, q(X)\}|X\right) + (k-h) \cdot \left(Q - \max\left\{Q^*_{ED}, q(X)\right\}\right)^+. \tag{21}$$

This completes the step *(i)* of the proof.

Step (ii) We first prove that Q^*_{ED} satisfies the equation

$$E_X \left[\left(R'(Q^*_{ED}|X) - (k-h)\right) \cdot I_{\{Q^*_{ED} < q(X)\}}\right] = \frac{k}{\beta} - k + h. \tag{22}$$

Defining $g(Q) = E_X \left[\left(R'(Q|X) - (k-h)\right) \cdot I_{\{Q < q(X)\}}\right]$, we wish to prove that $g(Q^*_{ED}) = \frac{k}{\beta} - k + h$. Since $R'(\cdot|X) = f(P_{x_1 x_2}) - \theta - 2 \cdot b \cdot q$, it is decreasing; further, $R'(q(X)|X) = k - h$. Hence,

$g(\cdot)$ is decreasing, and $g(q_{11}^{ED}) \leq g(q_{10}^{ED}) \leq g(q_{01}^{ED}) \leq g(q_{00}^{ED})$. Straightforward calculations show that $g(q_{11}^{ED}) = 0$; $g(q_{10}^{ED}) = P(1,1) \cdot [f(P_{11}) - f(P_{10})]$; $g(q_{01}^{ED}) = [P(1,1)f(P_{11}) + P(1,0)f(P_{10})] - (P(1,1) + P(1,0)) f(P_{01})$; and $g(q_{00}^{ED}) = P(1,1)f(P_{11}) + P(1,0)[f(P_{10}) + f(P_{01})] - (1 - P(0,0)) f(P_{00})$. To complete the proof of (22) , we consider four different ranges of parameter values, and show that the result holds for each range separately.

Case 1) $P(1,1) \cdot [f(P_{11}) - f(P_{10})] \geq \frac{k}{\beta} - k + h$: It is easy to check that $g(q_{10}^{ED}) \geq \frac{k}{\beta} - k + h > g(q_{11}^{ED})$. Thus, the solution to equation (22) is in the range $\left[q_{10}^{ED}, q_{11}^{ED}\right)$. In this range, $g(q) = P(1,1) \cdot [f(P_{11}) - 2b \cdot q - \theta - (k-h)]$, and $g(Q^*_{1,ED}) = \frac{k}{\beta} - k + h$, where $Q^*_{1,ED} = \frac{f(P_{11}) - (k+\theta-h) - \frac{\left(\frac{k}{\beta} - k + h\right)}{P(1,1)}}{2 \cdot b}$.

Case 2) $P(1,1) \cdot [f(P_{11}) - f(P_{10})] < \frac{k}{\beta} - k + h \leq [P(1,1)f(P_{11}) + P(1,0)f(P_{10})] - (P(1,1) + P(1,0)) f(P_{01})$: In this case, $g(q_{01}^{ED}) \geq \frac{k}{\beta} - k + h > g(q_{10}^{ED})$. Thus, the solution to equation (22) is in the range $\left[q_{01}^{ED}, q_{10}^{ED}\right)$. In this range, $g(q) = P(1,1)f(P_{11}) + P(1,0)f(P_{10}) - (P(1,1) + P(1,0)) \cdot (2b \cdot q + (k + \theta - h))$, and $g(Q^*_{2,ED}) = \frac{k}{\beta} - k + h$, where $Q^*_{2,ED} = \frac{\frac{P(1,1)f(P_{11}) + P(1,0)f(P_{10}) - \left(\frac{k}{\beta} - k + h\right)}{P(1,1) + P(1,0)} - (k+\theta-h)}{2 \cdot b}$.

Case 3) $[P(1,1)f(P_{11}) + P(1,0)f(P_{10})] - (P(1,1) + P(1,0)) f(P_{01}) < \frac{k}{\beta} - k + h \leq P(1,1)f(P_{11}) + P(1,0)[f(P_{10}) + f(P_{01})] - (1 - P(0,0)) f(P_{00})$: In this case, $g(q_{00}^{ED}) \geq \frac{k}{\beta} - k + h > g(q_{01}^{ED})$. Thus, the solution to equation (22) is in the range $\left[q_{00}^{ED}, q_{01}^{ED}\right)$. In this range, $g(q) = P(1,1)f(P_{11}) + P(1,0)[f(P_{10}) + f(P_{01})] - (1 - P(0,0)) \cdot (2b \cdot q + (k + \theta - h))$, and $g(Q^*_{3,ED}) = \frac{k}{\beta} - k + h$, where $Q^*_{3,ED} = \frac{\frac{P(1,1)f(P_{11}) + P(1,0)[f(P_{10}) + f(P_{01})] - \left(\frac{k}{\beta} - k + h\right)}{P(1,1) + 2P(1,0)} - (k+\theta-h)}{2 \cdot b}$.

Case 4) $P(1,1)f(P_{11}) + P(1,0)[f(P_{10}) + f(P_{01})] - (1 - P(0,0)) f(P_{00}) < \frac{k}{\beta} - k + h$: In this case, $g(q_{00}^{ED}) < \frac{k}{\beta} - k + h$. Thus, the solution to equation (22) is in the range $\left[0, q_{00}^{ED}\right)$. In this range, $g(q) = P(1,1)f(P_{11}) + P(1,0)[f(P_{10}) + f(P_{01})] + P(0,0)f(P_{00}) - (2b \cdot q + (k + \theta - h))$, and $g(Q^*_{4,ED}) = \frac{k}{\beta} - k + h$, where $Q^*_{4,ED} = \frac{P(1,1)f(P_{11}) + P(1,0)[f(P_{10}) + f(P_{01})] + P(0,0)f(P_{00}) - \left(\frac{k}{\beta} + \theta\right)}{2 \cdot b}$.

Routine algebra establishes that the solutions for the four different parameter ranges can be represented by the single expression $Q^*_{ED} = \max\left\{Q^*_{1,ED}, Q^*_{2,ED}, Q^*_{3,ED}, Q^*_{4,ED}\right\}$. Thus, $g(Q^*_{ED}) = \frac{k}{\beta} - k + h$, which proves equation (22).

Straightforward differentiation of equation (21) shows that, for $Q \neq q(X)$ and $Q > Q^*_{ED}$,

$$\Phi'(Q; X) = (k - h) + \left(R'(Q|X) - (k - h)\right) \cdot I_{\{Q < q(X)\}},$$

where $I_{\{A\}}$ is the indicator function that takes the value 1 if event A is true and 0 otherwise. Since $R'(\cdot|X)$ is decreasing and $R'(q(X)|X) = k - h$, $\Phi'(\cdot; X)$ is also decreasing in Q. Since in addition

$\Phi(\cdot;X)$ is continuous, it must also be concave. Thus, by equation (19), $\widetilde{v}(Q)$ is also concave in Q in the range (Q^*_{ED},∞). Further, $\widetilde{v}'(Q)$ is well-defined almost everywhere, and decreasing. For $Q < Q^*_{ED}$, $\widetilde{v}'(Q) = k$. For $Q > Q^*_{ED}$,

$$\begin{aligned}\widetilde{v}'(Q) &= \beta \cdot E_X\left[\Phi'(Q;X)\right] < \beta \cdot E_X\left[\Phi'(Q^*_{ED};X)\right] \\ &= \beta\cdot(k-h) + \beta\cdot E_X\left[\left(R'(Q^*_{ED}|X) - (k-h)\right)\cdot I_{\{Q^*_{ED}<q(X)\}}\right] \\ &= \beta\cdot(k-h) + \beta\cdot\left(\frac{k}{\beta} - k + h\right) \;= k \text{ (by equation (22))}.\end{aligned}$$

It follows that under the augmented policy,

$$\widetilde{v}'(Q) = k, \text{ for } Q < Q^*_{ED}, \text{ and} \tag{23}$$

$$\widetilde{v}'(Q) < k, \text{ for } Q > Q^*_{ED}. \tag{24}$$

We now prove the optimality of the augmented policy by backward induction. Assume that the augmented policy is followed for the production decision at the end of period 1 (after sales in period 1) and for production, shipment and inventory decisions from period 2 onwards for the rest of the horizon. We will show that the augmented policy is also optimal for the production policy in period 0, and the shipment and inventory policies in period 1, for *any* initial state (DC inventory). Thus, the augmented policy will be shown to be *unimprovable* and hence optimal.

