Overbooking with Endogenous Demand

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Abstract

Using airlines as a backdrop, we study optimal overbooking policies with endogenous customer demand, when customers internalize their expected cost of being bumped. We first consider the traditional setting in which compensation for bumped passengers is fixed and booking limits are the airline’s only form of control. We provide sufficient conditions under which demand endogeneity leads to lower overbooking limits in this case. We then consider the broader problem of joint control of ticket price, bumping compensation, and booking limit. We show that price and bumping compensation can act as substitutes, which reduces the general problem to a more tractable one-dimensional search for optimal overbooking compensation and effectively allows the value of flying to be decoupled from the cost of being bumped. Finally, we extend our analysis to the case of auction-based compensation schemes and demonstrate that these generally outperform fixed compensation schemes. Numerical experiments that gauge magnitudes suggest that fixed-compensation policies that account for demand endogeneity can significantly outperform those that do not and that auction-based policies bring smaller but still significant additional gains.

1 Introduction

Overbooking is the practice of selling more capacity than is available. It is commonly used by service companies whose capacity is perishable and whose customers sometimes fail to show up for service. Without overbooking, these no-show customers leave unutilized capacity that might have been sold to others. With overbooking, companies can serve more customers and increase revenues.

The practice is attractive to companies, and it is widely used in the air travel, hotel, and car rental industries. In the air travel industry – the focus of our analysis – its origins can be traced back to

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the 1940s, when it was discovered that overselling a flight, even by mistake, could be an effective way to address no-shows and be a viable money-making strategy (Mihm 2017). The benefits for airlines can be quite significant. Smith et al. (1992), for example, mention no-show rates of 15% for sold-out flights (absent overbooking), while Curry (1990) reports that overbooking can generate an additional 3-10% of gross passenger revenues for airlines.

At the same time, overbooking has a downside. When the number of no-shows is smaller than expected, the company doing the overbooking must refuse service to – or bump – some customers, and both the customers and the provider incur costs. Customers suffer the disutility of extra time spent waiting in the airport, a missed connection, a late arrival at the destination, the need to be “walked” to another hotel. The company that does the bumping typically needs to use substitute capacity – either its own or that of a competitor – to fulfill its service obligation, and it also often provides a voucher or some form of monetary compensation to those who are bumped.

In addition to the direct costs associated with the bumping of specific passengers, there also exist indirect costs associated with the practice: the prospect of this type of service failure reduces the value that customers expect to obtain from buying a ticket or making a reservation and can therefore adversely affect customer demand. What’s more, this indirect cost is especially relevant in today’s world in which customers can more freely obtain access to relevant information. The US Bureau of Transportation Statistics (BTS), for example, publishes the total numbers of boarded and bumped passengers for each major US-based airline on a monthly basis (BTS 2018). The resulting bumping probabilities – the summary measures of quality of service (QoS) – are, in turn, reported by both mainstream media outlets and by specialty sites devoted to air travel, such as thepointsguy.com (2018) and travelersunited.org (2018). For savvy travelers, specialized search engines, such as the KVS Availability Tool, let customers check availability and infer bumping probabilities for individual flight routes (kvstool.com 2018).

The now infamous 2017 case of a passenger being dragged off of a United Airlines flight has turned a spotlight on the practice of bumping and its costs. This incident drew outrage on social media, attracting the attention of more than 550 million users on the micro-blogging site Weibo (Hernandez and Li 2017), and had widespread consequences, prompting United and other airlines to rethink how to better manage this element of QoS. In response, United has increased the maximum compensation it offers to bumped passengers, now a $10,000 voucher, and it has developed an auction-based system to identify which passengers to bump, an analogue of a system that Delta Airlines has used for a number of years (Martin 2017, Zhang 2017). Both the increase in compensation and the change of
payment mechanism reflect the importance of secondary demand effects associated with overbooking.

Existing research on overbooking does not address demand effects associated with bumping in a realistic operational setting. While the focus of the traditional overbooking literature has been on deriving optimal overbooking policies in increasingly realistic (and complex) operational environments (Chatwin 1996, Karaesmen and van Ryzin 2004, Kunnumkal et al. 2012), we are not aware of any papers that account for demand endogeneity. This is all the more surprising given empirical evidence of airline customers’ strategic buying behavior (Li et al. 2014). Conversely, there have been several recent papers in the economics literature, such as Fu et al. (2012), Ely et al. (2017), and Sano (2017), that explicitly model demand effects associated with overbooking. These papers take a mechanism-design approach at the expense of using highly stylized models that render booking limits superfluous, however. In contrast, booking-limit control is broadly useful in our more detailed operational setting. Recent papers in the revenue management (RM) and operations management (OM) literature, such as Gallego et al. (2008), Gallego and Şahin (2010), Alexandrov and Lariviere (2012), and Cachon and Feldman (2018), consider related problems in advanced selling, though they only tangentially address traditional questions related to overbooking and do not seek to examine the differences between fixed and auction-based overbooking policies. We provide a more detailed review of the relevant literature in §2.

We consider the demand effect in the context of a model that captures important operational and customer details. Our approach follows the spirit of Dana and Petruzzi’s (Dana and Petruzzi 2001) analysis of inventory problems with QoS-sensitive demand. In our case, however, the inventory level is a fixed number of seats on an aircraft, and the control variables include the booking limit, the price, and the compensation for bumped customers. To squarely focus on customer response to overbooking, we consider a model in which customers are homogeneous in their valuation of the flight itself and heterogenous in their disutilities of being bumped. Section 3 defines our operational and customer model, describes the equilibria that emerge from bumping-sensitive demand, and defines the airline’s optimization problem. We then analyze and compare two distinct compensation schemes for bumped customers.

The first scheme is consistent with that traditionally found in the OM literature and assumes that all bumped customers receive the same compensation. Section 4.2 analyzes a setup in which ticket price and bumping compensation are fixed a priori, and the booking limit represents the airline’s only available control. We call these booking-limit policies and begin by characterizing the optimal booking limit and customer response in a traditional analysis, in which the airline ignores potential
demand effects of bumping. We then compare the results for this “myopic” policy to those for a setting in which the airline recognizes the demand effect that stems from bumping and calibrates its booking limit accordingly. We characterize conditions that are sufficient to imply that demand-dependent booking limits are no larger than those of traditional ones that ignore bumping-sensitive demand. These conditions are complex and depend on customers’ expected utility of purchasing a ticket without bumping, together with the form of the distribution of their disutility of being bumped.

Section 4.3 expands the analysis of the fixed-compensation scheme by considering a broader set of controls in which the airline sets the ticket price and bumping compensation along with the booking limit. We call these overbooking policies. Here, we demonstrate that, in fact, the ticket price and bumping compensation act as substitutes and that, for any given booking limit, there exists an infinite set of price-compensation pairs that obtain the same customer equilibrium and expected airline profit. These results have two important implications. First, when solving the broader overbooking problem, the airline need only consider a price that allows it to decouple the expected value customers obtain for the flight from the compensation they must be offered in the event of being bumped. In this case, the bumping compensation acts as a direct, rather than an indirect, control of customer demand. Second, for any given level of bumping compensation, the optimal booking limit is an analogue to that of the myopic booking-limit policy of §4.2 and can be found via closed-form expression. In turn, the optimal price, compensation, and booking limit can be found as via a simple line search over potential compensation levels.

In Section 5 we introduce an auction scheme for compensating bumped passengers and compare its performance to that of the optimal fixed-compensation schemes identified in §4.3. The auction can be viewed as a multi-unit auction with single-unit demand, a setting for which a so-called uniform-price scheme induces customers to truthfully reveal their preferences and is efficient, allocating seats to customers with the highest disutility of being bumped. For this scheme, the optimal price is, again, one that allows the airline to decouple the expected value customers obtain from the flight from their potential compensation. An upper-bound “cap” on auction payouts – such as the $10,000 limit publicized by United – provides the airline with a similar, direct control over the distribution of demand and allows us to compare straightforwardly the performance of the auction and the fixed-compensation schemes. We further show that, for any fixed-compensation scheme identified in §4, there is an analogous capped auction that performs at least as well. Finally, we identify conditions under which expected airline profits are increasing in the cap, so that the optimal auction-based overbooking policy has only a single active control, the booking limit.
In Section 6, we report the results of numerical experiments that assess the magnitude of the demand effect. We find that fixed-compensation policies that account for demand endogeneity can significantly outperform those that do not and that the use of auction-based policies brings smaller but significant additional gains. These numerical results suggest that the demand effect can have a first-order impact on both overbooking policies and expected revenues.

2 Literature Review

Our paper is related to the RM and economics literatures on overbooking, as well as to the OM literature on strategic consumers. We discuss each in turn.

The vast majority of the RM literature on overbooking does not focus on the issue of demand endogeneity. For overviews, see Chapter 4 in Talluri and van Ryzin (2004) and Chapter 5.2 in Belobaba et al. (2015). Here we describe only a few of the relevant papers, many of which themselves include useful references. Among the earliest works is a static single-fare-class model developed by Beckmann (1958) in which an airline minimizes lost revenue by reducing unused capacity or cost of overselling. Rothstein (1971) focuses on the dynamic aspect of the overbooking problem, deriving a policy that depends on time to flight and current reservations. Chatwin (1996), Karaesmen and van Ryzin (2004), Kunnumkal et al. (2012), and Lan et al. (2015) consider overbooking with multiple customer classes. Klophaus and Pölt (2010) study dynamic booking policies when customer willingness to pay can change over time. In contrast to these papers, our focus is on understanding how the airline’s overbooking policy can affect consumer demand \textit{ex ante}.

At the same time, empirical evidence of strategic consumer behavior has been well documented in the specific context of the airline industry. von Wangenheim and Bayón (2007, p. 36) document that “customers who experience negative consequences of revenue management significantly reduce the amount of their transactions with the airline.” Li et al. (2014) use a structural model to estimate the fraction of consumers in the air-travel industry who delay a purchase, anticipating lower future prices, and show that a non-trivial portion of customers engages in this strategic behavior.

More broadly, strategic consumer behavior has been widely studied in operations management, the most relevant stream of work focusing on the setting of inventory levels (Dana and Petruzzi 2001, Su and Zhang 2008, Cachon et al. 2018). Closest in spirit to our work, are Dana and Petruzzi (2001), who analyze the impact of demand endogeneity in a newsvendor-type model and show that a firm that recognizes strategic consumer behavior implements a higher inventory level because greater availability increases one’s willingness to pay. Also closely related is Alexandrov and Lariviere (2012), who allow
for overbooking when modeling strategic responses to restaurants’ reservation policies. Here, however, no-show behavior is a fluid function of the booking limit, and customers never need to be bumped.

Although the link between firm inventory levels and demand has been extensively studied in these contexts, we are not aware of analogous work that focuses specifically on the overbooking problem. Furthermore, we make relatively weak assumptions regarding consumers’ knowledge. Unlike most of the previously cited work, our model does not require the inventory level to be observable; that is, customers need not directly observe either the plane’s capacity or the airline’s booking limit. Customers in our model similarly need not be informed about the distributions of aggregate demand or bumping disutility.

The use of auctions to determine bumping compensation has become increasingly popular in practice, and a few recent papers in the economics literature use highly stylized models to analyze this mechanism. Fu et al. (2012) consider overbooking controls that exclude booking limits and rely only on price and bumping compensation. Similarly, Ely et al. (2017) consider initial price and refund policies for airlines when passengers are uncertain of their eventual willingness to pay at the time of ticketing, and Sano (2017) extends the analysis to multi-unit demand. These papers, like this one, argue in favor of auction-based compensation schemes. Unlike this paper, however, they do not do not capture essential operational details that affect booking limit controls and do not seek to characterize the effect and magnitude of demand endogeneity on the optimal overbooking policy.

3 Overbooking with Fixed Bumping Compensation

In this section we formally define the overbooking problem for the case in which the compensation paid to bumped customers is fixed. At the start of §5, we provide the details of an analogous model for an auction-based compensation scheme, and we note there the differences between that setup and the model we define here.

We first define our model’s primitives and the associated customer equilibrium, and we highlight important informational assumptions that we make. We then introduce the airline’s expected profit maximization problem and define relevant ranges for its policy parameters.

3.1 Model Primitives

We consider a monopolist airline that offers a flight with a single fare class and \( k \geq 1 \) available seats. The airline sets the ticket price, \( p \), a booking limit, \( b \), that acts as an upper bound on the number of tickets it will sell, and a compensation amount, \( c \), paid to each bumped customer. If more than \( k \)}
paying customers show up for the flight, the airline randomly selects a subset to bump and re-book on a subsequent flight. The expected cost of re-booking a customer on an alternative flight, \( r \), is exogenously defined. Hence, under the fixed compensation scheme, the total cost of bumping each customer is \( (c + r) \). We call the triple \((p, b, c)\) the airline’s overbooking policy.

Potential demand for a flight is uncertain. For example, the number of people who consider traveling from the flight’s origin to its destination on a particular date can be random, and we represent potential demand as a random variable, \( Q \). We denote the cumulative distribution function (CDF) of demand as \( F(q) \), with support \( 0 \leq q \leq Q \leq \infty \) and \( \int_0^Q dF(q) < \infty \). We analyze overbooking policies that use fixed compensation by differentiating relevant expressions, and for analytical convenience we therefore model \( Q \) as continuous, with density \( f(q) > 0 \) over its support. Thus, individual customers are infinitesimal.

Customers have three critical attributes: the value they derive from flying, their no-show probabilities, and the disutility they incur from being bumped. We assume that they are homogeneous along the first two dimensions. They share a common value from flying, \( v \), either on the original flight or, if bumped, on their re-booked flight. Customers who purchase tickets also have an identical no-show probability, \( \alpha \in (0, 1) \). The focus of our interest is the third dimension, customers’ disutilities of being bumped, which we call their hassle costs. We assume that hassle costs are heterogeneous across the population, and by sorting them from smallest to largest, we can model them as the cumulative distribution, \( G(w) \), of a random variable, \( W \), with support \( 0 \leq w \leq W \leq \infty \). Again, for analytical convenience we assume \( G(w) \) is continuous, with density \( g(w) > 0 \) over its support.

### 3.2 Model Equilibrium

Taken together, a plane with capacity \( k \), an airline overbooking policy, \((p, b, c)\), potential demand, \( Q \), and a set of customer attributes, \((v, \alpha, W)\), induce an equilibrium outcome. Individual customers decide whether or not to buy tickets, depending on the expected value of the purchase. In turn, each ticket holder shows up for the flight with probability \( 1 - \alpha \), and if the number of customers who do show is greater than the plane’s capacity, excess customers are bumped. In equilibrium there is a set of customers who decide to purchase, a complementary set who do not, and a corresponding probability that a customer who shows up for the flight is bumped.

To formally describe the equilibrium, we begin with the customer purchase decision. Suppose customers share a common belief regarding the endogenous probability of being bumped in equilibrium, and denote this probability by \( \beta \in [0, 1) \). Then the expected value obtained by a customer with hassle
cost $w$, drawn from $W$, who buys a ticket is

$$U(\beta, w) = -p + (1 - \alpha)v + (1 - \alpha)\beta(c - w).$$  \hspace{1cm} (1)$$

Here, the first term to the right of the equality is the ticket’s purchase price, and the second is the expected value of flying, given the no-show probability $\alpha$. The last term represents the expected value that the customer obtains from the possibility of being bumped: with probability $(1 - \alpha)\beta$ she shows up for the flight and is bumped; in turn, the value she obtains from being bumped is the compensation less her hassle cost, $c - w$. When $c > w$ the value of being bumped is a net reward, and when $c < w$ it is a net cost.

We assume that customers have an outside option whose value we normalized to zero. Given an equilibrium bumping probability, $\beta$, a passenger with hassle cost $w$ who considers purchasing a ticket will buy one if and only if the expected value of the purchase is non-negative: $U(\beta, w) \geq 0$.

An equilibrium $\beta$ then induces an equilibrium customer demand response via $U(\beta, w)$. For $\beta = 0$, which can happen for instance when $b \leq k$ and there is no bumping, $U(\beta, w) \geq 0$ if and only if $p \leq (1 - \alpha)v$. That is, with no bumping any customer will buy a ticket if and only if the price does not exceed the expected value of flying. For $\beta > 0$, $U(\beta, w)$ is strictly decreasing in $w$. In all cases, we can define the equilibrium threshold hassle cost, $\hat{w}$, at or below which a customer buys a ticket, and above which she does not, as follows:

$$\hat{w} = \begin{cases} 
  w, & \text{if } U(\beta, w) \leq 0; \\
  \{w \mid U(\beta, w) = 0\}, & \text{if } w < w < \overline{w}; \text{ and} \\
  \overline{w}, & \text{if } U(\beta, \overline{w}) \geq 0. 
\end{cases}$$  \hspace{1cm} (2)$$
Throughout the paper, we also refer to $\hat{w}$ using interchangeably the terms marginal customer’s hassle cost, customers’ equilibrium response, and customers’ response.

From (1) and (2) we see that $G(\hat{w})$ represents the fraction of potential customers who obtain non-negative value from purchasing a ticket. Thus, when $\hat{w} = w$, $G(\hat{w}) = 0$ and no one is willing to buy a ticket, and when $\hat{w} = \overline{w}$, $G(\hat{w}) = 1$ and everyone is willing to buy. We are often most interested in the interior case, in which $\hat{w} \in (w, \overline{w})$ so that $U(\beta, \hat{w}) = 0$ and $G(\hat{w}) \in (0, 1)$, but in some parts of our analysis the boundary cases, $U(\beta, w) \leq 0$ and $U(\beta, \overline{w}) \geq 0$, can also be important.

Having defined the manner in which an equilibrium $\beta$ induces an equilibrium $\hat{w}$, we turn to the mechanism by which an equilibrium $\hat{w}$ induces an equilibrium $\beta$. To this end, we first characterize the number of tickets sold.

Recall that the potential demand for a flight is a random variable, $Q$. As in Dana and Petruzzi (2001), we assume that, for any demand realization, $q$, the distribution of customers’ hassle costs

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follows $G(w)$. That is, we assume that $W$ is effectively independent of $Q$. Potential customers with hassle costs $w \leq \hat{w}$ then buy tickets, and those with hassle costs $w > \hat{w}$ do not, effectively thinning the potential demand. Given the booking limit $b$ and the potential demand realization $q$, we can define the number of tickets sold as $s = \min\{b, qG(\hat{w})\}$, where the term $qG(\hat{w})$ represents the “thinned” demand for the flight. In turn, we define the random variable

$$S = \min\{b, QG(\hat{w})\}$$  \hspace{1cm} (3)

as the equilibrium number of tickets sold.

From here, we can derive the bumping probability, $\beta$, in three steps. First, the sale of $s$ tickets results in a smaller number of customers who show up for the flight. We denote that random number, $N(s, \alpha) \in [0, s]$, as function of $s$ and $\alpha$, and for the moment, we leave the explicit dependence on $s$ and $\alpha$ undefined. In turn,

$$N \equiv N(S, \alpha)$$  \hspace{1cm} (4)

is the equilibrium number of customers who show up for the flight. Second, not all of those who show are bumped, and we let

$$(N - k)^+ \equiv (N(S, \alpha) - k)^+ = \max\{0, N(S, \alpha) - k\}$$  \hspace{1cm} (5)

denote the equilibrium number of customers who are bumped. Finally, we can use the numbers of shows and of bumped customers to calculate the bumping probability as the ratio of expectations

$$\beta = \frac{E[(N - k)^+]}{E[N]}.$$  \hspace{1cm} (6)

Thus, from $\hat{w}$ we obtain $\beta$.

Given potential demand, $Q$, and customer attributes, $(v, \alpha, W)$, an overbooking policy $(p, c, b)$ yields an equilibrium if there exist $\beta$ and $\hat{w}$ that simultaneously satisfy (2) and (6). In Section 3.3, we discuss the informational demands that such an equilibrium requires and discuss the use of $\beta$ as a measure of bumping probability. In §3.4 we then define the airline’s optimization problem and detail the parameter range to be considered for $(p, c, b)$.

### 3.3 Information Required to Obtain an Equilibrium

Our equilibrium model requires that, in choosing an overbooking policy $(p, b, c)$, the airline is aware of all relevant demand and customer data: the demand distribution, $F(q)$; the value customers derive from the flight, $v$, the no-show rate, $\alpha$, and the hassle-cost distribution, $G(w)$. The airline must also understand the calculation of $S$ and $N$, along with the equilibrium equations (2) and (6).
The informational requirements for customers are lower. In using (1) to decide whether or not to buy a ticket, each potential customer need know only her own attributes \((v, \alpha, w)\), the airline’s price and bumping compensation, \((p, c)\), and the bumping probability \(\beta\). Interestingly, we show in §4 and §5 that the problem can be reduced to a form that does not require the customer know or estimate \(\beta\). Furthermore, the customer need not know the distribution of demand, \(F(q)\), the hassle cost distribution, \(G(w)\), the flight’s capacity, \(k\), or the airline’s booking limit, \(b\), and she need not be able to calculate (6).

Rather, customers can obtain an estimate of (6) from statistics published by the US Bureau of Transportation Statistics (BTS 2018) and reported by many media outlets. More formally, suppose the airline runs a sequence of independent and identically distributed (i.i.d.) flights \(i \in \{1, \ldots, m\}\) with fixed overbooking policy \((p, b, c)\). Let \(N_i\) denote the random number of customers who show up for flight \(i\) and \((N_i - k)^+\) denote the random number of customers who are bumped that flight. Then, after \(m\) flights, the reported fraction of passengers who are bumped, \(\hat{\beta}\), is

\[
\hat{\beta} = \frac{\sum_{i=1}^{m} (N_i - k)^+}{\sum_{i=1}^{m} N_i} = \frac{1}{m} \sum_{i=1}^{m} \frac{(N_i - k)^+}{N_i} \xrightarrow{m \to \infty} \frac{E[(N - k)^+]}{E[N]} = \beta,
\]

by the law of large numbers. As the introduction notes, BTS reports the building blocks of \(\hat{\beta}\), \(\sum_{i=1}^{m} N_i\) and \(\sum_{i=1}^{m} (N_i - k)^+\), for each airline as a whole on a quarterly and annual basis, and while BTS does not report analogous data for specific routes, specialty sites can provide additional flight-specific data. This definition of \(\beta\) in is the \textit{ex post} fraction of passengers who are bumped, an analogue of the fill rate in inventory theory.

An alternative that could be considered is the \textit{ex ante} probability of bumping, \(\beta' = E[(N - k)^+]/N\), which is the expectation of the ratio of customers bumped, rather than the ratio of expectations and would be estimated as

\[
\hat{\beta}' = \frac{1}{m} \sum_{i=1}^{m} \frac{(N_i - k)^+}{N_i} \xrightarrow{m \to \infty} \frac{E[(N - k)^+]}{E[N]} = \beta'.
\]

Note that \(\beta\) and \(\beta'\) need not be the same. We believe that, in our specific context of airline overbooking, \(\beta\) is the measure that is more practically relevant and more realistically accessible to customers.

We briefly describe our reasoning here and provide supporting details and analysis in Appendix A. First, while \(\beta\) can be readily estimated from published \textit{aggregate} data, an analogous estimate of \(\beta'\) would require customers to obtain bumping fractions from individual flights, data that are not published and are not typically disclosed by airlines to their passengers. Second, suppose that, nevertheless, customers wished to estimate \(\beta'\) based their initial estimates of bumping on the publicly

\[1\] We defer the relevant discussion to those sections.
available statistic, $\beta$. Even if individual-flight data were available and customers used them to update their initial beliefs, differences between $\beta$ and $\beta'$ are small enough that a customer would have to take hundreds or (more typically) thousands of flights to distinguish the latter from the former. Third and finally, although differences between the $\beta$ and $\beta'$ are typically small – on the order of $10^{-3}$ or less – it can be shown that $\beta \geq \beta'$, a relationship that suggests that an equilibrium based on an initial estimate of $\beta$ will be stable: customers for whom $\beta$ is too high to fly will never fly and will not collect the data needed to change their initial estimate; conversely those for whom $\beta$ is low enough to fly will continue flying even if, after thousands of flights, their estimates slip from $\beta$ to $\beta'$.

### 3.4 Airline’s Optimization Problem

Having defined the model’s primitives and equilibrium expressions, we can now concisely formulate the airline’s associated optimization problem. Given an overbooking policy $(p, b, c)$ that yields a customer equilibrium, the airline earns revenue $p$ for each ticket sold and pays bumping compensation $c$ and rerouting cost $r$ for each customer bumped, netting profits

$$\Pi(p, b, c) = pS - (c + r)(N - k)^+.$$  

In principal, the airline chooses $p$, $c$, and $b$ to maximize expected profits subject to (1)–(6).

This problem statement is not complete, however, because it does not ensure the existence of a relevant equilibrium for each $(p, b, c)$. Below we define ranges for policy parameters that, in most cases, do induce equilibria, and in §4.1 we explicitly note remaining, boundary cases for which equilibria do not exist.

First, we discuss lower bounds. We assume that both $p$ and $c$ are non-negative, so the airline does not give potential customers money to fly, and it does not charge customers for the pleasure of being bumped. In turn, given $p \geq 0$, we assume that $b \geq k$, since the airline will not benefit by forcing itself to fly with empty seats.

Next we discuss upper bounds. As we noted in §3.2, for $\beta = 0$ all customers are willing to buy a ticket whenever $p \leq (1 - \alpha)v$, and no customer is willing to buy a ticket when $p > (1 - \alpha)v$, since the ticket price exceeds the expected value of flying. For cases in which $\beta > 0$, we similarly limit $p \leq (1 - \alpha)v$ to exclude cases in which customers only buy tickets because of a potential benefit of being bumped. Given $p \leq (1 - \alpha)v$ and $c \leq \overline{w}$, $U(\beta, \overline{w}) \geq 0$ for all $\beta \in [0, 1]$. Therefore, we can also require that $c \leq \overline{w}$ since, there is no need to consider higher levels of bumping compensation.

We call overbooking policies that fall within these bounds, *admissible* and summarize their properties as follows.
Definition 1. (Admissible Overbooking Policies)

Admissible overbooking policies have: (i) \(0 \leq p \leq (1 - \alpha)v\); (ii) \(b \geq k\); and (iii) \(0 \leq c \leq \bar{w}\).

We label the set of admissible policies \(\Xi\), and we call individual admissible policies \(\xi \in \Xi\). In §4.1 we characterize the equilibria of interest for \(\Xi\).

The airline then searches for an admissible overbooking policy that maximizes expected profits:

\[
\max_{\xi \in \Xi} E[\Pi(p, b, c)] \tag{10}
\]

subject to (1), (2), (3), (6).

4 Analysis of Fixed-Compensation Schemes

In this section, we analyze a set of schemes that pay bumped customers a fixed, pre-determined level of compensation, as defined in §3. We begin in §4.1 with a preliminary analysis that characterizes the expected number of bumped customers and the equilibria of interest. We then use this foundation to analyze two sets of overbooking policies of increasing complexity.

In §4.2, we analyze booking-limit policies, that is, single-control policies of the booking limit, \(b\), that assume an exogenously specified price, \(p\), and bumping compensation, \(c\). As a benchmark, we first analyze the traditional myopic policy considered in the RM literature, one that assumes the booking limit does not affect demand, and we develop a simple characterization of the optimal booking limit for this case. We then perform a more delicate analysis of booking-limit control that recognizes the endogeneity of demand, and we develop equilibrium conditions under which optimal demand-dependent booking limits are stricter than those suggested by the benchmark myopic policy.

Equation (1) highlights the fact that demand is in fact affected by all three controls, \((p, b, c)\), and in §4.3 we analyze overbooking policies' joint use of price, bumping compensation, and booking limit to maximize expected profit. The first part of our analysis shows that, in fact, price and bumping compensation act as substitutes and that it is sufficient to consider policies that set price equal to the expected value of flying and then use bumping compensation and booking limit as controls. Furthermore, in this setting the intensity of demand becomes a direct outcome of the bumping compensation and can be decoupled from the booking limit.

4.1 Preliminary Analysis

In this section, we provide two sets of preliminary results that we require to conduct a full analysis of the overbooking problem. First, we define the properties of a simple loss function that models the
conditional expectation of the number of bumped customers, given some realization of the number of tickets sold, and we show that commonly used distributions of numbers of customers who show up for the flight generate a loss function with the desired properties. Second, we characterize the set of equilibria to be considered in our analysis.

4.1.1 Modeling the Expected Number of Bumped Customers

Much of the analysis below requires that we differentiate expressions, such as (6), that include expected numbers of bumped customers, and we sometimes find it analytically convenient to work with the conditional expectation, given a sales realization $s$. To that end, we use the conditional expectation to characterize here both the expected number of customers who show up for a flight and the expected number of bumped customers.

Suppose $s$ tickets are sold. Given each customer has a no-show probability of $\alpha$, we call the indicator function of the event “customer $i$ shows up,” $\mathbb{1}\{i\text{ shows}\}$, and need make no additional assumptions to show that

$$E[N(s, \alpha)] = E\left[\sum_{i=1}^{s} \mathbb{1}\{i\text{ shows}\}\right] = (1 - \alpha)s,$$

so that $E[N] = (1 - \alpha)E[S]$.

If we further assume that individual customers’ no-show behavior is i.i.d., then $N(s, \alpha) \sim \mathcal{B}(s, 1 - \alpha)$, a binomially distributed random variable with probability of success $(1 - \alpha)$ and number of samples $s$. In turn, a normal distribution, $\mathcal{N}(\mu, \sigma)$, with mean $\mu = (1 - \alpha)s$ and standard deviation $\sigma = \sqrt{\alpha(1 - \alpha)s}$ represents a simple continuous approximation to the binomial $\mathcal{B}(s, 1 - \alpha)$.

In the same spirit, we can characterize the expected number of bumped customers by first conditioning on the number of tickets sold, and we let

$$\ell(s, k, \alpha) = E[(N(s, \alpha) - k)^+]$$

define a loss function that is a direct analogue of that used in inventory theory. To ease notational burden, we will sometimes write partial derivatives of this function using a prime symbol and the variable of interest: for example $\ell'(s) \equiv \frac{\partial \ell(s, k, \alpha)}{\partial s}$.

We make minimal assumptions regarding the loss function.

**Definition 2. (Loss Function)**

(i) $\ell''(s) \geq 0$;

(ii) $\ell(s, k, \alpha) = 0$ for all $s \leq k$ and $\ell'(s) = 0$ for all $s < k$;
(iii) \( \ell'(s) \in (0, 1 - \alpha) \) for all \( s \in [k, \infty) \); and
(iv) \( \lim_{s \to \infty} \ell'(s) = 1 - \alpha. \)

We note that the discrete analogues of \( \ell'(s) \geq 0 \) and \( \ell''(s) \geq 0 \) are \( \ell(s) - \ell(s-1) \geq 0 \) and \( \ell(s+1) - \ell(s) \geq \ell(s) - \ell(s-1) \), respectively.

Properties (i) and (ii) are common. Typically loss functions are convex, and loss can never be incurred when sales fall below the plane’s capacity \( k \). The upper limit in properties (iii) and (iv) follow the fact that each ticket sold has only a probability of \((1 - \alpha)\) of turning into a customer who shows up for the flight. Only for very large \( s \) do marginal sales lead to additional shows who will nearly certainly be lost, each with a show probability of \((1 - \alpha)\) for each new ticket sold.

As expected, the definition’s properties are generally satisfied by both the binomial and normal distributions described above.

**Lemma 1.** (Properties of Loss Function Satisfied)

For a plane with \( k \) seats and loss function \( \ell(s, k, \alpha) = (N(s, \alpha) - k)^+ \):

(i) \( N(s, \alpha) \sim B(s, 1 - \alpha) \) satisfies the discrete analogue of properties (i)–(iv) of Definition 2; and

(ii) \( N(s, \alpha) \sim N((1 - \alpha)s, \sqrt{\alpha(1 - \alpha)s}) \) satisfies properties (i), (iii), and (iv) of Definition 2.

The proof of this and all results can be found in Appendix B.

Note that, because of its infinite support below \( k \), the normal approximation does not satisfy the loss function’s property (ii). We emphasize that this does not affect our theoretical results, which only depend on the definition of \( \ell(\cdot) \) and not the specific distributional form of \( N \). We do use the normal approximation to the binomial in our numerical examples, however. When we do, we truncate the normal distribution at \( k \) and renormalize the probabilities over the support above \( k \) to sum to one.

### 4.1.2 Equilibria and Policies of Interest

Here, we characterize the set of equilibria we will consider when analyzing the airline’s problem (10). We also further characterize policies of interest: those that make positive expected profits.

We begin with the equilibria.

**Lemma 2.** (Existence and Uniqueness of Equilibria)

(i) For overbooking policies with \( p = (1 - \alpha)v \), \( b > k \), and \( c \leq w \) there is no equilibrium.

(ii) For all other overbooking policies \( \xi \in \Xi \), there exists at least one equilibrium.