We first derive the optimal distribution policy in period 1 for any information signal X. Let $\overline{Q}$ be the quantity on hand at the beginning of period 1. $\overline{Q}$ is the sum of the on-hand quantities at the end of period 0 (the start of the horizon) and the production in period 0 for period 1. Let Q be the shipment quantity to the output market, and Q_I the inventory carried forward to the next period. Then, the quantity sold in the spot market is $\overline{Q} - (Q + Q_I)$. The optimal distribution/inventory policy is given by the solution to the concave maximization problem

$$\max_{Q,Q_I} R(Q|X) + (k-h)\cdot\left(\overline{Q} - (Q+Q_I)\right) - h\cdot Q_I + \widetilde{v}(Q_I)$$

subject to the constraint $\overline{Q} \geq (Q + Q_I)$. The solution is simple to derive. The firm has three options: shipping to the output market, selling in the spot market and carrying inventory. The expected revenue from carrying inventory forward is $\widetilde{v}(Q_I) - h\cdot Q_I$; the expected marginal revenue of this inventory (from equations (23) and (24)) is $(k-h)$ for $Q_I < Q^*_{ED}$, and less than $(k-h)$ otherwise. The marginal revenue (= price) from the spot market is $(k-h)$. Since $R'(Q|X) >$

$(k-h)$ for $Q < q(X)$, shipping to the output market dominates the other options until the shipment quantity reaches $q(X)$. Since $R'(Q|X) < (k-h)$ for $Q > q(X)$, the spot market dominates shipping additional quantities above $q(X)$. So, the optimal quantity shipped never exceeds $q(X)$. The firm is *indifferent* between holding inventories and selling in the spot market as long as the inventories don't exceed Q^*_{ED}; above the inventory level of Q^*_{ED}, the spot market option dominates. Thus, if the available quantity $\overline{Q}$ is above $q(X)$, it is optimal to ship $q(X)$ to the output market; with regard to the remaining quantity, one optimal course of action is to carry inventory forward up to the level $[Q^*_{ED} - q(X)]^+$ (which is $< Q^*_{ED}$), and sell the rest in the spot market. Thus, the distribution/inventory policy specified by the augmented policy, is also optimal for period 1, for any available quantity $\overline{Q}$.

Given this period-1 shipment/inventory policy, the optimal production quantity Q^* in period 0, for an arbitrary on-hand inventory of Q, is the solution to the concave maximization problem

$$\max_{Q^*} \quad \{-k \cdot Q^* + \widetilde{v}(Q^* + Q)\},$$

where $Q^* \geq 0$. Since $\frac{\partial \widetilde{v}(Q^*+Q)}{\partial Q^*} = k$ for $(Q^* + Q) < Q^*_{ED}$, and $\frac{\partial \widetilde{v}(Q^*+Q)}{\partial Q^*} < k$ for $(Q^* + Q) > Q^*_{ED}$ (as shown previously), it is optimal to build up the warehouse inventory in period 1 to Q^*_{ED} when $Q < Q^*_{ED}$; i.e., to produce the quantity $Q^* = Q^*_{ED} - Q$, when $Q^*_{ED} > Q$. When $Q \geq Q^*_{ED}$, it is optimal to produce nothing, since the marginal cost k outweighs the expected marginal benefit. Thus, the optimal production quantity is $Q^* = (Q^*_{ED} - Q)^+$, which is identical to the production under the augmented policy.

Thus, we have shown that if the augmented policy is followed in future periods, it is optimal in the current period as well, for all initial states (DC's starting inventories). This shows that the augmented policy is unimprovable and hence optimal in the space $\widetilde{\Psi}$. This completes step *(ii)* of the proof.

Step (iii) Now consider the production and distribution/inventory policies proposed in the Theorem. When the starting inventory is $Q \leq Q^*_{ED}$, and the space of feasible policies is Ψ (i.e., there is no spot market available), the infinite horizon returns under the proposed policy are

$$\begin{aligned} v(Q) \quad = \quad & -k \cdot (Q^*_{ED} - Q) + \beta \cdot E_X \left[R\left(\min\{Q^*_{ED}, q(X)\} | X \right) - h \cdot (Q^*_{ED} - \min\{Q^*_{ED}, q(X)\}) \right] \\ & -\beta \cdot E_X \left[k \cdot \min\{Q^*_{ED}, q(X)\} \right] + \beta \cdot \widetilde{v}(Q^*_{ED}). \end{aligned}$$

It is clear that $v(Q) = k \cdot Q + v(0)$ for all $Q \leq Q^*_{ED}$. Simplifying, we get

$$v(0) = -k \cdot Q^*_{ED} + \frac{\beta}{1-\beta} \cdot E_X \left[R \left(\min\{Q^*_{ED}, q(X)\} | X \right) - h \cdot \left(Q^*_{ED} - \min\{Q^*_{ED}, q(X)\} \right) - k \cdot \min\{Q^*_{ED}, q(X)\} \right],$$

which is identical to $\widetilde{v}(0)$ under the augmented policy (compare with equations (19) and (20)). Since $\widetilde{\Psi} \supseteq \Psi$ and the augmented policy is optimal in $\widetilde{\Psi}$, the value function $\widetilde{v}(0)$ from the augmented policy is an upper bound on the value from any policy in Ψ. Since $v(0) = \widetilde{v}(0)$, the policy proposed in Theorem 1 attains this upper bound, and is hence optimal for Ψ. The myopia of the optimal policy follows by using the terminal value function $v_0(Q) = (k - h) \cdot Q$.

This completes the proof of Theorem **1**. ♦

Proof of Theorem 2:

The proof is very similiar to that of Theorem 1, and sketched below. The three steps of the proof are:

(i) Augment the space of feasible policies by introducing a "spot market" for the intermediate good. Derive the value function for a specific policy for the "augmented" problem.

(ii) Prove that the policy proposed in *(i)* is optimal for the "augmented" problem. Thus, its value function is an upper bound on the value obtainable in the original problem.

(iii) Show that the value function for the policy proposed in Theorem 2 is identical to that of the optimal policy for the augmented problem. This completes the proof since the augmented problem is a relaxation of the original problem.

Step (i): Let Ψ denote the space of all feasible policies for the firm. Let $\widetilde{\Psi}$ denote the augmented space, with a "spot market" available every period, in which the firm can sell any amount of its available intermediate good at the price $k - h$. Clearly, $\widetilde{\Psi} \supseteq \Psi$.

Now consider the following policy in the "augmented" problem:

Augmented policy:

(A) Production: Build up the warehouse inventory of the product each period to Q^*_{DD}. If the quantity available at the warehouse exceeds Q^*_{DD}, produce nothing.

(B) Shipment / Inventories: Conditional on the available quantity Q, and information $X = (x_1, x_2)$, ship up to the quantity $q(X) = q^{DD}_{x_1x_2} = \frac{\frac{f(P_{x_1x_2})+f(P_{x_2x_1})}{2}-(k+\theta-h)}{b}$ to the markets. Of the total shipment quantity $q^S = \min\{Q, q^{DD}_{x_1x_2}\}$, ship $\frac{q^S+d(x_1,x_2)}{2}$ units of product 1 and $\frac{q^S-d(x_1,x_2)}{2}$ units of product 2. *If there is any quantity left over after shipment, keep up to $[Q^*_{DD} - q(X)]^+ = Q^*_{DD} - \min\{Q^*_{DD}, q(X)\}$ in inventory. Sell the remaining quantity (if any; given by $[Q - \max\{Q^*_{DD}, q(X)\}]^+$) in the spot market.*

Observe that the "augmented" policy assumes that the total quantity shipped, $q^S \geq |d(x_1, x_2)|$. The proof that this will hold under this policy is trivial: the available quantity for shipping each period is at least Q^*_{DD}, and it is easily shown that $Q^*_{DD} = \max\left\{Q^*_{1,DD}, Q^*_{2,DD}, Q^*_{3,DD}\right\} \geq Q^*_{3,DD} \geq d(1,0)$. The induced conditional revenue function under the augmented policy is then given by

$$\begin{aligned} R(q|X_1 = x_1, X_2 = x_2) &= \left(f(P_{x_1x_2}) - b\cdot\left(\frac{q+d(x_1,x_2)}{2}\right)\right)\cdot\left(\frac{q+d(x_1,x_2)}{2}\right) \\ &\quad + \left(f(P_{x_2x_1}) - b\cdot\left(\frac{q-d(x_1,x_2)}{2}\right)\right)\cdot\left(\frac{q-d(x_1,x_2)}{2}\right) - \theta\cdot q \\ &= -\theta\cdot q + q\cdot\left(\frac{f(P_{x_1x_2})+f(P_{x_2x_1})}{2}\right) - \frac{b}{2}\cdot\left(q^2 - (d(x_1,x_2))^2\right). \end{aligned} \tag{25}$$