(iii) For the policies in part (ii), if \( g'(w) \leq 0, \forall w \in [c, \bar{w}] \), then \( \exists \) a unique equilibrium \( \{\beta, \bar{w}\} \).
The policies identified in part (i) of the lemma have a price, \( p = (1 - \alpha) v \), that leaves no consumer surplus, and a bumping compensation, \( c \leq w \), that adequately compensates no bumped customer. If there were no overbooking, so that \( b = k \), then \( \beta = 0 \) independently of \( \hat{\omega} \), any \( w \) would obtain \( U(0, w) = 0 \), and \( \hat{\omega} = \bar{w} \) would be consistent with \( \beta = 0 \). With \( b > k \), however, there is the potential for bumping customers, and there is no consistent \( (\beta, \hat{\omega}) \) pair: \( \beta = 0 \) induces \( \hat{\omega} = \bar{w} \), which in turn induces \( \beta > 0 \), which then induces \( \hat{\omega} = w \), and so on.

Part (ii) of the lemma shows that other admissible policies are better behaved and admit at least one equilibrium. While we cannot rule out the existence of multiple equilibria for any hassle-cost distribution, \( G(\cdot) \), part (iii) of the lemma provides a sufficient condition under which there is exactly one, namely when the density of the hassle cost is decreasing above \( c \), a property that is satisfied by uniform and exponential hassle-cost distributions, by normally distributed hassle-cost distributions whenever \( c \geq E[W] \), and more generally by decreasing failure rate (DFR) distributions.

Even if there do exist multiple equilibria associated with a given policy \( \xi \in \Xi \), the following lemma shows that they are well ordered and suggests that we have good reason to focus on the unique equilibrium that maximizes the airline’s expected profits.

**Lemma 3.** (Ordering of Equilibria)

Suppose an overbooking policy \( \xi \in \Xi \) induces multiple equilibria. Pick any two distinct equilibria from the set, and call them \( (\beta_1, \hat{\omega}_1) \neq (\beta_2, \hat{\omega}_2) \).

(i) Without loss of generality, we can order the two so that the second equilibrium has a strictly lower bumping probability and a strictly higher marginal hassle cost: \( \beta_1 > \beta_2 \) and \( \hat{\omega}_1 < \hat{\omega}_2 \).

(ii) Given the ordering in (i), the set of customers with \( w \leq \hat{\omega}_1 \) is a strict subset of those with \( w \leq \hat{\omega}_2 \), and the airline earns strictly higher expected profits in \( (\beta_2, \hat{\omega}_2) \).

Part (i) implies that we can order the equilibria from smallest to largest \( \hat{\omega} \) and that the largest of these is unique. Part (ii) further implies that the largest equilibrium maximizes the number of customers who buy tickets and obtain positive expected value, \( U(\beta, w) \). This largest equilibrium also maximizes the airline’s expected profits.

At the same time, a larger \( \hat{\omega} \) is not a Pareto improvement over a smaller one. In particular, customers with \( w < c \) – those who enjoy a net benefit from being bumped – see their \( U(\beta, w) \)’s decrease as the equilibrium bumping probability falls from \( \beta_1 \) to \( \beta_2 \). Nevertheless, when the equilibrium is \( (\beta_2, \hat{\omega}_2) \), even those customers obtain a positive expected value from purchasing tickets and remain in the market.

Thus, an airline whose overbooking policy can induce multiple equilibria has an interest in inducing
the largest of them, and if customers can be convinced that the low bumping probability that’s associated with the highest \( \hat{w} \) is the equilibrium of interest, they will willingly settle on the largest equilibrium as well. More importantly, in §4.3 we show that, even if an admissible policy induces multiple equilibria, we can also find an alternative that induces only the largest of them and earns the same expected profit. We will therefore assume that, if there are multiple possible equilibria, the largest of these is obtained.

Finally, when optimizing over policies \( \xi \in \Xi \), we will sometimes simplify our analysis by excluding equilibria that make no profit. To characterize these we recall from (11) that \( \mathbb{E}[N] = (1 - \alpha)\mathbb{E}[S] \) and from (6) that \( \mathbb{E}[(N - k)^+] = \beta\mathbb{E}[N] = (1 - \alpha)\beta\mathbb{E}[S] \). Therefore, we can substitute \( (1 - \alpha)\beta\mathbb{E}[S] \) for \( \mathbb{E}[(N - k)^+] \) in (10) and rearrange terms to show that the airline’s expected profits are

\[
\mathbb{E}[\Pi(p, b, c)] = [p - (1 - \alpha)\beta(c + r)] \mathbb{E}[S],
\]

where \( [p - (1 - \alpha)\beta(c + r)] \) is the expected margin per customer, and \( \mathbb{E}[S] \) is the expected number of units sold. We can then define a profit-making equilibrium as follows.

**Definition 3. (Profit-Making Equilibrium)**

Any profit-making equilibrium has \( [p - (1 - \alpha)\beta(c + r)] \mathbb{E}[S] > 0 \).

Note that \( G(w) = 0 \) implies that \( \mathbb{E}[S] = \mathbb{E}[\min\{b, QG(\hat{w})\}] = 0 \), so \( \hat{w} = w \) is never profit-making.

### 4.2 Optimal Booking Limits

In this section, we assume that the price, \( p \), and bumping compensation, \( c \), are exogenously defined and that the booking limit, \( b \), is the only form of control. The airline’s *booking-limit problem* is then

\[
\max_{b \geq k} \mathbb{E}\left[\Pi(p, b, c)\right]
\]

subject to (1), (2), (3), (6).

We first consider a benchmark case in §4.2.1, in which the airline myopically overlooks the effect of the booking limit on demand and treats \( \hat{w} \) as an exogenous factor. This is analogous to the traditional approach taken in the RM literature on overbooking. Subsequently in §4.2.2, we analyze the demand-dependent case in which the airline strategically takes customer behavior into account.

#### 4.2.1 Myopic Booking Limits

Suppose the airline does not recognize that the marginal customer’s hassle cost, \( \hat{w} \), is an equilibrium reaction to \( \beta \) and, in turn, \( b \). Rather it assumes that \( \hat{w} \) is fixed.
In this case, we can derive the optimal myopic booking limit, $b_m^*$, as the solution to the first order condition (FOC) with respect to $b$. That is,

$$\frac{dE[\Pi]}{db} \bigg|_{\text{myopic}} = \frac{\partial E[\Pi]}{\partial b} = 0. \tag{15}$$

On the revenue side, a marginal increase in $b$ yields the following change in unit sales:

$$\frac{\partial E[S]}{\partial b} = \frac{\partial E[\min\{b, QG(\hat{w})\}]}{\partial b} = P\{QG(\hat{w}) > b\}. \tag{16}$$

On the cost side,

$$E[(N-k)^+] = E[\ell(\min\{b, QG(\hat{w})\})] = \int_0^{\frac{b}{\ell'(\hat{w})}} \ell(qG(\hat{w})) f(q) dq + \int_{\frac{b}{\ell'(\hat{w})}}^{\infty} \ell(b) f(q) dq, \tag{17}$$

and differentiating (17) with respect to $b$ we have

$$\frac{\partial E[(N-k)^+]}{\partial b} = \frac{\partial E[\ell(\min\{b, QG(\hat{w})\})]}{\partial b} = \ell'(b) P\{QG(\hat{w}) > b\}. \tag{18}$$

We then use the results of (16) and (18) to find the FOC for (14), assuming expected profit is only affected through the booking limit, $b$, without an effect on $\hat{w}$. In particular, we see that $\frac{\partial E[\Pi]}{\partial b} = 0$ implies that

$$p P\{QG(\hat{w}) > b\} - (c + r) \ell'(b) P\{QG(\hat{w}) > b\} = 0 \Rightarrow p - \ell'(b) (c + r) = 0. \tag{19}$$

Note that, in this case, the FOC (19) defines the optimal booking limit by trading off the revenue, $p$, gained from the marginal customer, should she purchase a ticket, against the marginal expected bumping cost, $(c + r)$, induced by that marginal customer showing up. This is, in fact, the assumption that’s made in standard overbooking models, and it is completely independent of the demand distribution and the customer response $\hat{w}$.

**Proposition 1.** (Optimal Myopic Booking Limit)

*Given fixed, admissible $p$ and $c$, the optimal myopic booking limit, $b_m^*$, behaves as follows.*

(i) If $p - (1 - \alpha) (c + r) \geq 0$, then $b_m^* = \infty$, and the airline does not impose a booking limit.

(ii) If $p - (1 - \alpha) (c + r) < 0$, then there exists a unique optimal $b_m^* = \max \left\{ \ell'^{-1} \left( \frac{p}{c+r} \right), k \right\}$.

(iii) When $b_m^* \in (k, \infty)$, $\frac{\partial E[\Pi]}{\partial b} > 0$ for $b < b_m^*$ and $\frac{\partial E[\Pi]}{\partial b} < 0$ for $b > b_m^*$.

The results follow directly from the FOC (19). When $p - (1 - \alpha) (c + r) > 0$, the expected margin per customer continues to be positive, even when $\beta = 1$. In this case the airline is happy to bump customers, and the optimal booking limit is infinite. In contrast, when $p - (1 - \alpha) (c + r) < 0$, the
expected margin per customer \( p - (1 - \alpha) \beta (c + r) \) eventually becomes negative, as \( \beta \to 1 \), and the optimal myopic booking limit is finite. Because \( \ell(b) \) is increasing convex, the sign of \( \frac{\partial E[\Pi]}{\partial b} \) changes from positive to negative as \( b \) increases from \( b < b_m^* \) to \( b > b_m^* \), so that \( b_m^* \) is the unique solution to the FOC.

While the optimal myopic booking-limit policy is independent of the equilibrium it induces, customers nevertheless react to the policy to induce a specific equilibrium, \((\beta, \hat{w})\). We can use the convexity of the loss function, together with the FOC (19), to provide some insight into the nature of the equilibrium.

**Proposition 2.** (Optimal Myopic Booking Limit is Profit-Making)

(i) The equilibrium induced by any \( \xi \in \Xi \) obtains \( \ell'(b) > (1 - \alpha) \beta \).

Suppose \( p > 0 \).

(ii) If \( \beta = 0 \), or if either \( p < (1 - \alpha)v \) or \( c > \underline{w} \) or both, then \( b_m^* \) induces a profit-making equilibrium.

Recalling from (6) and (11) that \( E[(N - k)^+] = \beta E[N] = (1 - \alpha)\beta E[S] \), part (i) can equivalently be stated as \( \ell'(b)E[S] > E[(N - k)^+] \). To see that (ii) holds when \( \beta = 0 \), note the following. For \( \beta = 0 \), we have \( \hat{w} = \underline{w} \), so \( p, E[S] > 0 \) and \( E[\Pi(p, b_m^*, c)] = pE[S] - (c + r)E[(N - k)^+] = pE[S] > 0 \). To see that (ii) holds when \( p < (1 - \alpha)v \) or \( c > \underline{w} \), take the FOC (19), use the inequality from part (i) to substitute for \( \ell'(b_m^*) \) and show that the expected margin per customer is strictly positive: \( [p - (1 - \alpha)\beta(c + r)] > 0 \).

In §4.3 we address the cases in which \( p = (1 - \alpha)v \) and \( c \leq \underline{w} \).

4.2.2 Strategic Booking Limits

Now suppose the airline is aware that the marginal customer’s hassle cost, \( \hat{w} \), is in fact an equilibrium outcome, and call the booking limit that optimally takes the demand effect into account the strategic booking limit, \( b_s^* \). Given \( b_s^* \) is the airline’s optimal response to customers’ actions, the expected profits it generates will be trivially (weakly) greater than those induced by \( b_m^* \).

What is perhaps less obvious is how the two equilibrium booking limits, \( b_s^* \) and \( b_m^* \), compare to each other. If an airline is aware of the fact that its overbooking policy will affect customer demand through \( \hat{w} \) – with some customers potentially enjoying a net benefit of being bumped, while others incurring a net cost – should it overbook more or less than in the myopic case?

To answer this question, we first consider policy parameters and equilibria that allow us to develop relevant FOCs. These include policies for which \( p \in (0, (1 - \alpha)v) \), \( c \in (0, \underline{w}) \), and \( b_m^* \in (k, \infty) \). For the same reason, we will assume that the policy \((p, b_m^*, c)\) obtains an interior equilibrium \( U(\beta, \hat{w}) = 0 \).
for which \( \hat{w} \in (\underline{w}, \overline{w}) \). In §4.3 we will explicitly consider boundary cases.

As before, we consider the FOC. Compared to the myopic FOC in (15), differentiation of expected profits in the strategic case yields extra, complicating terms:

\[
\left. \frac{dE[\Pi]}{db} \right|_{\text{strategic}} = \frac{\partial E[\Pi]}{\partial b} + \frac{\partial E[\Pi]}{\partial \hat{w}} \frac{d\hat{w}}{db} = \left. \frac{dE[\Pi]}{db} \right|_{\text{myopic}} + \frac{\partial E[\Pi]}{\partial \hat{w}} \frac{d\hat{w}}{db} = 0. \tag{20}
\]

From Proposition 1 part (iii), we know that \( \frac{dE[\Pi]}{db} \big|_{\text{myopic}} = \frac{\partial E[\Pi]}{\partial b} \) is positive for \( b < b_m^* \), zero for \( b = b_m^* \), and negative for \( b > b_m^* \). Thus, to answer the question of how \( b_s^* \) compares to \( b_m^* \), it suffices to characterize the sign of the product \( \frac{\partial E[\Pi]}{\partial \hat{w}} \frac{d\hat{w}}{db} \) as a function of \( b \).

We begin with the first term, \( \frac{\partial E[\Pi]}{\partial \hat{w}} \), which can be obtained by differentiating (3) and (17) with respect to the (interior) \( \hat{w} \). For the revenues, we have

\[
\frac{\partial E[S]}{\partial \hat{w}} = \int_0^{b_G(\hat{w})} g(\hat{w}) q f(q) dq, \tag{21}
\]

and for the costs, we have

\[
\frac{\partial E[(N - k)^+]}{\partial \hat{w}} = \int_0^{b_G(\hat{w})} \ell'(q G(\hat{w})) g(\hat{w}) q f(q) dq, \tag{22}
\]

so that

\[
\frac{\partial E[\Pi]}{\partial \hat{w}} = \int_0^{b_G(\hat{w})} \left[ p - (c + r) \ell'(q G(\hat{w})) \right] g(\hat{w}) q f(q) dq. \tag{23}
\]

From part (ii) of Proposition 1 we know that \( \frac{p}{c + r} = \ell'(b_m^*) \), and together with the fact that \( \ell(b) \) is increasing and convex, it implies that, like \( \frac{\partial E[\Pi]}{\partial b} \), the partial derivative \( \frac{\partial E[\Pi]}{\partial \hat{w}} \) is positive and increasing for \( b \leq b_m^* \) and decreasing for \( b > b_m^* \).

It follows that it is the sign of the last term in (20), \( \frac{d\hat{w}}{db} \), that dictates how \( b_m^* \) compares to \( b_s^* \).

As the proof of the following proposition (in Appendix B) shows, the expression for \( \frac{d\hat{w}}{db} \) is complex. Nevertheless, we can use it to demonstrate the following relationship between \( b_m^* \) and \( b_s^* \).

**Proposition 3.** (Optimal Strategic Booking Limit)

*Suppose \( \exists p \in (0, (1 - \alpha)v) \) and \( c \in (0, \overline{w}) \) for which \( b_m^* \in (k, \infty) \) induces a profit-making equilibrium \( \hat{w} \in (\underline{w}, \overline{w}) \). Then we have the following.*

(i) For any given \( b > k \), if \( \beta \geq \sqrt{(v - \frac{p}{1 - \alpha}) \frac{\ell' G(\hat{w})}{G(\hat{w})}} \), then \( \frac{d\hat{w}}{db} < 0 \).

(ii) In turn, if \( \beta > \sqrt{(v - \frac{p}{1 - \alpha}) \frac{\ell' G(\hat{w})}{G(\hat{w})}} \) for all \( b > k \), then \( b_s^* < b_m^* \).

Proposition 3 shows that, when bumping probabilities are high enough, an airline that accounts for customers’ equilibrium response to its policy overbooks less, as compared to the myopic alternative.
The proposition’s sufficient condition is complex, however. In addition to depending on problem parameters and distributional assumptions, it requires that a relationship between the equilibrium quantities $\beta$ and $\hat{w}$ is satisfied for any $b > k$.

Despite this difficulty, we observe that, as $p \to (1 - \alpha)v$, the proposition’s sufficient condition is in fact trivially satisfied for all $b$, an insight that motives our analysis below. In particular, in §4.3 we will show that, in the full profit-maximization problem described in (10), this boundary case is optimal at the same time that it allows us to reduce the complexity of our analysis.

4.3 Optimal Overbooking Policies

We now consider the full profit-maximization problem as described in (10). To derive optimal admissible overbooking policies $\xi \in \Xi$, we could naively compute and jointly analyze the first order conditions for all three decision variables, $(p, b, c)$. As we have seen in the equilibrium analysis of §4.2.2, however, even a standard analysis of the first order conditions for $b$ is delicate. Rather, in this section we will show that we can reduce the complexity of the analysis by exploiting structural properties of the problem. We begin in §4.3.1 by characterizing the myopic policy, which ignores the demand effects induced by its parameter choices. Then in §4.3.2, we develop the structural properties that allow us to reduce the problem, and in §4.3.3 we characterize optimal strategic policies.

4.3.1 Myopic Overbooking Policies

As in §4.2.1, we begin by considering an airline that does not recognize that the marginal customer’s hassle cost, $\hat{w}$, is an equilibrium reaction to $\beta$ and, in turn, to $(p, b, c)$. Instead it believes that $\hat{w}$ is fixed, and it uses a myopic overbooking policy that maximizes the objective function of (10) without considering the problem’s equilibrium constraints. We call the associated optimal myopic solution $(p^*_m, c^*_m, b^*_m)$.

In this case, we find that the airline charges the maximum price and chooses not to compensate bumped passengers at all. More formally we have the following.

**Proposition 4.** (Optimal Myopic Overbooking Policy)

A myopic airline sets $p^*_m = (1 - \alpha)v$ and $c^*_m = 0$. When $v < r$, it selects a finite optimal booking limit $b^*_m = \max \left\{ \ell^{\ell-1} \left( \frac{(1-\alpha)v}{r} \right), k \right\}$. Otherwise, $b^*_m$ is infinite.

The rationale for the policy is as follows. For a fixed $\hat{w}$, any given choice of $b$ determines sales $S = \min\{b, QG(\hat{w})\}$, without regard to the value of $p$ or $c$, and we see from (13) that, by maximizing
minimizing $c$, the airline maximizes per-customer contribution and expected profit. In turn, the optimal myopic booking limit is the same as that in Proposition 1 for $p = (1 - \alpha)v$ and $c = 0$.

The customer equilibrium obtained from the policy, of course, differs. From part (i) of Lemma (2) we see that, when $p = (1 - \alpha)v$ and $c = 0$, myopic solutions that recommend $b^*_m > k$ will not generate a customer equilibrium. In contrast, those that set $b^*_m = k$ obtain $\hat{w} = \bar{w}$ and earn expected profits of $E[\Pi((1 - \alpha)v, k, 0)] = (1 - \alpha)v E[\min\{k, QG(\bar{w})\}]$.

4.3.2 Problem Reduction

In contrast to the myopic policy, above, a strategic overbooking policy considers customer response when solving (10). In this case, we can show that there exist optimal strategic policies that set $p = (1 - \alpha)v$, a result that greatly simplifies our analysis and affords a number of insights. The result holds straightforwardly for overbooking policies that induce interior equilibria, for which $U(\beta, \hat{w}) = 0$, and we begin by first eliminating boundary cases from consideration.

**Lemma 4.** (Boundary Equilibria Not Optimal)

Any optimal strategic overbooking policy induces a customer equilibrium with $U(\beta, \hat{w}) = 0$.

To demonstrate the result, we need to rule out two cases. The first, when the equilibrium is $U(\beta, w) < 0$, induces expected sales of zero and is not profit making. Therefore, it is dominated by any policy that charges a positive price and does not overbook. The second, more interesting case, occurs when the equilibrium is $U(\beta, \bar{w}) > 0$, so that $G(\bar{w}) = 1$ and all customers receive a strictly positive surplus. The lemma’s proof shows that, when $U(\beta, \bar{w}) > 0$, we must also have $p < (1 - \alpha)v$, and the airline can raise the price a bit without affecting unit demand and thereby increase expected profits. Thus neither boundary case can be optimal, and when searching for optimal strategic overbooking policies, we can consider only policies that induce customer equilibria with $U(\beta, \hat{w}) = 0$.

Conversely, suppose the airline employs an admissible policy $(p, c, b)$ for which $U(\beta, \hat{w}) = 0$. Then we can use (1)-(2) to rewrite the equilibrium condition as

$$p = (1 - \alpha)v + (1 - \alpha)\beta(c - \hat{w}).$$

(24)

We see that, for a fixed equilibrium pair $(\beta, \hat{w})$, there is an infinite set of $(p, c)$ pairs that satisfy the equilibrium equation (24). Recalling that $N = (1 - \alpha)\min\{b, QG(\bar{w})\}$, $(N - k)^+ = \ell(\min\{b, QG(\bar{w})\})$, and $\beta = E[N]/E[(N - k)^+]$, we see that a given $b$ and $\hat{w}$ uniquely define $\beta$, without regard to $p$ or $c$.

Now consider an airline with an overbooking policy $(p, b, c)$ that obtains equilibrium $(\beta, \hat{w})$. If the airline maintains the same $b$, then there will be an infinite set of price-bumping-compensation pairs,
$(p', c')$ – including the original $(p, c)$ – that will satisfy the equilibrium equation (24) and maintain the same equilibrium $(\beta, \hat{w})$. Furthermore, if we use (24) to substitute for $p$ in the profit expression (13), we see that for any $p'$ and $c'$ that satisfies (24), including the original $p$ and $c$,

$$E[\Pi(p', b, c')] = [(1 - \alpha)v + (1 - \alpha)\beta(c' - \hat{w}) - (1 - \alpha)\beta(c' + r)] E[S] = [(1 - \alpha)v - (1 - \alpha)\beta(\hat{w} + r)] E[S] = E[\Pi((1 - \alpha)v, b, \hat{w})].$$

In particular, the substitution of $(1 - \alpha)v$ for $p$ and $\hat{w}$ for $c$ obtains the same equilibrium and expected profit. Thus, we have the following result.

**Lemma 5.** (Multiple Equivalent Policies)

For any admissible policy $(p, b, c)$ for which $\beta > 0$ and $U(\beta, \hat{w}) = 0$, there exists an infinite set of alternative policies with the same booking limit, $b' \equiv b$, and alternative price and bumping compensation,

$$p' \in \max \{0, (1 - \alpha)(v - \hat{w}\beta)\}, (1 - \alpha)v \quad \text{and} \quad c' = \left(\hat{w} - \frac{v}{\beta}\right) + \left(\frac{p'}{(1 - \alpha)\beta}\right),$$

with the same equilibrium $(\beta, \hat{w})$ and expected profits $E[\Pi(p, b, c)] = E[\Pi(p', c', b')] = E[\Pi((1 - \alpha)v, b, \hat{w})].$

Lemma 4 shows that an optimal policy induces an interior equilibrium, and Lemma 5 in turn shows that the airline can match the performance of such a policy using an alternative with price $p = (1 - \alpha)v$. Together, the two imply we need only look at the following sub-class of admissible policies.

**Proposition 5.** (Problem Reduction)

If there exists an optimal strategic overbooking policy $\xi \in \Xi$, then there exists an optimal strategic policy that sets $p^* = (1 - \alpha)v$, induces an interior equilibrium $U(\beta, \hat{w})$, and optimizes (10).

Thus, in the search for an optimal strategic policy, the airline need only consider $p = (1 - \alpha)v$. With this insight, we continue our analysis for $p = (1 - \alpha)v$ below.

### 4.3.3 Strategic Overbooking Policies

Proposition 5 allows the airline to reduce the complexity of its profit maximization problem in two ways. It can fix the decision variable $p = (1 - \alpha)v$ in (10) and optimize over only $(b, c)$. In addition, as (24) shows, given $p = (1 - \alpha)v$, any interior equilibrium must have $c \equiv \hat{w}$. This fact, in turn, has four additional implications.

First, whatever $c$ the airline chooses uniquely determines $\hat{w}$, without regard to the booking limit $b$, thereby eliminating the effect of $b$ on $\hat{w}$. From (20), we recall that $\frac{d\hat{w}}{db}$ is a source of significant complication in the analysis of strategic booking limits, and with $\frac{d\hat{w}}{db} = 0$ we eliminate this difficulty.
Second, given \( p = (1 - \alpha)v \) and a booking limit, \( b \), the choice of \( c \equiv \hat{w} \) effectively sets a unique equilibrium because \( E[S], E[(N - k)^+] \), and \( \beta = (1 - \alpha)E[S]/E[(N - k)^+] \) are uniquely determined by \( b \) and \( \hat{w} \). Thus, the sufficient conditions of Lemma 2 part (ii) are no longer necessary: when \( p = (1 - \alpha)v \) the uniqueness of interior equilibria holds for any hassle-cost-distribution \( G(\cdot) \).

Third, from (24) we also see that any customer with hassle-cost \( w \leq c \equiv \hat{w} \) is willing to buy a ticket – no matter how high the equilibrium \( \beta \) – and is, in fact happy to be bumped, should she be bumped. Conversely, any customer with hassle-cost \( w > c \equiv \hat{w} \) will never buy a ticket as long as \( \beta > 0 \), no matter how low. (When \( b = k \) then \( \beta = 0, \hat{w} = \overline{w} \).) Thus when \( p = (1 - \alpha)v \), customers do not need to know or carefully estimate \( \beta \) when making purchase decisions. In addition to their own preference and no-show information, \((v, w, \alpha)\), they need only know the ticket price, \( p \), whether or not the airline overbooks – whether \( \beta = 0 \) or \( \beta > 0 \) – and, if it overbooks, what bumping compensation, \( c \), the airline offers, information that the airline can credibly communicate to its customers.

Fourth, we can tighten the lower bound for admissible values of \( c \). On the one hand, because \( G(w) = 0 \) effectively shuts down demand, we need never consider \( c < \overline{w} \). On the other hand, because there is no need for overbooking when \( \overline{q}G(c) \leq k \), we similarly never need consider \( c < G^{-1}(k/\overline{q}) \). Here, the relevant lower bound would still be \( c = 0 \) for \( \overline{q} = \infty \). Together with the fact that we are fixing \( p \), we define a subclass of \( \Xi \), which we call reduced admissible policies, \( \Xi_R \subseteq \Xi \) for which \( p = (1 - \alpha)v \), \( \max\{w, G^{-1}(k/\overline{q})\} \leq c \overline{w} \), and \( k \leq b \). We know that there exists an optimal policy \( \xi \in \Xi_R \).

Therefore, with \( p = (1 - \alpha)v \) and \( c \equiv \hat{w} \), we can further simplify the airline’s original optimization problem (10). Given \( \hat{w} \equiv c \) we define unit sales and loss as direct functions of \( b \) and \( c \),

\[
S = \min\{b, QG(c)\} \quad \text{and} \quad (N - k)^+ = (N(S, \alpha) - k)^+ ,
\]

so that the airlines profits are

\[
\Pi((1 - \alpha)v, b, c) = (1 - \alpha)v S - (c + r) (N - k)^+ .
\]

In turn, the airline can optimize

\[
\max_{\xi \in \Xi_R} \mathbb{E}[\Pi((1 - \alpha)v, b, c)] ,
\]

without explicit equilibrium constraints, to identify an optimal strategic overbooking policy.

Given an optimal strategic price, \( p^*_s = (1 - \alpha)v \), we can differentiate (28) with respect to \( b \),

\[
\frac{\partial \mathbb{E}[\Pi]}{\partial b} = (1 - \alpha)v P\{QG(c) > b\} - (c + r) \ell'(b)P\{QG(c) > b\} ,
\]

and with respect to \( c \),

\[
\frac{\partial \mathbb{E}[\Pi]}{\partial c} = -\mathbb{E}[(N - k)^+] + g(c) \int_0^b \left[(1 - \alpha)v - (c + r) \ell'(qG(c))\right] qf(q) \, dq ,
\]

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and use their first order conditions to identify optimal strategic booking limit, $b_s^*$, and bumping compensation, $c_s^*$.

The first order condition with respect to $b$ is precisely that in (19) but with $p = (1 - \alpha)v$, and for a given $c$ the results of Proposition 1 hold here as well. The following proposition describes how, as the value $c_s^*$ increases, $b_s^*$ systematically decreases from $\infty$ down to $k$.

**Proposition 6.** (Booking Limit for Optimal Strategic Overbooking Policy)

(i) If $c_s^* \leq v - r$, then $b_s^* = \infty$.

(ii) If $v - r < c_s^* < \frac{(1-\alpha)v}{v'(k)} - r$, then $b_s^* = \ell^{-1} \left( \frac{(1-\alpha)v}{c_s^* + r} \right)$.

(iii) If $c_s^* \geq \frac{(1-\alpha)v}{v'(k)} - r$, then $b_s^* = k$.

Part (i) of the proposition follows directly from part (i) of Proposition 1, and parts (ii) and (iii) of the proposition follow from part (ii) of Proposition 1. When $b_s^* = k$, we also know that $c_s^* = \bar{w}$, since no one is bumped, and the high compensation ensures maximum demand. Note that $\ell'(k) < (1 - \alpha)$, so the ordering of $c_s^*$ in parts (i)-(iii) is well defined.

Proposition 6 is interesting for two reasons. First, it reflects that fact that, for $p_s^* = (1 - \alpha)v$ and a given $c_s^*$, the optimal booking limit is simply calculated as the myopic optimal booking limit for that $p_s^*$ and $c_s^*$. In turn, it shows that an optimal overbooking policy may be found by a line search over potential values of $c$.

5 Overbooking with a Bumping Auction

We now consider overbooking policies in which the airline compensates bumped customers using an auction. In this case, whether a passenger is bumped and how much she is compensated depend on the magnitude of her hassle cost, $w$. We define the primitives of the auction model and format in §5.1.

Our analysis of the auction-based compensation scheme yields three sets of insights. In §5.2 we show that, given use of an auction without a pre-set limit on bumping compensation, customers are always happy to be bumped, and as with the fixed-compensation model, there exist effective auction-based schemes in which the ticket price equals the expected value of flying. Then in §5.3 we show that, as expected, the use of auction-based bumping compensation can discriminate effectively among customers with lower and higher hassle costs to lower expected bumping compensation, and an auction with a pre-announced upper limit on its potential bumping compensation, though not necessarily optimal, always increases the airline’s expected profit in comparison to the analogous fixed-compensation scheme of §4.3. In cases in which the auction’s expected total bumping compensation
is convex in the number of tickets sold, we can characterize the behavior of specific policy parameters. In §5.4 we consider these convex cases, we characterize optimal policy parameters, and we identify an optimal auction-based overbooking policy.

5.1 Primitives for the Auction Model

In most respects, the primitives of our model of overbooking with auction-based bumping compensation parallel those of the fixed-compensation model defined in §4. The airline operates a flight with \( k \) seats. It sells tickets up to booking limit, \( b \), at price, \( p \). If \( b > k \) and customers are bumped, it pays rerouting cost, \( r \), for each customer it bumps. The airline may decide to impose an upper bound, \( c_a \), on the compensation that it is willing to pay to bumped passengers, an analogue of the fixed compensation, \( c \), paid in §4. In the context of an auction, we will call the upper bound, \( c_a \), a cap.

Customer attributes also remain the same. Potential customers are homogenous in the value they obtain from the flight, \( v \), as well as in their no-show probability, \( \alpha \), and they are heterogenous in the hassle cost, \( w \). We continue to model hassle costs as i.i.d. samples drawn from a common random variable, \( W \), over support \( 0 \leq w < \infty \), with CDF \( G(w) \) and density \( g(w) > 0 \) over its support.

As before, potential demand, \( Q \), is random, with support \( 0 \leq q < \infty \) and CDF \( F(q) \). In §5.2 and §5.3, in which we provide preliminary results regarding auction-based overbooking policies, we assume that the support of \( Q \), \( S \), and \( N \) is integral. In §5.4, in which we further characterize the optimal booking limit, \( b^*_a \), and compensation cap, \( c^*_a \), we return to §4’s assumption that these random variables are continuous, and we define an analogous continuous approximation for the expected auction-based bumping compensation. In both cases, our original definition of \( S \), \( N \), and \((N-k)^+\) in (3)–(5) continue to hold.

The auction model differs from that in §4 in that, when the number of customers who show up for the flight, \( n \), exceeds the flight’s capacity, \( k \), the airline does not choose \((n-k)\) customers to bump at random. Rather, it chooses which passengers to bump using an auction.

5.2 Auction with No Cap

We begin our analysis by analyzing an auction for which the compensation paid to bumped customers is not limited by a pre-determined cap. As in the fixed-compensation model of §4, the airline must also decide on a ticket price and booking limit.