The rest of the derivation of the value function for the augmented policy under DD is identical to that for ED, except that Q^*_{DD} replaces Q^*_{ED}, and the conditional revenue function $R(q|X)$ is now given by (25). Thus, we get:

$$\widetilde{v}(Q) = \begin{cases} k\cdot Q + \widetilde{v}(0), & \text{if } Q \leq Q^*_{DD}; \\ \beta\cdot E_X\left[\Phi(Q;X)\right] + k\cdot Q^*_{DD} + \widetilde{v}(0), & \text{if } Q > Q^*_{DD}; \end{cases} \tag{26}$$

where

$$\widetilde{v}(0) = -k\cdot Q^*_{DD} + \frac{\beta}{1-\beta}\cdot E_X\left[R\left(\min\{Q^*_{DD}, q(X)\}|X\right) - h\cdot\left(Q^*_{DD} - \min\{Q^*_{DD}, q(X)\}\right) - k\cdot\min\{Q^*_{DD}, q(X)\}\right], \tag{27}$$

and

$$\Phi(Q;X) = R\left(\min\{Q, q(X)\}|X\right) - R\left(\min\{Q^*_{DD}, q(X)\}|X\right) + (k-h)\cdot\left(Q - \max\{Q^*_{DD}, q(X)\}\right)^+. \tag{28}$$

This completes step *(i)* of the proof.

Step (ii) We first derive the optimal allocation rule (among the two products) for a total shipment quantity q. Suppose q_1 and q_2 are the quantities of each product shipped $(q_1 + q_2 = q)$. When the information signals received in a period are $X_1 = x_1$ and $X_2 = x_2$, the expected revenues from each product are $(f(P_{x_1x_2}) - b \cdot q_1) \cdot q_1 - \theta \cdot q_1$ and $(f(P_{x_2x_1}) - b \cdot q_2) \cdot q_2 - \theta \cdot q_2$. To maximize the total revenues, the firm solves the constrained optimization problem

$$\begin{array}{ll} \max\limits_{q_1} & (f(P_{x_1x_2}) - b \cdot q_1) \cdot q_1 - \theta \cdot q_1 + (f(P_{x_2x_1}) - b \cdot (q - q_1)) \cdot (q - q_1) - \theta \cdot (q - q_1) \\ \text{such that} & q \geq q_1 \geq 0. \end{array}$$

The first-order conditions from the Lagrangian, and the complementary slackness conditions give us:

$$\begin{aligned} f(P_{x_1x_2}) - f(P_{x_2x_1}) + 2b \cdot (q - 2 \cdot q_1) + \mu_1 - \mu_2 &= 0, \text{ and} \\ \mu_1 \cdot q_1 = \mu_2 \cdot (q - q_1) &= 0, \end{aligned}$$

where $\mu_1, \mu_2 \geq 0$. There are three possible solutions, depending on the value of q:

(i) $q < d(x_1, x_2)$: In this case, $q_1 = q$, $q_2 = 0$, $\mu_1 = 0$, and $\mu_2 > 0$.

(ii) $q < -d(x_1, x_2)$ $(= d(x_2, x_1))$: In this case, $q_1 = 0$, $q_2 = q$, $\mu_1 > 0$, and $\mu_2 = 0$.

(iii) $q \geq |d(x_1, x_2)|$: In this case, $q_1 = \frac{q+d(x_1,x_2)}{2}$, $q_2 = \frac{q-d(x_1,x_2)}{2}$, and $\mu_1 = \mu_2 = 0$.

Thus, the optimal allocation policy of any shipment quantity q $(\geq |d(x_1, x_2)|)$ in any period is to ship $\frac{q+d(x_1,x_2)}{2}$ of product 1 and $\frac{q-d(x_1,x_2)}{2}$ of product 2. This necessary condition for optimality in $\widetilde{\Psi}$ is satisfied by the augmented policy. Plugging these solutions back into the objective function given by (29), the expected revenues conditional on the information $(X_1 = x_1, X_1 = x_2)$ are

$$R(q|X_1 = x_1, X_2 = x_2) = \begin{cases} -\theta \cdot q + q \cdot f(P_{x_1x_2}) - bq^2, & \text{if } q < d(x_1, x_2); \\ -\theta \cdot q + q \cdot \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2}\right) - \frac{b}{2} \cdot \left(q^2 - (d(x_1, x_2))^2\right), & \text{if } q \geq |d(x_1, x_2)|; \\ -\theta \cdot q + q \cdot f(P_{x_2x_1}) - bq^2, & \text{if } q < -d(x_1, x_2). \end{cases} \quad (29)$$

From equation (25),

$$R'(q|X_1 = x_1, X_2 = x_2) = \begin{cases} -\theta + f(P_{x_1x_2}) - 2bq, & \text{if } q < d(x_1, x_2), \\ -\theta + \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2}\right) - bq, & \text{if } q \geq |d(x_1, x_2)|, \text{ and} \\ -\theta + f(P_{x_2x_1}) - 2bq, & \text{if } q < -d(x_1, x_2), \end{cases} \quad (30)$$

which is continuous, differentiable almost everywhere (except at the points $\pm d(x_1, x_2)$), and decreasing in q for every (X_1, X_2). Since, in addition, $R(q|X)$ is continuous, it is concave.

Next, we prove that Q^*_{DD} satisfies the equation

$$E_X\left[\left(R'(Q^*_{DD}|X)-(k-h)\right)\cdot I_{\left\{Q^*_{DD}<q(X)\right\}}\right]=\frac{k}{\beta}-k+h. \tag{31}$$

Defining $g(Q) = E_X\left[(R'(Q|X)-(k-h))\cdot I_{\{Q<q(X)\}}\right]$, we wish to prove that $g(Q^*_{DD}) = \frac{k}{\beta} - k + h$. From equation (30), $R'(\cdot|X)$ is decreasing, and $R'(q(X)|X) = k-h$. Hence, $g(\cdot)$ is decreasing, and $g(q^{DD}_{11}) \le g(q^{DD}_{10}) = g(q^{DD}_{01}) \le g(q^{DD}_{00})$. Straightforward calculations show that $g(q^{DD}_{11}) = 0$; $g(q^{DD}_{10}) = P(1,1)\cdot\left[f(P_{11}) - \frac{(f(P_{10})+f(P_{01}))}{2}\right]$; and $g(q^{DD}_{00}) = P(1,1)\cdot f(P_{11}) + P(1,0)\cdot(f(P_{10})+f(P_{01})) - [P(1,1)+2\cdot P(1,0)]\,f(P_{00})$. To complete the proof of (31), we consider three different ranges of parameter values, and show that the result holds for each range separately.

Case 1) $P(1,1)\cdot\left[f(P_{11}) - \frac{(f(P_{10})+f(P_{01}))}{2}\right] \ge \frac{k}{\beta} - k + h$: It is easy to check that $g(q^{DD}_{10}) \ge \frac{k}{\beta} - k + h > g(q^{DD}_{11})$. Thus, the solution to equation (31) is in the range $\left[q^{DD}_{10}, q^{DD}_{11}\right)$. In this range, $g(q) = P(1,1)\cdot[f(P_{11}) - b\cdot q - \theta - (k-h)]$, and $g(Q^*_{1,DD}) = \frac{k}{\beta} - k + h$, where $Q^*_{1,DD} = \frac{f(P_{11})-(k+\theta-h)-\frac{\left(\frac{k}{\beta}-k+h\right)}{P(1,1)}}{b}$.

Case 2) $P(1,1)\cdot\left[f(P_{11}) - \frac{(f(P_{10})+f(P_{01}))}{2}\right] < \frac{k}{\beta}-k+h \le P(1,1)\cdot f(P_{11})+P(1,0)\cdot(f(P_{10})+f(P_{01})) - [P(1,1)+2\cdot P(1,0)]\,f(P_{00})$: In this case, $g(q^{DD}_{00}) \ge \frac{k}{\beta} - k + h > g(q^{DD}_{10})$. Thus, the solution to equation (31) is in the range $\left[q^{DD}_{00}, q^{DD}_{10}\right)$. In this range, $g(q) = P(1,1)f(P_{11}) + P(1,0)\cdot(f(P_{10})+f(P_{01})) - (P(1,1)+2\cdot P(1,0))\cdot(b\cdot q+(k+\theta-h))$, and $g(Q^*_{2,DD}) = \frac{k}{\beta} - k + h$, where $Q^*_{2,DD} = \frac{\left(\frac{P(1,1)f(P_{11})+P(1,0)\cdot(f(P_{10})+f(P_{01}))-\left(\frac{k}{\beta}-k+h\right)}{P(1,1)+2\cdot P(1,0)}\right)-(k+\theta-h)}{b}$.

Case 3) $P(1,1)\cdot f(P_{11})+P(1,0)\cdot(f(P_{10})+f(P_{01}))-[P(1,1)+2\cdot P(1,0)]\,f(P_{00}) < \frac{k}{\beta}-k+h$: In this case, $\frac{k}{\beta} - k + h > g(q^{DD}_{00})$. Thus, the solution to equation (31) is in the range $\left[0, q^{DD}_{00}\right)$. In this range, $g(q) = P(1,1)\cdot f(P_{11}) + P(1,0)\cdot(f(P_{10})+f(P_{01})) + P(0,0)f(P_{00}) - (b\cdot q + (k+\theta-h))$, and $g(Q^*_{3,DD}) = \frac{k}{\beta} - k + h$, where $Q^*_{3,DD} = \frac{P(1,1)\cdot f(P_{11})+P(1,0)\cdot(f(P_{10})+f(P_{01}))+P(0,0)f(P_{00})-\left(\frac{k}{\beta}+\theta\right)}{b}$.

Routine algebra establishes that the solutions for the three different parameter ranges can be represented by the single expression $Q^*_{DD} = \max\left\{Q^*_{1,DD}, Q^*_{2,DD}, Q^*_{3,DD}\right\}$. Thus, $g(Q^*_{DD}) = \frac{k}{\beta} - k + h$, which proves equation (31).

Using the above results (as in the proof for ED), it can be established that under the augmented policy for DD,

$$\begin{aligned}\widetilde{v}'(Q) &= k, \text{ for } Q < Q^*_{DD}, \text{ and}\\ \widetilde{v}'(Q) &< k, \text{ for } Q > Q^*_{DD}.\end{aligned} \tag{32}$$

The rest of the proof of optimality uses backward induction. Assume that the augmented policy is followed for the production decision at the end of period 1 (after sales in period 1) and for production, shipment and inventory decisions from period 2 onwards for the rest of the horizon. We will show that the augmented policy is also optimal for the production policy in period 0, and the shipment and inventory policies in period 1, for *any* initial state (DC inventory). Thus, the augmented policy will be shown to be *unimprovable* and hence optimal.

We first derive the optimal distribution policy in period 1 for any information signal X. Let $\overline{Q}$ be the quantity on hand at the beginning of period 1. Let Q be the shipment quantity to the output market, and Q_I the inventory carried forward to the next period. Then, the quantity sold in the spot market is $\overline{Q} - (Q + Q_I)$. The optimal distribution/inventory policy is given by the solution to the concave maximization problem

$$\max_{Q,Q_I} R(Q|X) + (k-h) \cdot \left(\overline{Q} - (Q + Q_I) \right) - h \cdot Q_I + \widetilde{v}\,(Q_I)$$

subject to the constraint $\overline{Q} \geq (Q + Q_I)$. Using equations (32), (32), and (30), and by arguments similiar to those for the ED proof, the distribution/inventory policy specified by the augmented policy, is also optimal for period 1, for any available quantity $\overline{Q}$.

Given this period-1 shipment/inventory policy, the optimal production quantity Q^* in period 0, for an arbitrary on-hand inventory of Q, is the solution to the concave maximization problem

$$\max_{Q^*} \quad \{-k \cdot Q^* \; + \widetilde{v}(Q^* \; + Q)\},$$

where $Q^* \geq 0$. Since $\frac{\partial \widetilde{v}(Q^*+Q)}{\partial Q^*} = k$ for $(Q^* + Q) < Q^*_{DD}$, and $\frac{\partial \widetilde{v}(Q^*+Q)}{\partial Q^*} < k$ for $(Q^* + Q) > Q^*_{DD}$ (from equations (32) and (32)), the optimal production quantity is $Q^* = (Q^*_{DD} - Q)^+$, which is identical to the production under the augmented policy.

Thus, by induction, the augmented policy is optimal in the space $\widetilde{\Psi}$. This completes step *(ii)* of the proof.

Step (iii) Now consider the production and distribution/inventory policies proposed in the Theorem. When the starting inventory is $Q \leq Q^*_{DD}$, and the space of feasible policies is Ψ (i.e., there is no spot market available), the value function under the proposed policy, for all $Q \leq Q^*_{DD}$, is given by

$v(Q) = k \cdot Q + v(0)$, where

$$v(0) = -k \cdot Q^*_{DD} + \frac{\beta}{1-\beta} \cdot E_X \left[R\left(\min\{Q^*_{DD}, q(X)\}|X\right) - h \cdot \left(Q^*_{DD} - \min\{Q^*_{DD}, q(X)\}\right) - k \cdot \min\{Q^*_{DD}, q(X)\}\right],$$

which is identical to $\widetilde{v}(0)$ under the augmented policy (compare with equations (26) and (27)). Thus, $v(0)$ attains the upper bound $\widetilde{v}(0)$ on the value of any policy in Ψ, is hence optimal. The myopia of the optimal policy follows by using the terminal value function $v_0(Q) = (k-h) \cdot Q$. This completes the proof of Theorem **2**. ♦

Proof of Theorem 3:

The proof of the relationship between the ship-up-to levels for the two structures is trivial. To prove the second part of the Theorem, recall that the expressions for Q^*_{ED} and Q^*_{DD} depend on the ranges of the parameter values (We derived the parameter ranges corresponding to the build-up-to levels in the proofs of Theorems 1 and 2. These are reproduced below.)

Range *(i)* $P(1,1) \cdot (f(P_{11}) - f(P_{10})) \geq \frac{k}{\beta} - k + h$:Here, $Q^*_{DD} = Q^*_{1,DD} \in [q^{DD}_{10}, q^{DD}_{11})$ and $Q^*_{ED} = Q^*_{1,ED} \in [q^{ED}_{10}, q^{ED}_{11})$.

Range *(ii)* $P(1,1) \cdot (f(P_{11}) - f(P_{10})) < \frac{k}{\beta} - k + h \leq P(1,1) \cdot \left(f(P_{11}) - \frac{f(P_{01})+f(P_{10})}{2}\right)$: $Q^*_{DD} = Q^*_{1,DD} \in [q^{DD}_{10}, q^{DD}_{11})$ and $Q^*_{ED} = Q^*_{2,ED} \in [q^{ED}_{01}, q^{ED}_{10})$.

Range *(iii)* $P(1,1) \cdot \left(f(P_{11}) - \frac{f(P_{01})+f(P_{10})}{2}\right) < \frac{k}{\beta} - k + h \leq P(1,1) \cdot f(P_{11}) + P(1,0) \cdot f(P_{10}) - (P(1,1) + P(1,0))\, f(P_{01})$: $Q^*_{DD} = Q^*_{2,DD} \in [q^{DD}_{00}, q^{DD}_{10})$ and $Q^*_{ED} = Q^*_{2,ED} \in [q^{ED}_{01}, q^{ED}_{10})$.

Range *(iv)* $P(1,1) \cdot f(P_{11}) + P(1,0) \cdot f(P_{10}) - (P(1,1) + P(1,0))\, f(P_{01}) < \frac{k}{\beta} - k + h \leq P(1,1) \cdot f(P_{11}) + P(1,0) \cdot [f(P_{10}) + f(P_{01})] - (P(1,1) + 2 \cdot P(1,0))\, f(P_{00})$: $Q^*_{DD} = Q^*_{2,DD} \in [q^{DD}_{00}, q^{DD}_{10})$ and $Q^*_{ED} = Q^*_{3,ED} \in [q^{ED}_{00}, q^{ED}_{01})$.