The auction format is a reverse form of a so-called uniform price, multi-unit auction with single-unit demand and works as follows. If the number of people who show up for the flight, \( n \), exceeds the flight’s
capacity, \( k \), then each of the \( n \) potential passengers is asked to reveal the minimum compensation, \( \varpi \), she would require to give up her seat and be rerouted on another flight. We call \( \varpi \) a bid and assume that all \( n \) passengers submit their bids simultaneously and independently. The airline observes the \( n \) bids and orders them from smallest to largest. We describe them using the notation of order statistics:

\[ \varpi_{1:n} \leq \varpi_{2:n} \leq \ldots \leq \varpi_{n:n}. \]

The airline then bumps the customers with the \( n - k \) lowest bids and pays each of bumped passenger \( \varpi_{n-k+1:n} \), the lowest bid among those passengers the airline allows to board the flight.

We have not yet described which customers decide to buy tickets, or not, so we do not yet know the distribution of the \( w \)'s of those who show up for the flight. Nevertheless, we can show that this auction format motivates customers who paid for a ticket and show up for the flight to truthfully bid their hassle costs. This fact, in turn, allows us to characterize which customers buy tickets and the airline's optimal price, which we denote as \( p^*_a \).

**Proposition 7.** (Properties of the Auction with No Cap)

Suppose that, when \( n > k \) customers show up for a flight, the airline runs a reverse, uniform price, multi-unit auction. Then we have the following.

(i) Customers’ optimal bids match their underlying hassle costs: \( \{ \varpi_{1:n} = w_{1:n}, \ldots, \varpi_{n:n} = w_{n:n} \} \).

(ii) All customers are willing to purchase tickets, irrespective of their hassle cost \( w \in [\underline{w}, \overline{w}] \).

(iii) The airline’s optimal price is \( p^*_a = (1 - \alpha)v \).

Part (i) of the proposition is well-known. For example, Section 13.4.2 in Krishna (2010) notes the dominance of truthful bidding and resulting efficiency for this auction format, and for completeness we provide an explicit proof of the former in the appendix.

For part (ii) we note that, given the optimality of customers’ bidding their true hassle costs, the airline offers compensation of \( w_{n-k+1:n} \) to each of the \( (n-k) \) customers who it bumps, an amount that is, by definition, at least as great as any of their hassle costs. Thus, for any customer, the expected value of being bumped is always non-negative and, given the opportunity, any customer will purchase a ticket. This is an analogue of the equilibrium \( \hat{w} \equiv \overline{w} \) in the fixed-compensation scheme.

We note that, because customers are happy to be bumped, they need not estimate the chances of being bumped – a fact we pointed out in the model in §4 – and we need not define or evaluate analogues of the equilibrium expressions, (1)–(2). Rather, as in (26) we can define \( S = \min\{b, QG(\overline{w})\} = \min\{b, Q\} \) and \( (N(S, \alpha) - k)^+ \) as direct functions of \( b \) and \( \overline{w} = \overline{w} \).

Part (iii) of the proposition then follows \( \hat{w} \equiv \overline{w} \). In particular, the fact that numbers of passengers ticketed, \( S \), and bumped, \( (N(S, \alpha) - k)^+ \), are independent of the price, implies that, to maximize
expected profit, the airline should simply increase the price to its maximum, \( p_a^* = (1 - \alpha)v \).

Given the same optimal price in both the fixed-compensation and auction-compensation models, \( p_a^* = (1 - \alpha)v = p_a^* \), it is natural to ask how the performance of two classes of overbooking policies compare. On the one hand, the auction format should lower expected bumping compensation by choosing low-cost customers to bump. On the other hand, when the optimal level of fixed bumping compensation, \( c_s^* \), falls strictly below \( \overline{w} \), there exists the possibility of obtaining sample realizations, for which \( w_{n-k+1:n} > c_s^* \), a fact that may drive \( \mathbb{E}[w_{n-k+1:n}] \) to exceed \( c_s^* \). In the next section, we will consider a capped version of the auction scheme that allows us to make a direct comparison.

5.3 Auction with a Cap

An airline that runs the auction scheme described §5.2 may potentially end up paying very high total bumping compensation, depending on the hassle cost distribution and its sample realizations. One way to limit the payout is to place a cap on the compensation paid to each customer who is bumped. For example, both United and Delta Airlines have upper bounds that they publicize, United offering up to $10,000 and Delta up to $9,950 in vouchers that can be applied to the ticket price of future flights (Martin 2017, Tuttle 2017).

In this section we consider this form of cap, which works as follows. The airline offers tickets for a flight at price \( p \) and publicly discloses that, when bumping customers, it uses an auction with cap, \( c_a \), on the maximum compensation paid. In the event that \( n > k \) passengers show up for the flight, the airline runs an auction, as before, but now limits the compensation per bumped customer to \( \min\{ c_a, w_{n-k+1:n} \} \). We note that the uncapped auction of §5.2 is equivalent to a capped auction with cap \( c_a = \overline{w} \).

The presence of a cap \( c_a < \overline{w} \) has the potential to eliminate the dominance of truthful bidding and to render an analysis of the auction unmanageable. For the special case of a price \( p = (1 - \alpha)v \) – which we already know is optimal for overbooking policies with fixed-compensation and for uncapped auction-based policies – the analysis remains tractable, however. In particular, we have the following.

**Proposition 8.** (Properties of the Auction with a Cap)

*Suppose that the airline sets the price \( p = (1 - \alpha)v \) and \( b > k \). When \( n > k \) customers show up for a flight, it runs a reverse, uniform price, multi-unit auction with compensation cap \( c_a \leq \overline{w} \). Then we have the following.*

(i) *Customers are willing to purchase tickets, if and only if their hassle costs are \( w \leq c_a \).*

(ii) *Customers’ optimal bids match their underlying hassle costs: \( \{ w_{1:n}^1 = w_{1:n}, \ldots, w_{n:n} = w_{n:n} \} \).*
The proposition’s results follow the logic of the fixed-compensation scheme with \( p = (1 - \alpha)v \). In the auction setting, a price of \( p = (1 - \alpha)v \), together with a pre-announced cap of \( c_a \), ensures that customers with \( w > c_a \) have a surplus of \(-p + (1 - \alpha)v = 0\) on every sample path on which they are not bumped and a surplus of \(-p + (1 - \alpha)v + (c_a - w) < 0\) on every sample path on which they are, so the expected value of their purchasing a ticket is negative.

This demonstrates the “only if” statement of part (i). If we then consider a truncated hassle-cost distribution \( W(c_a) \) with upper bound \( \overline{w} = c_a \) and CDF \( G_{c_a}(w) = G(w)/G(c_a) \), we return to the setting in §5.2 of an auction with no cap, one in which every customer with hassle cost \( w \in [\underline{w}, \overline{w}] \) is willing to purchase a ticket. The “if” statement of part (i) and the statement of part (ii) then follow from parts (i) and (ii) of Proposition 7.

The cap on the auction’s bumping compensation lets us directly compare this auction format’s expected profits to those of analogous fixed-compensation policies. As in (26) we can define the number of units sold, \( S = \min\{b, QG(c_a)\} \), and in turn number of bumped customers, \( (N(S, \alpha) - k)^+ \), as functions of the cap, \( c_a \). To define expected bumping compensation, we let \( P_N(n|s) \) denote the probability that \( n \) among \( s \) ticketed passengers show up for the flight, and we let \( w(c_a)_{n-k+1:n} \) denote the \( n - k + 1 \)st ordered hassle cost from a truncated hassle-cost distribution \( W(c_a) \). We then define the expected total bumping compensation, given \( s \) tickets are sold, as

\[
C(s, c_a) = \begin{cases} 
0 & \text{if } s \leq k, \\
\sum_{n=k}^{s} (n-k) E[w(c_a)_{n-k+1:n}] P_N(n|s) & \text{otherwise},
\end{cases}
\]

a conditional expectation that is an analog to \( c\ell(s) \), the conditional expectation of total bumping costs in the fixed-compensation scheme. The resulting marginal expression for the expected bumping compensation, which randomizes (31) over \( S \), becomes \( C(S, c_a) \).

With these quantities defined, we can denote auction-based profits for price \( p = (1 - \alpha)v \) as

\[
\Pi_a((1 - \alpha)v, b, c_a) = (1 - \alpha)v S - r (N - k)^+ - C(S, c_a),
\]

where revenues equal the ticket price times number of tickets sold, re-routing costs equal \( r \) times the number of passengers bumped, and total bumping compensation equals \( C(S, c_a) \).

Now consider an admissible overbooking policy with \( p = (1 - \alpha)v \), \( b \geq k \), and fixed compensation \( \underline{w} < c \leq \overline{w} \). Given discrete distributions for \( Q \), \( S \), and \( N \) in (26), we can directly compare the objective functions (27) and (32) to show that auction-based bumping compensation dominates fixed-compensation policies.
Proposition 9. (Auction with Cap Dominates Fixed Compensation)

Given any fixed-compensation policy with \( p = (1 - \alpha)v, b > k, w < c \leq \overline{w}, \) and equilibrium \( \beta > 0, \) an auction-based policy with the same price, \( p = (1 - \alpha)v, \) the same booking limit, \( b, \) and an analogous cap, \( c_a = c, \) earns strictly higher expected profits:

\[
E[\Pi_a((1 - \alpha)v, b, c_a)] > E[\Pi((1 - \alpha)v, b, c)].
\]

The fixed-compensation and auction-based policies have the same expected revenues, \( (1 - \alpha)vE[S], \) and the same expected rerouting costs, \( rE[(N - k)^+] \). While both policies have the same numbers of bumped passenger, \( (N - k)^+, \) in any realization for which \( n > k, \) the auction scheme’s compensation is weakly lower by construction: \( w(c)_{n-k+1:n} \leq c. \) Given \( g(w) > 0 \) there further exists a positive probability that \( w(c)_{n-k+1:n} < c, \) a strict inequality that carries over to expected profits.

Thus, any strategic overbooking policy with price \( p = (1 - \alpha)v \) is outperformed by the analogous auction-based policy. This includes strategic overbooking policies for which \( b \) and \( c \) are optimized for the price \( p = (1 - \alpha)v. \) Proposition 5’s results, that there exist optimal policies with \( p = (1 - \alpha)v \) for continuously distributed \( S \) and \( (N - k)^+, \) suggest that, in fact, there exists an auction-based compensation scheme that outperforms any fixed-compensation scheme.

5.4 Optimal Policy Parameters for Auctions

In §5.2 and §5.3 we were able to use relatively elementary arguments to provide two insights of interest regarding overbooking with auctions. First, as with overbooking policies with fixed bumping compensation, the optimal price for auctions without a cap on bumping compensation is \( p = (1 - \alpha)v. \) Second, given an additional cap on auction-based bumping compensation, \( c_a, \) any fixed-compensation policy with price \( p = (1 - \alpha)v \) is outperformed by an analogous auction-based format. When there is a cap, the potential complexity of passenger bidding behavior when \( p < (1 - \alpha)v \) prevents us from providing a sharper characterization, however. Nevertheless, by using the same type of continuous approximations we employed in §4, we can differentiate critical expressions to provide additional insight.

5.4.1 Primitives for the Continuous Approximation

As in §4, we assume that \( Q \) is a continuous random variable with density \( f(q) > 0 \) over its support, that \( S \) and \( N(s, \alpha) \) are likewise continuous, and that the loss function \( \ell(s) = E[(N(s, \alpha) - k)^+] \) is characterized by Definition 2. In addition to these assumptions, we define an analogous continuous approximation to the conditional expectation of the auction-based bumping compensation, given the number of potential passengers, \( n. \)
We continue to let \( E[w(c_a)_{n-k+1:n}] \) represent the expected value of the order statistic that determines bumping compensation. Now, however, we assume that the expectation varies continuously with a continuously-defined \( n \). Accordingly we let \( P_N(n|s) \) denote the CDF of the conditional distribution of \( N \), given \( s \), with support \([0, s]\) and density \( p_N(n|s) > 0 \) over its support, so that

\[
C(s, c_a) = \begin{cases} 
0 & \text{if } s \leq k, \text{ and} \\
\int_k^s (n - k) E[w(c_a)_{n-k+1:n}] p_N(n|s) \, dn & \text{otherwise,}
\end{cases}
\]

with analogous total bumping compensation \( C(S, c_a) \), as in (31).

We note that a continuous approximation for \( E[w(c_a)_{n-k+1:n}] \) can be created in a number of ways. One longstanding method that works well for relatively large \( n \) uses the inverse CDF of the hassle-cost distribution to map back from the relevant fractile to its \( w \) value (Arnold et al. 2008).

Recalling from §5.3 that the conditional distribution of the hassle cost, given a cap \( c_a \), is \( G_{c_a} \), we have

\[
E[w(c_a)_{n-k+1:n}] \approx G_{c_a}^{-1} \left( \frac{n-k+1}{n+1} \right).
\]

For notational simplicity, we define \( \tilde{G}(c_a, n) \equiv G_{c_a}^{-1} \left( \frac{n-k+1}{n+1} \right) \).

We emphasize that, as with the loss function, our proofs do not depend on the particular distribution of the number of customers who show up. We do require, however, that \( N(s, \alpha) \) is stochastically increasing and convex (SICX) in \( s \). (See Section 6.A.1 in Shaked and Shanthikumar (1994).) In particular, this means that \( E[\psi(N(s, \alpha))] \) is increasing in \( s \) for all increasing \( \psi(\cdot) \) and increasing convex in \( s \) for all increasing convex \( \psi(\cdot) \). The binomial distribution, for example, is SICX in \( s \).

Similarly, our proofs do not depend on the particular form of the approximation for \( E[w(c_a)_{n-k+1:n}] \), only on its differentiability with respect to \( n \), the resulting differentiability of \( C(s, c_a) \) with respect to \( s \), and the properties of the latter’s derivatives.

As with \( \ell(\cdot) \) we sometimes write partial derivatives with respect to one argument by using a prime symbol and the variable of interest: for example we let \( C'(s) \equiv \frac{\partial C(s, c_a)}{\partial s} \). With this notation in hand we can state a first result, that expected total bumping costs inherit the convexity properties of the per-passenger expected bumping cost.

**Lemma 6.** (Convexity of Auction-Based Expected Bumping Cost)

Suppose \( N(s, \alpha) \) is SICX in \( s \).

(i) If \( \partial E[w(c_a)_{n-k+1:n}]/\partial n \geq 0 \), then \( C'(s) > 0 \).

(ii) If in addition \( \partial^2 E[w(c_a)_{n-k+1:n}]/\partial n^2 > 0 \), then \( C''(s) > 0 \).

The fact that expected bumping cost per customer, \( E[w(c_a)_{n-k+1:n}] \) is itself increasing in the number of customers bumped is not surprising. It can, in fact, be proven for any distribution using
the original, discrete representation of order statistics and a sample-path argument. Given this fact, the propositions below will not state \( C'(s) > 0 \) as an explicit assumption.

The convexity of the expected order statistic, however, depends more specifically on the form of the hassle-cost distribution, \( G \), and we can use the approximation, \( \tilde{G} \), defined above to provide some insight into the types of distributions for which it holds.

**Lemma 7.** (Convexity of the Approximation \( \tilde{G} \))

Suppose \( N(s, \alpha) \) is SICX in \( s \) and we use the specific approximation \( E[w(c_a)_{n-k+1:n}] \approx \tilde{G}(c_a, n) \).

If for any \( c_a \in (w, \bar{w}] \), \( g'(w) < 0 \) for all \( w \in [G^{-1}(\frac{G(c_a)}{k+1}), c_a] \), then \( C''(s) \geq 0 \).

Thus, a sufficient condition for the convexity of the approximation \( \tilde{G} \) is a decreasing hassle-cost density. This is the same condition that part (iii) of Lemma 2 showed is sufficient for the uniqueness of a customer equilibrium. As with Lemma 2, we note that DFR distributions satisfy the condition.