Range *(v)* $P(1,1) \cdot f(P_{11}) + P(1,0) \cdot [f(P_{10}) + f(P_{01})] - (P(1,1) + 2 \cdot P(1,0))\, f(P_{00}) < \frac{k}{\beta} - k + h$: $Q^*_{DD} = Q^*_{3,DD} \in [0, q^{DD}_{00})$ and $Q^*_{ED} = Q^*_{4,ED} \in [0, q^{ED}_{00})$.

It is straight forward to check that $2 \cdot Q^*_{ED} = Q^*_{DD}$ for the ranges (i), (iv) and (v). Under conditions of range (ii), $2 \cdot Q^*_{ED} - Q^*_{DD} = \left(\frac{P(1,0)}{b \cdot P(1,1) \cdot (P(1,1)+P(1,0))}\right) \cdot \left[\left(\frac{k}{\beta} - k + h\right) - P(1,1) \cdot (f(P_{11}) - f(P_{10}))\right]$, and in range (iii), $2 \cdot Q^*_{ED} - Q^*_{DD} = \frac{P(1,0) \cdot \left[P(1,1).f(P_{11}) + P(1,0).f(P_{10}) - (P(1,1)+P(1,0)) f(P_{01}) - \left(\frac{k}{\beta} - k + h\right)\right]}{b \cdot (P(1,1)+P(1,0)) \cdot (P(1,1)+2P(1,0))}$. Clearly, $2 \cdot Q^*_{ED} \geq Q^*_{DD}$ for all the ranges. ♦

Proof of Theorem 4:

(i) Since $P(1,0) = P(0,1)$ and $Q^*_{ED} \leq q^{ED}_{11}$ always, equation (6) simplifies to

$$I_{ED} = 2 \cdot P(1,0) \cdot \left[\left(Q^*_{ED} - q^{ED}_{10} \right)^+ + \left(Q^*_{ED} - q^{ED}_{01} \right)^+ \right] + 2 \cdot P(0,0) \cdot \left(Q^*_{ED} - q^{ED}_{00} \right)^+ . \tag{33}$$

Since $q^{DD}_{10} = q^{DD}_{01}$ and $Q^*_{DD} \leq q^{DD}_{11}$, equation (7) simplifies to

$$I_{DD} = 2 \cdot P(1,0) \cdot \left(Q^*_{DD} - q^{DD}_{10} \right)^+ + P(0,0) \cdot \left(Q^*_{DD} - q^{DD}_{00} \right)^+ . \tag{34}$$

From equations (33) and (34) , the reduction in inventory achieved through delayed differentiation is

$$\begin{aligned} I_{ED} - I_{DD} &= 2 \cdot P(1,0) \cdot \left[\left(Q^*_{ED} - q^{ED}_{10} \right)^+ + \left(Q^*_{ED} - q^{ED}_{01} \right)^+ - \left(Q^*_{DD} - q^{DD}_{10} \right)^+ \right] \\ &\quad + P(0,0) \cdot \left[2 \cdot \left(Q^*_{ED} - q^{ED}_{00} \right)^+ - \left(Q^*_{DD} - q^{DD}_{00} \right)^+ \right], \end{aligned} \tag{35}$$

which can be further simplified using the results of Theorems 1 and 2. As was discussed in the proof of Theorem 3, depending on the parameter values, Q^*_{DD} and Q^*_{ED} can belong to one of five possible ranges, with their values determined by Theorems 1 and 2. Plugging these values, and the known values of the ship-up-to levels, into equation (35), it can be shown that $I_{ED} - I_{DD} = 2 \cdot Q^*_{ED} - Q^*_{DD}$ in each of the five ranges. *(ii)* The proof of the second part of the Theorem is straightforward from equation (8) and the result of the first part. ♦

Proof of Theorem 5:

(i) DD under aggregate forecasts: Suppose a total quantity q of the intermediate good is refined and shipped (as either or both of the two products), and the information signals received in any period are (x_1, x_2), garbled so that the signals are not separable across products. Suppose q_1 and q_2 are the quantities of each product shipped under DD. Thus, $q_2 = q - q_1$. The expected revenues from each product (using aggregate forecasts) are then $\left(\frac{f(P_{x_1 x_2}) + f(P_{x_2 x_1})}{2} - b \cdot q_1 \right) \cdot q_1 - \theta \cdot q_1$

and $\left(\frac{f(P_{x_1x_2})+f(P_{x_2x_1})}{2} - b \cdot q_2\right) \cdot q_2 - \theta \cdot q_2$. To maximize the total revenues for a total shipment of q, the firm solves the constrained optimization problem

$$\max_{q_1} \quad \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2} - b \cdot q_1\right) \cdot q_1 - \theta \cdot q_1 + \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2} - b \cdot (q - q_1)\right) \cdot (q - q_1)$$
$$-\theta \cdot (q - q_1)$$
$$\text{such that} \qquad q \geq q_1 \geq 0.$$

This simplifies to

$$\max_{q_1} \quad \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2} - \theta\right) \cdot q - b \cdot \left[q_1^2 + (q - q_1)^2\right]$$
$$\text{such that} \qquad q \geq q_1 \geq 0.$$

The solution is, clearly, $q_1 = q_2 = \frac{q}{2}$. Plugging this solution back into the objective function, the expected revenues under forecast aggregation, conditional on the information set (x_1, x_2), are

$$R(q|x_1, x_2) = -\theta \cdot q + q \cdot \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2}\right) - \frac{b}{2} \cdot q^2.$$

Differentiating this with respect to q gives the expected marginal revenues

$$R'(q|x_1, x_2) = -\theta + \left(\frac{f(P_{x_1x_2}) + f(P_{x_2x_1})}{2}\right) - bq, \tag{36}$$

which is identical to the expected marginal revenues for DD under disaggregate forecasts when $q \geq |d(x_1, x_2)|$, as given by equation(30) in the proof of Theorem 2. The rest of the proof follows that of Theorem 2.

(ii) ED under aggregate forecasts: Suppose the information signals received in any period are (x_1, x_2), garbled so that the signals are not separable across products. Consider (say) product 1. The expected revenues from shipping quantity q of the product (using aggregate forecasts) are

$\left(\frac{f(P_{x_1x_2})+f(P_{x_2x_1})}{2} - b\cdot q\right)\cdot q - \theta\cdot q$, and hence the expected marginal revenues are $R'(q|x_1,x_2) = -\theta + \left(\frac{f(P_{x_1x_2})+f(P_{x_2x_1})}{2}\right) - 2\cdot bq$; comparing this with the marginal revenues for DD under aggregate forecasts, given by equation (36) , we see that they are identical except for scaling the demand slope parameter b by a factor of 2. The rest of the arguments for the proof of the optimal policy for DD under aggregate forecasts apply here as well; because of the multiplier, it is easily seen that Q^*_{ED} (for each product) $=\frac{Q^*_{DD}}{2}$ and $q^{ED}_{x_1x_2} = \frac{q^{DD}_{x_1x_2}}{2}$, obtained by replacing b by $2\cdot b$ in the aggregate forecasts expressions for Q^*_{DD} and $q^{DD}_{x_1x_2}$.

This completes the proof of the Theorem. ♦

<u>Proof of Theorem 6:</u>

These results follow from the value functions derived for the respective policies in the proofs of optimality. See equations (19) and (20) of Theorem 1, and equations (26) and (27) of Theorem 2. ♦

Figure 1: Firm's Supply Chain

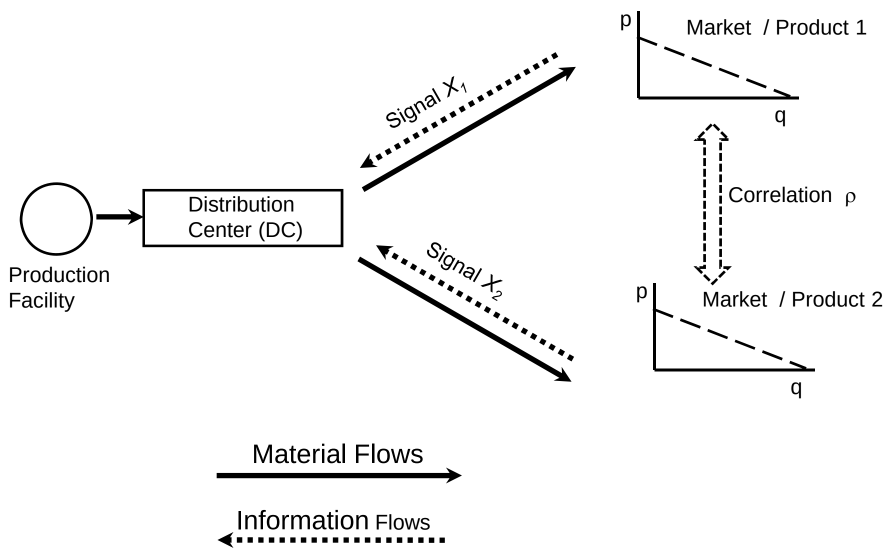

Figure 2: Timeline of Events

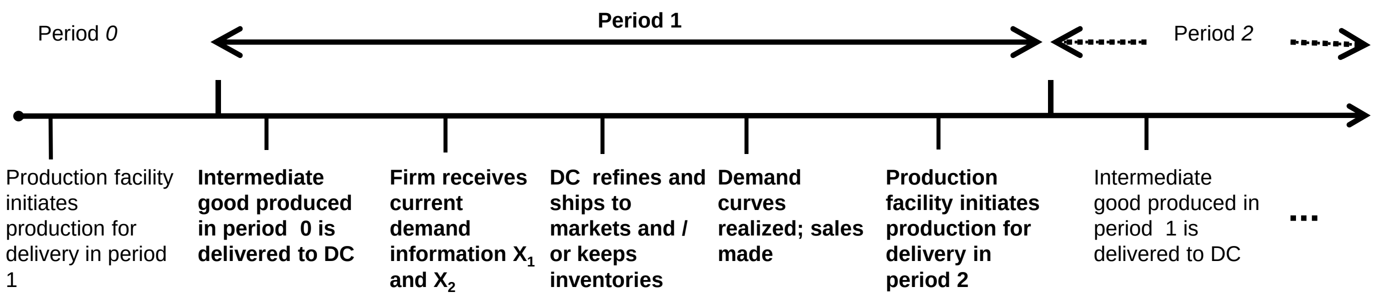

Figure 3: Profits under Early Differentiation

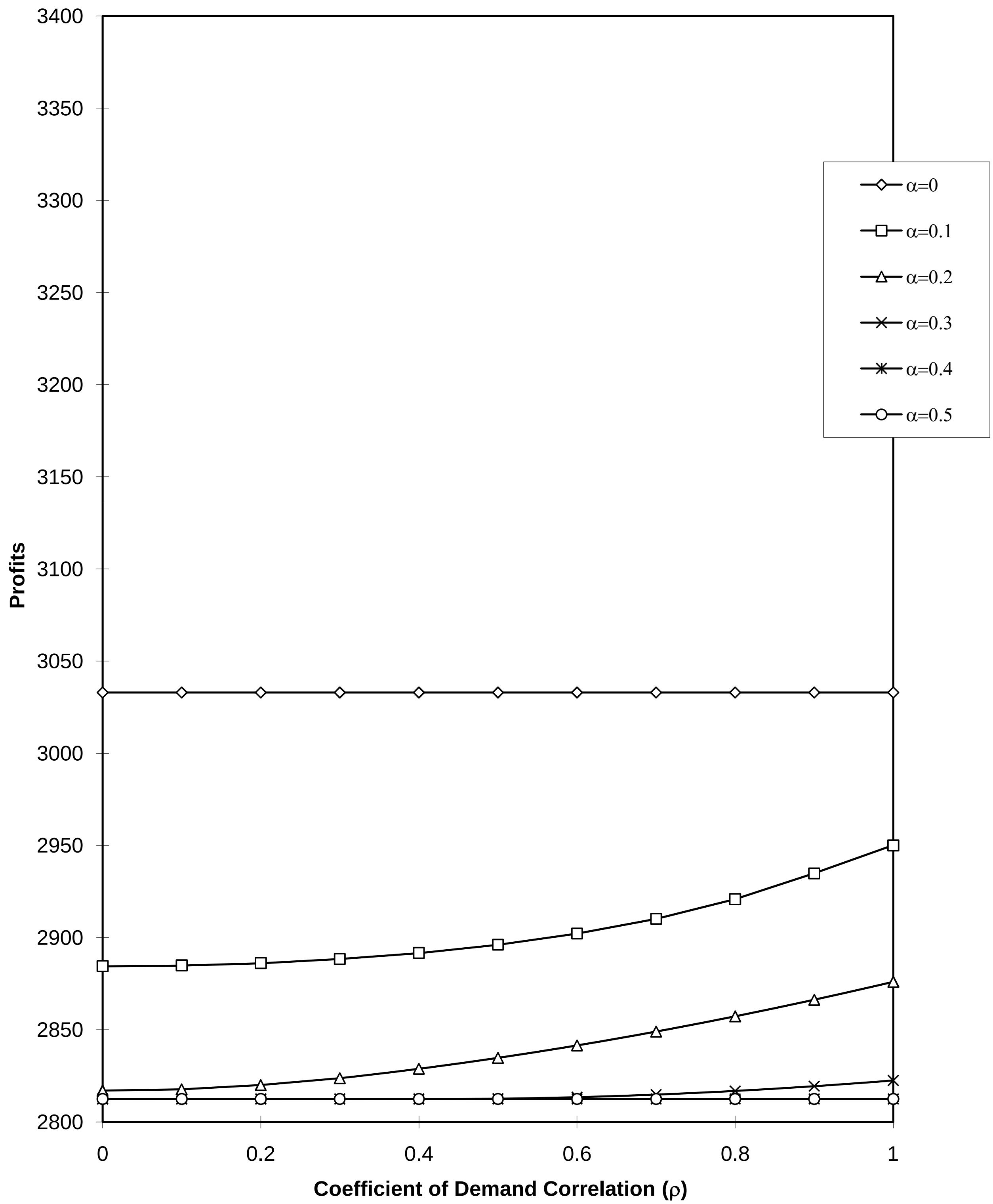