### 5.4.2 Properties of the Booking Limit

When the airline sets a price of \( p = (1 - \alpha)v \), auction profits (32) are well defined, and we can differentiate expected profits with respect to relevant policy parameters. To develop the required FOC we begin by differentiating the expectation of (32) with respect to \( b \).

\[
\frac{dE[\Pi_a]}{db} = (1 - \alpha)v \frac{dE[S]}{db} - r \frac{dE[(N-k)^+]}{db} - \frac{dE[C(S)]}{db} = 0. \tag{34}
\]

Differentiating the expectation of (33) with respect to \( b \) we have

\[
\frac{dE[C(S)]}{db} = C'(b)P\{Q > b\}. \tag{35}
\]

and from (16), (18), and (35) we can write the FOC in (34) as

\[
\frac{dE[\Pi_a]}{db} = P\{Q > b\} \left[(1 - \alpha)v - r \ell'(b) - C'(b)\right] = 0. \tag{36}
\]

To ensure that \( b_a^* \) is a local maximum, we need also to examine the second-order condition (SOC)

\[
\frac{d^2E[\Pi_a]}{db^2} = f(b) \left[(1 - \alpha)v - r \ell'(b) - C'(b)\right] + P\{Q > b\} \left[-r\ell''(b) - C''(b)\right]. \tag{37}
\]

We now have the machinery needed to characterize the optimal booking limit, \( b_a^* \). From the FOC we see that \( (1 - \alpha)v - r \ell'(b) - C'(b) = 0 \), which implies that the SOC is negative whenever \( [-r\ell''(b) - C''(b)] < 0 \). Recalling Definition 2, we know that \( \ell''(s) > 0 \), so a sufficient condition for \( \frac{d^2E[\Pi_a]}{db^2} \) to be negative is \( C''(b) > 0 \). Finally, from Lemma 6 we know that \( C''(b) > 0 \) whenever \( \frac{\partial^2E[w(c_a)_{n-k+1:n}]}{\partial n^2} \geq 0 \).
Proposition 10. (Optimal Booking Limit for the Auction)

Suppose \( p = (1 - \alpha)v \) and \( C''(s) > 0 \). Then there exists a unique optimal booking limit, \( b^*_a \), with the following properties.

(i) If \( \ell'(k) \geq \frac{(1-\alpha)v}{r} \) then \( b^*_a = k \).

(ii) If \( \ell'(k) < \frac{(1-\alpha)v}{r} \) and \( \exists b \in (k, \infty) \) s.t. \( C'(b) \geq (v - r)(1 - \alpha) \), then \( (36) \) determines \( b^*_a \in (k, \infty) \).

(iii) If \( C'(b) < (v - r)(1 - \alpha) \) for all \( b \geq k \) then \( b^*_a = \infty \).

The proposition’s results follow from the FOC \( (36) \) and the fact that \( \lim_{s \to \infty} \ell(s) = (1 - \alpha) \). As with the myopic booking limit characterized in Proposition 1, the higher the rerouting and bumping costs, the smaller the booking limit. When expected marginal rerouting costs are high enough, there is no overbooking. With moderate levels, there is a finite level of overbooking. Finally, given the upper limit of \((1 - \alpha)r\) on the marginal expected rerouting cost, small enough marginal bumping costs can lead to infinite booking limits.

5.4.3 Properties of the Cap

As the airline changes the cap on the maximum bumping compensation, \( c_a < \bar{w} \), it systematically changes the hassle-cost distribution. In particular, if \( c_a^1 < c_a^2 \), then \( \mathbb{P}\{W(c_a^1) \leq w\} \geq \mathbb{P}\{W(c_a^2) \leq w\} \) for all \( w \in [\bar{w}, \overline{\bar{w}}] \). That is, \( W(c_a^2) \) is larger than \( W(c_a^1) \) in the so-called usual stochastic order.

This difference helps us to characterize the effect of potential changes to the cap. To begin, the stochastic order immediately implies that, for any fixed \( n \), expected bumping compensation, \( \mathbb{E}[w(c_a)_{n-k+1:n}] \) increases with the cap. (See Theorem 1.A.3(b) in Shaked and Shanthikumar (1994).)

In turn, for any underlying hassle-cost distribution, \( G \), the expected total bumping cost grows more quickly with \( s \).

Lemma 8. (Bumping Compensation Grows with the Cap)

For \( c_a \in (\bar{w}, \overline{\bar{w}}) \) and \( s > k \), (i) \( \partial C(s,c_a)/\partial c_a > 0 \), and (ii) \( \partial^2 C(s,c_a)/\partial s\partial c_a > 0 \).

In addition, when \( C(s,c_a) \) is convex in \( s \), we can use Lemma 8 to characterize how the optimal booking limit in Proposition 10 and expected profits change with the cap.

Proposition 11. (Optimal Auction Parameters)

For fixed \( p \), let \( b^*_a(c_a) \) be the optimal booking limit induced by \( c_a \). Suppose \( p = (1 - \alpha)v \), \( w < c_a < \bar{w} \), \( k < b^*_a(c_a) < \infty \) and \( C''(s) > 0 \). Then we have the following.

(i) The optimal booking limit, \( b^*_a \), is decreasing in \( c_a \).

(ii) The resulting expected profit, \( \mathbb{E}[\Pi_a] \), is increasing in \( c_a \).
Part (i) of the proposition follows from the FOC and the fact that \( C'(s) \) is increasing in the cap. It implies that the optimal booking limit for the auction with no cap is the smallest among those for all auction policies we have considered. Part (ii) implies that, among all auctions-based policies with price \( p = (1 - \alpha)v \), an auction with no cap – which in the context of the proposition is the maximal cap – maximizes the airline’s expected profits.

With analogous continuous-distribution models for both the fixed-compensation and auction-based overbooking policies, we can now make a direct comparison of the two classes of policy to conclude the following.

**Proposition 12.** (Optimality of Overbooking Policy)

(i) The optimal overbooking policy uses an auction to determine customers’ bumping compensation.

When \( N(s, \alpha) \) is SICX, \( C''(s) > 0 \), and the auction-based overbooking policy sets \( p = (1 - \alpha)v \), we also have the following.

(ii) The optimal cap on bumping compensation is effectively unbounded: \( c^*_a = \bar{w} \).

(iii) The optimal booking limit \( b^*_a \) is defined as in Proposition 10.

While we have not ruled out the possibility that an auction with cap \( c_a < \bar{w} \) and price \( p < (1 - \alpha)v \) could outperform the auctions we have considered, we do know that there exists a capped auction with price \( p = (1 - \alpha)v \) that outperforms all fixed-compensation policies. Thus, as part (i) states, there is some auction scheme that is optimal, a result that does not depend on the price or the convexity of \( C(s) \). For \( p = (1 - \alpha)v \), SICX \( N(s, \alpha) \) and convex \( C(s) \), parts (ii) and (iii) highlight that the optimal overbooking policy sets \( c^*_a = \bar{w} \) and requires only a minimal line search for the optimal \( b \).

6 Numerical Experiments

Having analyzed how demand endogeneity affects overbooking under both fixed and auction schemes, a natural question that arises is whether or not the magnitude of the demand effect is significant. To address this question, we run two sets of numerical experiments that span a wide range of problem parameters and are meant to capture some typical values for flight capacities \( k \), no-show rates \( \alpha \), customer valuations \( v \), compensation amounts \( c \), and rebooking costs \( r \). The results are reported in Table 1 and Table 2.

The first set of experiments considers the demand effect for the narrower set of booking-limit policies considered in §4.2. Here, price, \( p \), and fixed bumping compensation, \( c \), are taken as given, and the airline chooses a booking limit to maximize expected revenues. In these experiments, we quantify
the benefit provided by the strategic booking-limit policies of §4.2.2, in which the airline recognizes that its booking limit affects customer demand, beyond that of the benchmark myopic setting of §4.2.1, in which it does not.

The 720 experiments cover parameter ranges designed to test a wide array of contexts, and we describe their details in Appendix C. For each of the 720 problem instances, we find the optimal myopic booking limit, $b_{m}^*$, and the optimal strategic booking limit, $b_{s}^*$. Customers react optimally to each booking limit, and the resulting bumping probabilities, $\beta_{m}^*$ and $\beta_{s}^*$, as well as the resulting expected profits, $E[\Pi_{m}^*]$ and $E[\Pi_{s}^*]$, reflect the resulting customer equilibrium. For each of the 720 problem instances, we compare the results obtained from the use of the myopic and the strategic booking limits by calculating relevant ratios, and for each ratio we sort the results of the 720 problem instances from smallest to largest and report relevant distributional statistics.

Table 1 reports the summary statistics. The first column reports those for the ratio of the optimal booking limits, the second displays statistics for the ratio of equilibrium bumping probabilities, and the third results for the ratio of the expected profits. The table shows that the demand effect has a significant impact in all three cases. As suggested by Proposition 3, the optimal strategic booking limit is never higher than the myopic analog, and furthermore the myopic booking limit is on average 1.64 times higher (64% higher) than its strategic counterpart. The median ratio for the bumping probabilities is 1.60, and the mean is unbounded due to cases in which the strategic airline does not overbook, so that $\beta_{s}^* = 0$. Finally, the mean and median ratios for expected profits are 1.27, suggesting the airline’s ability to account for demand effects can have a significantly positive impact for its revenue management.

The second set of experiments considers full control of price, bumping compensation, and booking limit and compares the optimal strategic fixed-compensation schemes of §4.3.3 to the optimal auction schemes of §5.2. In these auction schemes there is no cap on bumping compensation. We construct 360 problem instances that systematically vary relevant parameters, and again we describe their details in Appendix C. In addition to the statistics reported in Table 1, Table 2 reports analogous summary
statistics for the ratio of the optimal fixed-compensation value to the expected bumping cost associated with the optimal auction policy, $c^*_s / E[C]$.

From Table 2 we see that optimal expected bumping cost per passenger is significantly lower for the auction-based scheme (mean ratio of 4.54) and that, in turn, this allows the airline to significantly increase booking limits (mean ratio of 1.09) and bumping probabilities (mean ratio of 6.24). Optimal expected profits are always higher for the auction-based policies, and while the improvement in expected profits is relatively modest (mean ratio of 1.02) when compared to the analogous improvements seen in Table 1, we note that a 2% improvement itself can be significant in the RM context.

7 Managerial Implications and Limitations

Overbooking is widely used in practice and studied in the RM literature. Existing models of overbooking do not explicitly account for the demand effects that can accompany the bumping of passengers, however. One possible reason for this absence may be that, despite the wide adoption of overbooking, bumping probabilities tend to be low, on the order of tenths of a percent, and managers may assume that there is limited room for revenue increases from policy improvements. Nevertheless, the potential importance of these demand effects has been highlighted by recent events in the air travel industry, such as the widely publicized 2017 incident of a passenger who was unwillingly dragged off United Express Flight 3411.

Our numerical experiments confirm that fixed-compensation policies that account for demand endogeneity can, in fact, significantly outperform those that do not and that the use of auction-based policies brings smaller but still-significant additional gains. These results suggest that significant benefits may accrue to airlines that incorporate demand effects into their overbooking models and move to more profitable and customer-friendly auction-based compensation schemes.

Our paper also provides a theoretical basis to support these shifts in policy. In particular, our analytical results suggest that, in both the fixed and auction-based compensation settings, an effective means of managing overbooking is to ensure that customers who buy tickets are always fairly compensated for being bumped. This approach ensures that an airline maintains the good will of bumped customers and allows customers to decouple their initial purchase decisions from the possibility of being bumped: when deciding whether to purchase tickets, they need not know the bumping probability. These policies also allow the airline to significantly reduce the complexity of its overbooking policies.

These results are based on a relatively rich model of the operational context. We make limited distributional assumptions regarding customer demand and customer disutility from being bumped.
Similarly, our results hold under relatively limited informational requirements on the part of customers, who need only know their own preferences, observable statistics on the part of the airline and, potentially, an estimate of the bumping probability. Again, in the context of the families of policies described above, customers who are assured of fair bumping compensation need not estimate the bumping probability.

At the same time, our model also makes some limiting assumptions that may be interesting to relax in subsequent work. In particular, our focus on the disutility of being bumped has motivated us to assume that relevant customer heterogeneity is captured by a hassle cost distribution and that customers otherwise share a common valuation \( v \) for the flight. It would be valuable to study the empirical relationship between value and hassle cost to see how well our assumption fits with practice and to provide a more refined characterization of the relationship between the two.

Furthermore, while our results suggest that each airline, when considered in isolation, would do well to move to an auction-based policy, we do not directly model competitive factors that might drive airlines to choose other overbooking schemes (Netessine and Shumsky 2005). Similarly, in practice airlines’ use of multiple fare classes can affect consumer behavior (Cohen et al. 2019), and it would be useful to extend our analysis to this broader setting.

References


Appendix

A Ex Post versus Ex Ante Bumping Probability

Section 3.3, describes our rationale for modeling the customer’s estimate $\beta$ as the \textit{ex post} fill rate rather than the \textit{ex ante} bumping probability, $\beta'$. There are three main reasons behind this choice:

First, while aggregate \textit{ex post} statistics such as (7) are published by BTS and widely cited in the news and travel media, the estimation of \textit{ex ante} statistics such as (8) require reporting of the fractions of customers bumped from individual flights $((N_i - k)/N_i)$. These data are not reported by BTS and airlines do not make them available to passengers.

In fact, anecdotal evidence suggests the opposite. In direct discussions with one of us, a former airline employee noted that her employer prefer to hide the magnitude of passenger bumping at the gate, to avoid generating additional customer ill will. This assertion is all the more plausible, given the public outrage that occurred in the aftermath of the United Express Flight 3411 incident in 2017.

Second, suppose nevertheless a frequent flyer could observe data on the fraction of customers bumped on flights, $\{(N_i - k)/N_i \mid i = 1, 2, \ldots \}$ and wished to estimate the \textit{ex ante} statistic, $\beta'$. Practically speaking, bumping probabilities are generally low enough that it would require samples from many hundreds or thousands of flights for the customer to estimate the probability accurately. For example, among the 12 major US airlines tracked in the BTS statistics (BTS 2018), the 2017 annual statistics for $\beta$ range from 0.00007 to 0.00138, with a weighted average of 0.00054.

Similarly, we have calculated the $\beta$ and $\beta'$ associated with the optimal fixed-compensation policies evaluated §6 and, in absolute terms, differences between the two tend not to be large. In the 360 fixed-compensation examples evaluated in Table 2, the absolute difference between $\beta$ and $\beta'$ ranged from 0.023% to 0.319%, with a median of 0.079% and a mean of 0.095%; this difference would be difficult to infer from a limited numbers of samples.

To give a sense of the number of samples required to differentiate between the two statistics, we estimate the sample sizes needed to differentiate an alternative hypotheses, $H1 = \beta'$, from the null hypothesis, $H0 = \beta$, using one-sided, fixed-sample tests with Type I error of 0.1 and Type II error of 0.5 (Chow et al. 2008). Note that these are low-significance, low-powered tests that tend to minimize the required sample size.

Table 3 shows the distribution of results, which range from the low thousands to the several tens of thousands. If a frequent flyer observed the fraction of customers bumped from one of our example flights every day, it would take her at least a few years – and possibly several decades – to disambiguate $\beta'$ from $\beta$. 
Third, if we informally consider the equilibrium modelled in this paper as the one-stage stationary outcome of repeated customer purchase decisions, we can see that, if customers begin with an initial estimate of bumping that conforms to the published estimate of $\beta$, they will tend to stick with the paper’s equilibrium model. That’s because, although the equilibrium $\beta$ and $\beta'$ tend to be quite close to each other, one can show that $\beta \geq \beta'$, a fact we demonstrate at the bottom of this appendix.

More specifically, suppose customers use $\beta$ as an initial estimate of $\beta'$. Those customer for whom $U(\beta, w) < 0$ will not buy a ticket, even if $U(\beta', w) \geq 0$. Conversely, those for whom $U(\beta, w) \geq 0$ do buy tickets and would continue buying as their estimates slip from $\beta$ to $\beta'$ and their utility increases. Of course, a detailed analysis of sample-path fluctuations would show that some customers for whom $U(\beta, w) \geq 0$ would erroneously conclude that $U(\beta', w) < 0$ and stop buying tickets. (These customer-statisticians might use more highly powered tests than ours, tests that require larger sample sizes than those we estimate in Table 3.) and the At the same time, these are second order effects and this type of analysis is well beyond the scope of our current paper.

Finally we demonstrate that $\beta \geq \beta'$. We begin by recalling the following result.

**Lemma 9.** (Wijsman (1985), Theorem 2)

Let $\mu$ be a measure on the real line $\mathbb{R}$ and let $f_i, g_i \ (i = 1, 2)$ be four Borel-measurable functions: $\mathbb{R} \to \mathbb{R}$ such that $f_2 \geq 0$, $g_2 \geq 0$ and $\int |f_i g_i| \, d\mu < \infty \ (i, j = 1, 2)$. If $f_1/f_2$ and $g_1/g_2$ are monotonic in the same direction, then $\int f_1 g_1 \, d\mu \int f_2 g_2 \, d\mu \geq \int f_1 g_2 \, d\mu \int f_2 g_1 \, d\mu$.

Let $\mu$ be the cumulative distribution function of $N$ and take $f_1(N) = \frac{(N-k)^+}{N}$, $f_2(N) = (N-k)^+$, $g_1(N) = \frac{N}{(N-k)^+}$, and $g_2(N) = 1$. Then $f_2 \geq 0$, $g_2 \geq 0$, $\int |f_1 g_1| \, d\mu = \int f_1 \, d\mu = 1 < \infty$, $\int |f_1 g_2| \, d\mu = \int \frac{(N-k)^+}{N} \, d\mu < \int 1 \, d\mu = 1 < \infty$, $\int |f_2 g_1| \, d\mu = \int N \, d\mu = \mathbb{E}[N] \leq \mathbb{E}[Q] < \infty$ and $\int |f_2 g_2| \, d\mu = \int (N-k)^+ \, d\mu = \mathbb{E}[(N-k)^+] < \mathbb{E}[N] < \infty$. We also see that both $f_1/f_2 = 1/N$ and $g_1/g_2 = \frac{N}{(N-k)^+}$ are monotonically decreasing in $N$. We then apply Lemma 9 and get

$$1 \cdot \mathbb{E}[(N-k)^+] = \int f_1 g_1 \, d\mu \int f_2 g_2 \, d\mu \geq \int f_1 g_2 \, d\mu \int f_2 g_1 \, d\mu = \mathbb{E}\left[\frac{(N-k)^+}{N}\right] \cdot \mathbb{E}[N]$$

and thus

$$\beta = \frac{\mathbb{E}[(N-k)^+]}{\mathbb{E}[N]} \geq \mathbb{E}\left[\frac{(N-k)^+}{N}\right] = \beta'.$$

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<td>36,405</td>
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Table 3: Percentiles of Sample Size Needed to Distinguish $H_1 = \beta'$ from $H_0 = \beta$
B Proofs

Lemma 1. (Properties of Loss Function Satisfied)

For a plane with \( k \) seats and loss function \( \ell(s, k, \alpha) = (N(s, \alpha) - k)^+ \):

(i) \( N(s, \alpha) \sim B(s, 1 - \alpha) \) satisfies the discrete analogue of properties (i)–(iv) of Definition 2; and

(ii) \( N(s, \alpha) \sim N((1 - \alpha)s, \sqrt{\alpha(1 - \alpha)s}) \) satisfies properties (i), (iii), and (iv) of Definition 2.

Proof. Recall that Definition 2 states that (i) \( \ell''(s) \geq 0 \); (ii) \( \ell(s, k, \alpha) = 0 \) for all \( s \leq k \) and \( \ell'(s) = 0 \) for all \( s < k \); (iii) \( \ell'(s) \in (0, 1 - \alpha) \) for all \( s \in [k, \infty) \); (iv) \( \lim_{s \to \infty} \ell'(s) = 1 - \alpha \).

Part (i). We first consider \( N(s, \alpha) \sim B(s, 1 - \alpha) \) and verify the properties in the following order: (i) and (ii), (iv), and (iii).

For properties (i) and (ii) we have the following. As in Section 5.4.1 we note that \( N(s, \alpha) \sim B(s, 1 - \alpha) \) is stochastically increasing and convex (SICX) in \( s \). (See Section 6.A.1, including Example 6.A.2, in Shaked and Shanthikumar (1994)). This implies that \( E[N(s, \alpha)] \) is increasing in \( s \) for all increasing \( \psi(\cdot) \) and increasing, convex in \( s \) for all increasing, convex \( \psi(\cdot) \). We let \( \psi(x) = (x - k)^+ \), the maximum of two increasing, convex functions, 0 and \( x - k \), which implies that \( \psi(x) \) is increasing and convex. It then follows that \( \ell(s) = E[(N(s, \alpha) - k)^+] = E[\psi(N(s, \alpha))] \) is increasing and convex in \( s \),

Now let's show the discrete analogue of property (iv). Note that

\[
\ell'(s) = \ell(s + 1) - \ell(s) = E[(N(s + 1, \alpha) - k)^+] - E[(N(s, \alpha) - k)^+] = (E[N(s + 1, \alpha) - k] + E[(k - N(s + 1, \alpha))^+]) - (E[N(s, \alpha) - k] + E[(k - N(s, \alpha))^+]) = (s + 1)(1 - \alpha) - k + E[k - N(s + 1, \alpha)|N(s + 1, \alpha) < k] P(N(s + 1, \alpha) < k) - (s(1 - \alpha) - k) - E[k - N(s, \alpha)|N(s, \alpha) < k] P(N(s, \alpha) < k) = 1 - \alpha + E[k - N(s + 1, \alpha)|N(s + 1, \alpha) < k] P(N(s + 1, \alpha) < k) - E[k - N(s, \alpha)|N(s, \alpha) < k] P(N(s, \alpha) < k).
\]

By Chebyshev’s inequality, \( P(|N(s, \alpha) - \mu| > z\sigma) \leq 1/z^2 \) where in our case \( \mu = (1 - \alpha)s \) and \( \sigma = \sqrt{\alpha(1 - \alpha)s} \). Therefore, \( P(N(s, \alpha) < \mu - z\sigma) = P(\mu - N(s, \alpha) > z\sigma) \leq P(|N(s, \alpha) - \mu| > z\sigma) \leq 1/z^2 \).

Let \( k = \mu - z\sigma \). Then \( z = \frac{\mu - k}{\sigma} = \frac{(1 - \alpha)s - k}{\sqrt{\alpha(1 - \alpha)s}} \). Therefore,

\[
P(N(s, \alpha) < k) \leq \frac{s\alpha(1 - \alpha)}{(1 - \alpha)s - k)^2} \to 0 \text{ as } s \to \infty. \tag{39}
\]

Similarly \( P(N(s + 1, 1 - \alpha) < k) \to 0 \) as \( s \to \infty \). Therefore, by (38), we have \( \lim_{s \to \infty} \ell'(s) = 1 - \alpha \).

Last, by property (iv) and the fact that \( \ell'(s) > 0 \), we have property (iii).

Part (ii). We now derive properties of interest for \( \ell(s) \) under \( N(s, \alpha) \sim N((1 - \alpha)s, \sqrt{\alpha(1 - \alpha)s}) \) in the order (iv), (i), (iii).

For sales of \( s \) and a capacity of \( k \), it is well known that the expected loss – in our case, the expected number of lost customers – equals \( \sigma L(z) \), where \( L(z) = \phi(z) - z(1 - \Phi(z)) \) is the standard normal loss function, \( \sigma = \sqrt{\alpha(1 - \alpha)s} \) and \( z = \frac{k - (1 - \alpha)s}{\sqrt{\alpha(1 - \alpha)s}} \). (See §12.5 in Cachon and Terwiesch (2013).)
There is no equilibrium.

Therefore,

\[ \ell(s) = \sigma(s) L(z(s)) = \sqrt{\alpha(1-\alpha)s} L \left( \frac{(k-(1-\alpha)s)}{\sqrt{\alpha(1-\alpha)}} \right). \]  

(40)

Noting that \( L'(z) = \Phi(z) - 1, \sigma'(s) = \frac{\sigma(s)}{2s}, \) and \( z'(s) = -\left[ \frac{1-\alpha}{\sigma(s)} + \frac{z(s)}{2s} \right], \) we can differentiate with respect to \( s \) and collect terms to obtain

\[
\ell'(s) = \sigma'(s) L(z(s)) + \sigma(s) L'(z(s)) z'(s)
\]

\[
= \frac{\sigma(s)}{2s} (\phi(z(s)) - z(s)(1 - \Phi(z(s)))) + \sigma(s)(1 - \Phi(z(s))) \left( \frac{1-\alpha}{\sigma(s)} + \frac{z(s)}{2s} \right)
\]

\[
= \frac{\sigma(s)}{2s} \phi(z(s)) + (1-\alpha)(1-\Phi(z(s))).
\]

(41)

Note, for any \( s > 0, \) we have \( \lim_{s \to \infty} \ell'(s) = (1-\alpha), \) property (iv).

We can then differentiate (41) again with respect to \( s \) to find \( \ell''(s). \)

\[
\ell''(s) = \frac{2s \sigma'(s) - 2\sigma(s)}{4s^2} \phi(z(s)) + \frac{\sigma(s)}{2s} \phi'(z(s)) z'(s) - (1-\alpha) \phi(z(s)) z'(s)
\]

(42)

We next recall that \( \phi'(z) = -z \phi(z). \) We use this identity, along with those for \( \sigma'(s) \) and \( z'(s), \) to substitute out terms with derivatives in (42). Collecting terms, we have

\[
\ell''(s) = \left[(\sigma(s)z(s) + 2s(1-\alpha))^2 - \sigma^2(s)\right] \frac{\phi(z(s))}{4s^2 \sigma(s)}.
\]

(43)

To ensure that \( \ell''(s) > 0, \) we need \( \sigma(s)z(s) + 2s(1-\alpha) - \sigma(s) > 0, \) which is equivalent to \( k - (1-\alpha)s + 2s(1-\alpha) > \sqrt{\alpha(1-\alpha)s}. \) Squaring both sides, this is \( k^2 + 2ks(1-\alpha) + s^2(1-\alpha)^2 - \alpha(1-\alpha)s > 0. \) The same inequality can be rearranged as

\[
\left[k + \frac{(2k-\alpha)(1-\alpha)}{2k}\right]^2 + s^2(1-\alpha)^2 \left[1 - \frac{(2k-\alpha)^2}{4k^2}\right] > 0.
\]

(44)

A sufficient condition for (44) to hold is \( 1 - \frac{(2k-\alpha)^2}{4k^2} \geq 0, \) which can be simplified to \( k \geq \frac{\alpha}{4}. \) We have thus shown that \( \ell''(s) > 0 \) when \( k \geq \frac{\alpha}{4}, \) which practically holds given \( k \geq 1. \) We have thus shown that property (i) holds.

Last, by (41), we know that \( \ell'(s) > 0. \) Since \( \ell''(s) > 0 \) and \( \lim_{s \to \infty} \ell'(s) = (1-\alpha), \) we must have \( \ell'(s) \in (0,1-\alpha) \) for all \( s \in [k,\infty), \) property (iii).

\( \square \)

**Lemma 2.** (Existence and Uniqueness of Equilibria)

(i) For overbooking policies with \( p = (1-\alpha)v, b > k, \) and \( c \leq w \) there is no equilibrium.

(ii) For all other overbooking policies \( \xi \in \Xi, \) there exists at least one equilibrium.

(iii) For the policies in part (ii), if \( g'(w) \leq 0, \forall w \in [c,\bar{w}], \) then \( \exists \) a unique equilibrium \( \{\beta,\tilde{w}\}. \)

**Proof.**

**Part (i).** The existence of equilibrium means that there exists at least one \( \tilde{w} \) such that

\[ U(\beta(\tilde{w}),\tilde{w}) = -p + (1-\alpha)v + (1-\alpha)\beta(\tilde{w})(c - \tilde{w}) \geq 0, \]  

(45)
where, from (3)-(6) and (12),

$$
\beta(w) = \frac{\mathbb{E}[\ell(\min\{b, QG(w)\})]}{\mathbb{E}[\min\{b, QG(w)\}]}.
$$

(46)

Consider a policy with \( c \leq w, b > k \) and \( p = (1 - \alpha)v \). Suppose there is an equilibrium \((\bar{w}, \beta(\bar{w}))\). Then we know that \( U(\beta(\bar{w}), \bar{w}) = 0 + (1 - \alpha)\beta(\bar{w})(c - \bar{w}) \leq 0 \) regardless of the value of \( \beta(\bar{w}) \) because \( \bar{w} \geq w \geq c \). From (2) we know that this implies \( \bar{w} = w \). Now we show by l'Hôpital's rule that \( \beta(w) = 0 \).

By (46),

$$
\lim_{w \to w} \beta(w) = \lim_{w \to w} \frac{\mathbb{E}[\ell(\min\{b, QG(w)\})]}{(1 - \alpha)\mathbb{E}[\min\{b, QG(w)\}]} = \frac{\lim_{w \to w} \mathbb{E}[\ell(\min\{b, QG(w)\})]}{\lim_{w \to w} (1 - \alpha)\mathbb{E}[\min\{b, QG(w)\}]},
$$

(47)

by (21) and (22). The denominator of (47) is positive and finite, since \( \mathbb{E}[Q] < \infty \) and \( g(w) > 0 \) over its support, and the numerator equals 0. Thus, \( \lim_{w \to w} \beta(w) = 0 \) in this policy does not yield a consistent \( \bar{w} \).

**Part (ii).** We divide the space of admissible overbooking policies into four partitions: 1) \( b = k \), 2) \( b > k, w < c \leq \bar{w} \), 3) \( b > k, c \leq w \) and \( p < (1 - \alpha)v \), 4) \( b > k, c \leq w \) and \( p = (1 - \alpha)v \). Recall that in part (i) we have shown that policies in 4) obtain no equilibrium. We will now show the existence of \( \bar{w} \) that satisfies (45) for all policies in 1), 2) and 3).

1) When \( b = k \), the equilibrium bumping probability is \( \beta = 0 \). By Definition 1, customers obtain non-negative utility from buying a ticket. The corresponding equilibrium \( \bar{w} \) is \( \bar{w} \) and thus \( \bar{w} = \bar{w} \). Since \( w \neq \bar{w} \), this policy does not yield a consistent \( \bar{w} \).

2) Next consider policies with \( b > k \) and \( w < c \leq \bar{w} \). We know that \( U(\beta(w), w) \geq 0 \) for all \( w \in [w, c] \). If \( U(\beta(\bar{w}), \bar{w}) \geq 0 \), then \( \bar{w} = \bar{w} \). If \( U(\beta(\bar{w}), \bar{w}) < 0 \), then since \( U(\beta(c), c) \geq 0 \), by the Intermediate Value Theorem, there exists at least one \( \bar{w} \in (c, \bar{w}) \) such that \( U(\beta(\bar{w}), \bar{w}) = 0 \).

3) Finally, consider policies with \( b > k \) and \( c \leq w \) and \( p < (1 - \alpha)v \), we have \( c - \bar{w} \leq 0 \) and \( -p + (1 - \alpha)v > 0 \). When \( w = \bar{w} \), \( \beta(w) = 0 \) and thus \( U(\beta(w), w) \geq 0 \). If \( U(\beta(w), w) \geq 0 \) for all \( w \in (w, \bar{w}) \), then \( \bar{w} = \bar{w} \). Otherwise, by the Intermediate Value Theorem, there exists at least one \( \bar{w} \in (w, \bar{w}) \) such that \( U(\beta(\bar{w}), \bar{w}) = 0 \).

**Part (iii).** Let \( U(\beta(w), w) = -p + (1 - \alpha)v + (1 - \alpha)h(w) \) where \( h(w) = \beta(w)(c - w) \) and \( \beta(w) \) is defined as in (46). Note, if \( U(\beta(w), w) \geq 0 \) for all \( w \), then the unique equilibrium \( \bar{w} \) is \( \bar{w} \). Otherwise, by contradiction suppose that there are multiple solutions (zeros of \( U(\beta(w), w) \)) and \((\bar{w}_1, \beta_1)\), with \((\bar{w}_2, \beta_2)\) being two of them, \((\bar{w}_1, \bar{w}_2) \in [c, \bar{w}]\). Because \( U(\beta(\bar{w}_1), \bar{w}_1) = U(\beta(\bar{w}_2), \bar{w}_2) = 0 \), there must exist some \( w \in [\bar{w}_1, \bar{w}_2] \) such that \( \frac{dU(\beta(w), w)}{dw} = (1 - \alpha)\frac{dh(w)}{dw} = 0 \). Therefore, we can show that, if
If \( \frac{g'(w)}{g(w)} \neq 0 \), then \( U(\beta(w), w) \) decreases in \( w \) and therefore has at most one zero. In this case, \( \min_{w \in [c, \bar{w}]} U(\beta(w), w) = U(\beta(\bar{w}), \bar{w}) \) and \( \max_{w \in [c, \bar{w}]} U(\beta(w), w) = U(\beta(c), c) \). Note that \( U(\beta(c), c) \geq 0 \). If \( U(\beta(\bar{w}), \bar{w}) \geq 0 \), then \( \hat{w} = \bar{w} \). Otherwise, there exists exactly one \( w \) that satisfies \( U(\beta(w), w) = 0 \), and that \( w \) is \( \hat{w} \).

The uniqueness condition derived from as (48) can be rewritten as

\[
\psi(w) \equiv w - \frac{G'(w)}{g(w)} \leq c.
\]

Taking the derivative of \( \psi(w) \) w.r.t. \( w \), we have \( \frac{\partial \psi(w)}{\partial w} = \frac{G'(w)}{g^2(w)} g'(w) \). Since \( \psi(c) = c - \frac{g(c)}{G(c)} < c \), as long as \( g'(w) \leq 0, \frac{\partial \psi(w)}{\partial w} \leq 0 \) for \( w \in [c, \bar{w}] \), and (49) is satisfied.