Figure 4: Profits under Delayed Differentiation

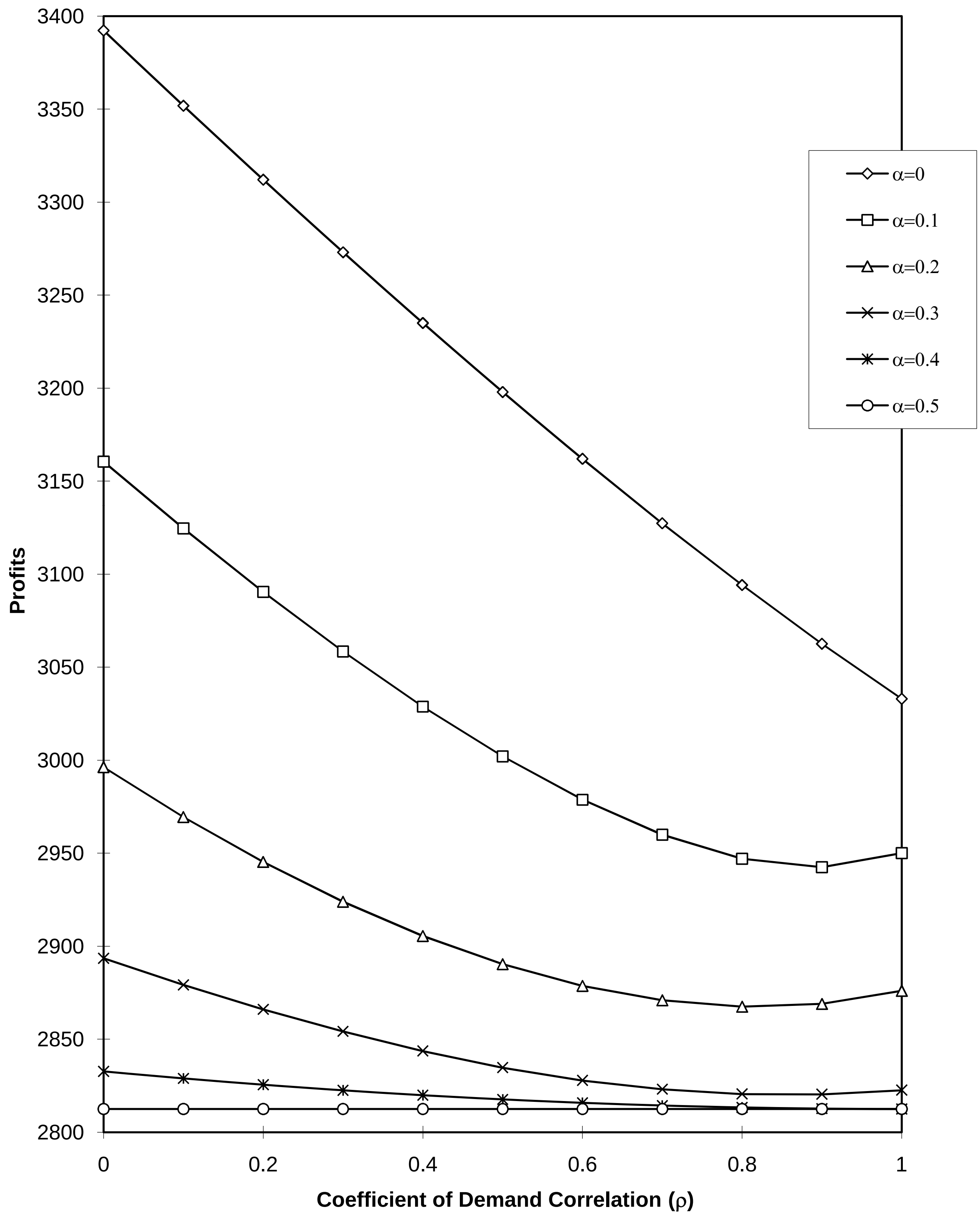

Figure 5: Value of Postponement vs. Demand Correlation

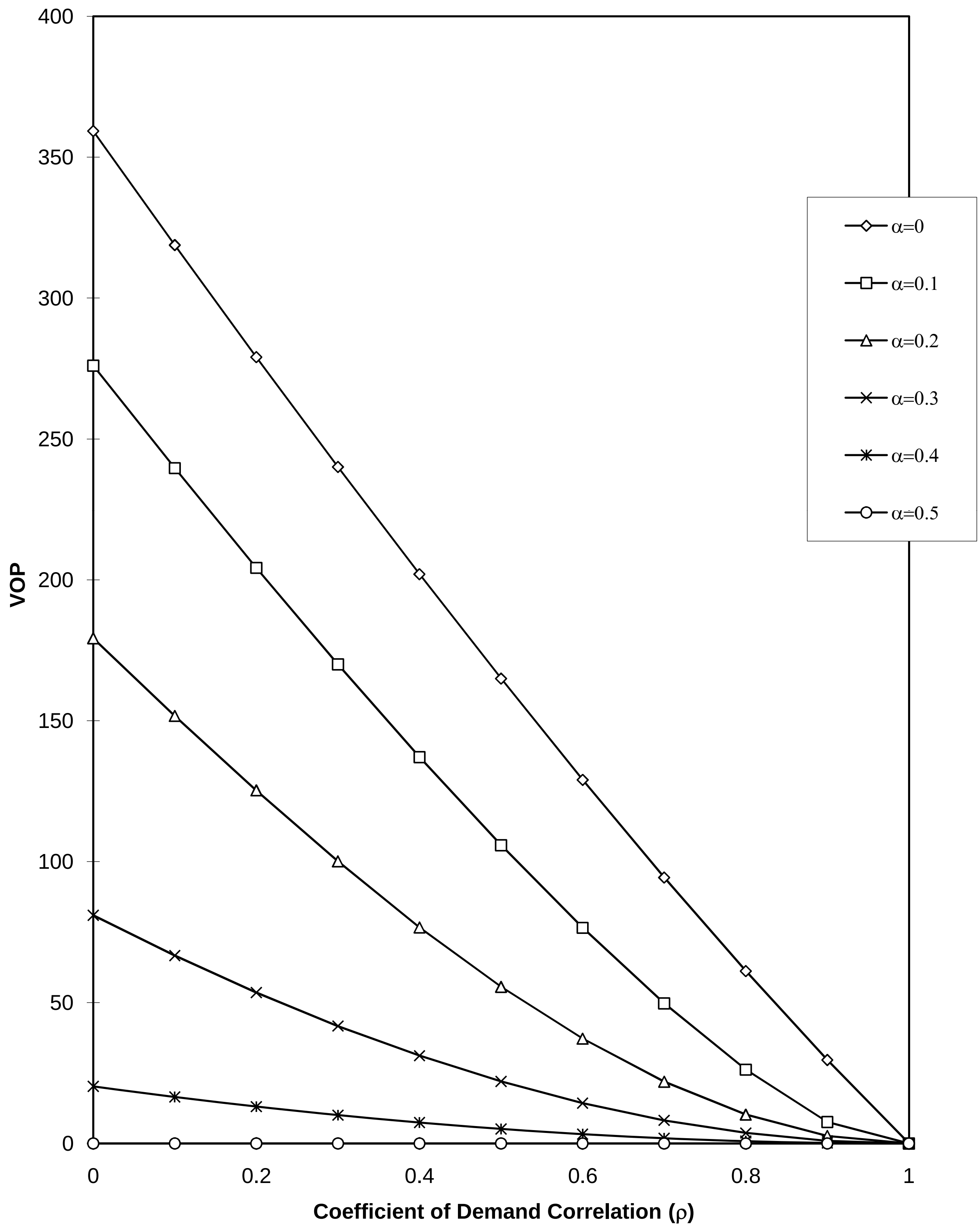

Figure 6: Value of Postponement vs. Demand Variability

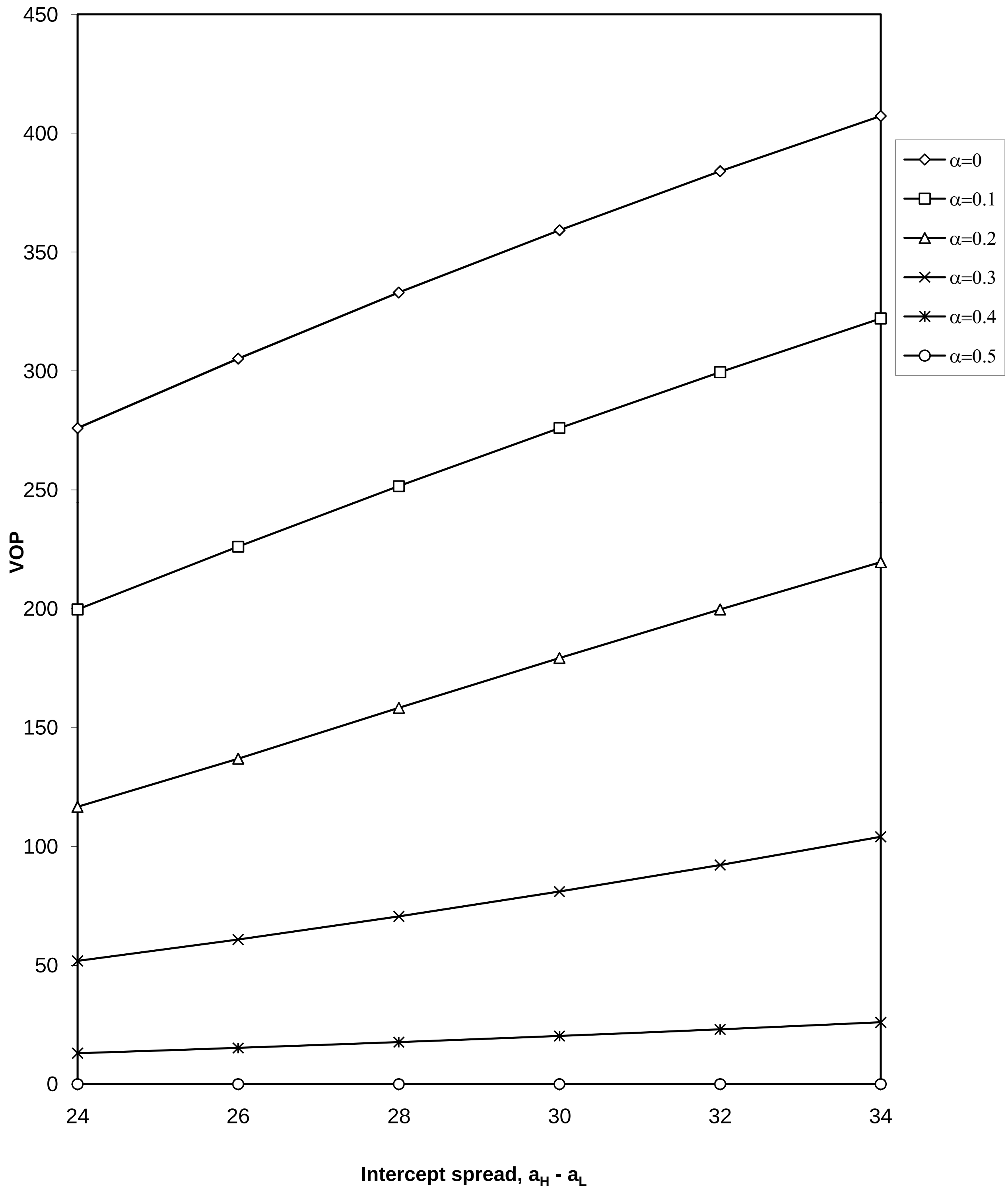

Figure 7: Value of Postponement vs. Demand Priors

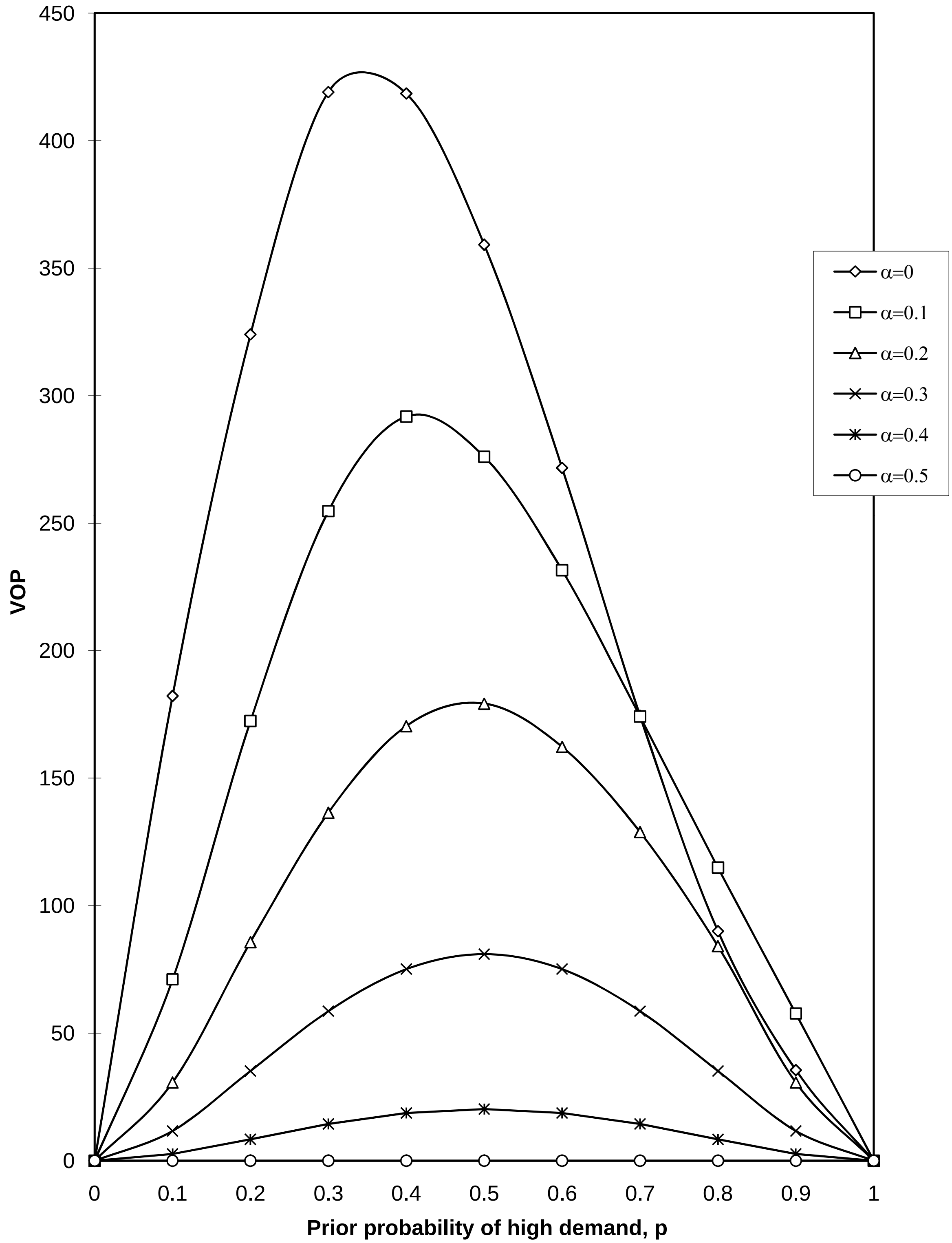

Figure 8: Value of Postponement vs. Holding Costs

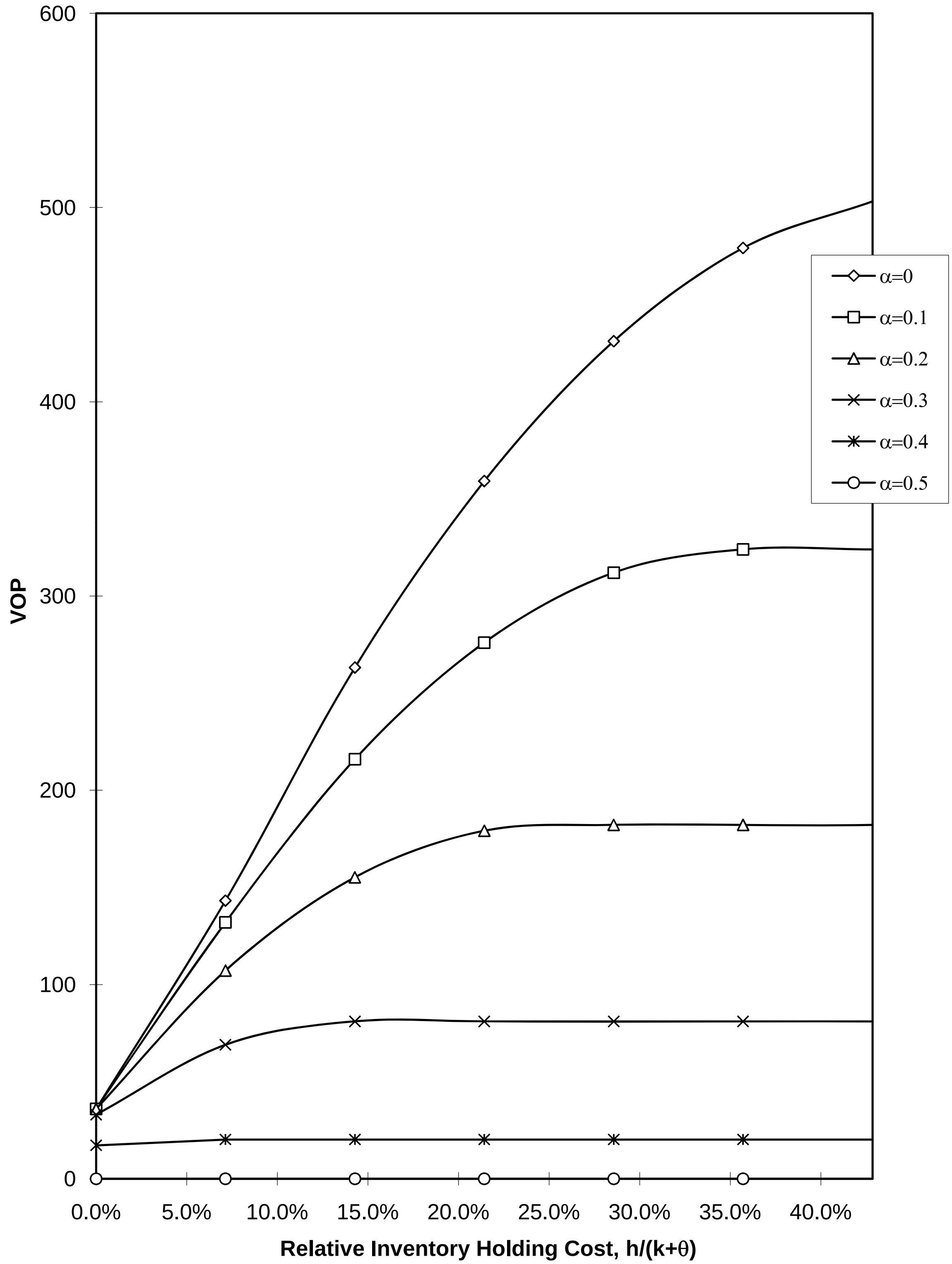

Figure 9: Cost-Benefit Tradeoff of Postponement