**Specific Distributions**

If the hassle cost is uniformly distributed, then \( g'(w) = 0 \) and \( \frac{\partial \psi(w)}{\partial w} = 0 \) and (49) always holds. Equation (49) also holds for exponentially distributed hassle cost because \( g'(w) < 0 \). Now consider normally distributed hassle cost, \( W \sim \mathcal{N}(\mu, \sigma^2) \). When \( w > \mu \), \( g'(w) < 0 \) and thus \( \psi'(w) < 0 \). (49) is automatically satisfied since \( c > c - \frac{g(c)}{G(c)} = \psi(c) \geq \psi(w) = w - \frac{g(w)}{G(w)} \) for all \( w \geq c \). When \( w < \mu \), \( g'(w) > 0 \) and thus \( \psi'(w) > 0 \). Thus, (49) holds if \( c > \psi(\mu) = \mu - \frac{g(\mu)}{G(\mu)} \) because \( \psi(\mu) \geq \psi(w) \) for all \( w \geq c \). Therefore, we conclude that, for normally distributed \( W \), the uniqueness condition holds if \( c > \mu - \frac{g(\mu)}{G(\mu)} \).

**Lemma 3.** (Ordering of Equilibria)

Suppose an overbooking policy \( \xi \in \Xi \) induces multiple equilibria. Pick any two distinct equilibria from the set, and call them \( (\beta_1, \hat{w}_1) \neq (\beta_2, \hat{w}_2) \).

(i) Without loss of generality, we can order the two so that the second equilibrium has a strictly lower bumping probability and a strictly higher marginal hassle cost: \( \beta_1 > \beta_2 \) and \( \hat{w}_1 < \hat{w}_2 \).

(ii) Given the ordering in (i), the set of customers with \( w \leq \hat{w}_1 \) is a strict subset of those with \( w \leq \hat{w}_2 \), and the airline earns strictly higher expected profits in \( (\beta_2, \hat{w}_2) \).

**Proof.**

**Part (i).** Our proof proceeds in five steps.

First, suppose \( (\beta_1, \hat{w}_1) \) and \( (\beta_2, \hat{w}_2) \) are distinct equilibria. Then without loss of generality, we can assume \( \hat{w}_1 < \hat{w}_2 \). This is because \( \beta \) can be expressed as a function of \( \hat{w} \), as in (46), so \( \hat{w}_1 = \hat{w}_2 \) implies \( \beta_1 = \beta_2 \).
Second, we show that $U(\beta, \hat{w}_1) = 0$. Since $\underline{w} \leq \hat{w}_1 < \hat{w}_2 \leq \overline{w}$, we have $\hat{w}_1 < \overline{w}$, and from (2) we know this implies $U(\beta_1, \hat{w}_1) \leq 0$. By contradiction, suppose that $U(\beta_1, \hat{w}_1) < 0$. Then from (2) we also know that $\hat{w}_1 = \overline{w}$, and as in (47) we can show that this implies $\beta_1 = 0$. At the same time, since $\xi \in \Xi$ is admissible, $p \leq (1 - \alpha)v$, and since $\beta_1 = 0$, (2) implies that $U(\beta_1, \hat{w}_1) = -p + (1 - \alpha)v + (1 - \alpha)\beta_1(c - \hat{w}_1) = -p + (1 - \alpha)v \geq 0$, a contradiction.

Third, we show that, if there exist multiple equilibria, then we must have $\beta_1 > 0$. By contradiction, suppose not. Then we have $\underline{w} \leq \hat{w}_1 < \hat{w}_2 \leq \overline{w}$ and, because $\beta_1 = 0$, (1) shows that $U(\beta_1, w) = -p + (1 - \alpha)v + (1 - \alpha)\beta_1(c - w) \geq 0$, for all $w \in [\underline{w}, \overline{w}]$, including $w > \hat{w}_1$. Thus $\hat{w}_1 < \overline{w}$ is not an equilibrium threshold customer response, so $U(\beta_1, \hat{w}_1) = 0$ is not an equilibrium, a contradiction.

Fourth, we show that, if $\beta_1 > 0$ then $\beta_2 > 0$ as well. If we look at the definition of $\beta$ in (46), we see that the numerator, $E[\ell(\min\{b, QG(\hat{w})\})$, is increasing in $\overline{w}$. Thus, we have $0 < E[\ell(\min\{b, QG(\hat{w})\}] < E[\ell(\min\{b, QG(\hat{w})\})].$ While we do not know how the ratio in (46) that determines $\beta$ changes, we do know that $\beta_2 > 0$.

Finally, we now have $p \leq (1 - \alpha)v$, $\underline{w} \leq \hat{w}_1 < \hat{w}_2 \leq \overline{w}$, $\beta_1, \beta_2 > 0$, and

$$0 = U(\beta_1, \hat{w}_1) = -p + (1 - \alpha)v + (1 - \alpha)\beta_1(c - \hat{w}_1) \leq -p + (1 - \alpha)v + (1 - \alpha)\beta_2(c - \hat{w}_2) = U(\beta_2, \hat{w}_2),$$

so that $\beta_1(c - \hat{w}_1) \leq \beta_2(c - \hat{w}_2)$. Note that $\hat{w}_1 < \hat{w}_2$ implies that $c - \hat{w}_1 > c - \hat{w}_2$. Furthermore, $U(\beta_1, \hat{w}_1) = 0$ and $p \leq (1 - \alpha)v$ imply that $c - \hat{w}_1 \leq 0$, so we have $c - \hat{w}_2 < c - \hat{w}_1 \leq 0$. Given $c - \hat{w}_2 < 0$, $\beta_1, \beta_2 > 0$, and $\beta_1(c - \hat{w}_1) \leq \beta_2(c - \hat{w}_2)$, we then have

$$\frac{\beta_2}{\beta_1} \leq \frac{c - \hat{w}_1}{c - \hat{w}_2} = \frac{\hat{w}_1 - c}{\hat{w}_2 - c} < 1.$$ 

Thus for any two distinct equilibria, we can order them so that $\beta_1 > \beta_2$ and $\hat{w}_1 < \hat{w}_2$.

Part (ii). Let $\Pi_i = pE[S_i] - (c + r)E[(N_i - k)^+] = E[S_i](p - (1 - \alpha)(c + r)\beta_i)$ for $i = 1, 2$. $\hat{w}_1 < \hat{w}_2$ leads to $E[N_1] < E[N_2]$ and $\beta_1 > \beta_2$ gives $-(c + r)\beta_1 < -(c + r)\beta_2$. Therefore, $\Pi_1 < \Pi_2$.

**Proposition 1.** (Optimal Myopic Booking Limit)

Given fixed, admissible $p$ and $c$, the optimal myopic booking limit, $b^*_m$, behaves as follows.

(i) If $p - (1 - \alpha)(c + r) \geq 0$, then $b^*_m = \infty$, and the airline does not impose a booking limit.

(ii) If $p - (1 - \alpha)(c + r) < 0$, then there exists a unique optimal $b^*_m = \max \left\{ \ell'^{-1} \left( \frac{p}{c + r} \right), k \right\}$.

(iii) When $b^*_m \in (k, \infty)$, $\frac{\partial \Pi_1}{\partial b} > 0$ for $b < b^*_m$ and $\frac{\partial \Pi_1}{\partial b} < 0$ for $b > b^*_m$.

**Proof.** Recall the FOC given by (19): $p - \ell'(b)(c + r) = 0$.

Part (i). By Definition 2 part (iii), $p - \ell'(b)(c + r) > p - (1 - \alpha)(c + r)$. Therefore, when $p - (1 - \alpha)(c + r) \geq 0$, the marginal increase in profit is always positive and the airline is incentivized to overbook as much as possible.

Part (ii). When $p - (1 - \alpha)(c + r) < 0$, if $p - \ell'(k)(c + r) > 0$, the FOC (19) has a solution by the Intermediate Value Theorem. By Definition 2 part (iii), $\ell'(b)$ increases monotonically in $b$ for $b \geq k$ from 0 to $1 - \alpha$, hence the solution is unique. If $p - \ell'(k)(c + r) \leq 0$, then $b^*_m = k$.

Part (iii). We know that $\frac{\partial \Pi_1}{\partial b} = \left[ p - \ell'(b)(c + r) \right] P \{ QG(\hat{w}) > b \}$ and $p - \ell'(b^*_m)(c + r) = 0$. By Definition 2 part (i), $p - \ell'(b)(c + r) > 0$ for $b < b^*_m$ and $p - \ell'(b)(c + r) < 0$ for $b > b^*_m$. 

□
Proposition 2. (Optimal Myopic Booking Limit is Profit-Making)

(i) The equilibrium induced by any \( \xi \in \Xi \) obtains \( \ell'(b) > (1-\alpha)\beta \).

Suppose \( p > 0 \).

(ii) If \( \beta = 0 \), or if either \( p < (1-\alpha)v \) or \( c > w \) or both, then \( b_m^* \) induces a profit-making equilibrium.

Proof.

Part (i). Because \( \beta = \frac{\mathbb{E}[(N-k)^+]}{\mathbb{E}[N]} \) and \( \ell(b) > (1-\alpha)\beta \), it is sufficient to show that \( \ell'(b) \mathbb{E}[S] > \mathbb{E}[(N-k)^+] \). Expanding this expression, this is equivalent to showing

\[
\ell'(b) \left( \int_0^b qG(\hat{w}) f(q) dq + \int_b^\infty b f(q) dq \right) > \int_0^b \ell(qG(\hat{w})) f(q) dq + \int_b^\infty \ell(b) f(q) dq. \tag{50}
\]

Rearranging (50) gives

\[
\int_0^b [qG(\hat{w}) \ell'(b) - \ell(qG(\hat{w}))] f(q) dq + \int_b^\infty [b \ell'(b) - \ell(b)] f(q) dq > 0. \tag{51}
\]

Note that \( \ell(b) = \int_0^b \ell'(t)dt < \int_0^b \ell'b^pdt = b\ell'(b) \) for all \( b > k \) and \( \ell(b) = b\ell'(b) = 0 \) for all \( b \leq k \). Therefore, the second integral of (51) is positive. To see that the first integral of (51) is non-negative, note that by the convexity of \( \ell(\cdot) \), \( qG(\hat{w}) \ell'(b) \geq qG(\hat{w})\ell'(qG(\hat{w})) \) for \( q \leq \frac{b}{G'(w)} \). Therefore, \( qG(\hat{w})\ell'(b) - \ell(qG(\hat{w})) \geq qG(\hat{w})\ell'(qG(\hat{w})) - \ell(qG(\hat{w})) \) for \( q \leq \frac{b}{G'(w)} \). Again by \( \ell(b) \leq b\ell'(b) \) for any \( b \), we know that \( qG(\hat{w})\ell'(qG(\hat{w})) - \ell(qG(\hat{w})) \geq 0 \).

Part (ii). For \( \beta = 0 \), we have \( \hat{w} = w \), so \( p, \mathbb{E}[S] > 0 \) and \( \mathbb{E}[\Pi(p, b_m^*, c)] = p\mathbb{E}[S] - (c + r)\mathbb{E}[(N-k)^+] = p\mathbb{E}[S] > 0 \). Otherwise, taking the FOC in (19) \( p - \ell'(b_m^*)(c + r) = 0 \) and using the inequality from part (i) to substitute for \( \ell'(b_m^*) \), we immediately have \( [p - (1-\alpha)\beta(c + r)] > 0 \), where \( \beta \) is the equilibrium bumping probability induced by \( b_m^* \). Since \( c > w \) or \( p < (1-\alpha)v \), the \( \hat{w} \) induced by \( b_m^* \) must be greater than \( w \) and thus \( \mathbb{E}[S] = \int_0^{b_m^*} qG(\hat{w}) f(q) dq + \int_{b_m^*}^\infty b_m^* f(q) dq > 0 \). By Definition 3, the result follows.

Proposition 3. (Optimal Strategic Booking Limit)

Suppose \( \exists p \in (0, (1-\alpha)v) \) and \( c \in (0, \overline{w}) \) for which \( b_m^* \in (k, \infty) \) induces a profit-making equilibrium \( \hat{w} \in (w, \overline{w}) \). Then we have the following.

(i) For any given \( b > k \), if \( \beta \geq \sqrt{(v - \frac{p}{1-\alpha})\frac{\bar{g}(w)}{G(w)}} \), then \( \frac{d\hat{w}}{db} < 0 \).

(ii) In turn, if \( \beta > \sqrt{(v - \frac{p}{1-\alpha})\frac{\bar{g}(w)}{G(w)}} \) for all \( b > k \), then \( b_s^* < b_m^* \).

Proof. We continue to consider policy parameters and equilibria that allow us to develop relevant FOCs. These include policies for which \( p \in (0, (1-\alpha)v) \), \( c \in (0, \overline{w}) \), and \( b \in (k, \infty) \). For the same reason, we will assume that the policy \( (p, b, c) \) obtains an interior equilibrium \( U(\beta, \hat{w}) = 0 \) for which \( \hat{w} \in (w, \overline{w}) \).

Part (i). Since \( U(\beta, \hat{w}) = 0 \), by (1) we can express the break-even hassle cost as

\[
\hat{w} = c + \left( v - \frac{p}{1-\alpha} \right) \frac{1}{\beta}. \tag{52}
\]
Note that since \( b > k \), we have \( \beta > 0 \) and thus \( 1/\beta \) is well-defined.

Differentiating \( \hat{w} \) with respect to \( b \) according to (52),

\[
\frac{d\hat{w}}{db} = \left( v - \frac{p}{1 - \alpha} \right) \frac{E[(N - k)^+] \frac{dE[N]}{db} - E[N] \frac{dE[(N - k)^+]}{db}}{E[(N - k)^+]^2} \\
= \left( v - \frac{p}{1 - \alpha} \right) \frac{E[(N - k)^+] \frac{\partial E[N]}{\partial b} - E[N] \frac{\partial E[(N - k)^+]}{\partial b}}{E[(N - k)^+]^2} \\
+ \left( v - \frac{p}{1 - \alpha} \right) \frac{E[(N - k)^+] \frac{\partial E[N]}{\partial w} - E[N] \frac{\partial E[(N - k)^+]}{\partial w}}{E[(N - k)^+]^2} \frac{d\hat{w}}{db}.
\]

Rearranging (53) and applying (6), (16), (18), (21) and (22), we have

\[
\frac{d\hat{w}}{db} = \left( v - \frac{p}{1 - \alpha} \right) \left[ (1 - \alpha) \beta \frac{\partial E[S]}{\partial b} - \frac{\partial E[(N - k)^+]}{\partial b} \right] \\
\beta^2 E[N] - \left( v - \frac{p}{1 - \alpha} \right) \left[ (1 - \alpha) \beta \frac{\partial E[S]}{\partial w} - \frac{\partial E[(N - k)^+]}{\partial w} \right] \\
= \frac{\beta^2 E[N] - \left( v - \frac{p}{1 - \alpha} \right) g(\hat{w}) \int_0^b \left[ (1 - \alpha) \beta - \ell'(qG(\hat{w})) \right] qf(q) dq}{P\{QG(\hat{w}) > b\} \ [(1 - \alpha) \beta - \ell'(b)]} \\
\beta^2 E[N] + \left( v - \frac{p}{1 - \alpha} \right) g(\hat{w}) \int_0^b \left[ \ell'(qG(\hat{w})) - (1 - \alpha) \beta \right] qf(q) dq \]

\[
= \frac{\beta^2 E[N] + \left( v - \frac{p}{1 - \alpha} \right) g(\hat{w}) \int_0^b \left[ \ell'(qG(\hat{w})) - (1 - \alpha) \frac{E[(N - k)^+]}{E[N]} \right] qf(q) dq}{P\{QG(\hat{w}) > b\} \ [(1 - \alpha) \beta - \ell'(b)]} \\
= \frac{\beta^2 E[N] + \left( v - \frac{p}{1 - \alpha} \right) g(\hat{w}) \int_0^b \left[ E[N] \ell'(qG(\hat{w})) - (1 - \alpha) E[(N - k)^+] \right] qf(q) dq}{P\{QG(\hat{w}) > b\} \ [(1 - \alpha) \beta - \ell'(b)]}.
\]

The numerator of (54) is negative by Proposition 2 part (i), therefore \( \frac{d\hat{w}}{db} < 0 \) if and only if the denominator of (54) is positive. Note that the integral in the denominator

\[
\int_0^b \left( E[N] \ell'(qG(\hat{w})) - E[(N - k)^+] \right) (1 - \alpha) qf(q) dq
\]

\[
= \int_0^b \left( E[N] \ell'(b) - E[(N - k)^+] \right) (1 - \alpha) qf(q) dq \\
+ \int_0^b \left( E[N] \ell'(qG(\hat{w})) - E[N] \ell'(b) \right) qf(q) dq,
\]

and the first integral of (55) is positive by Proposition 2 part (i). Therefore, it suffices to have

\[
\frac{E[(N - k)^+]^2}{E[N]} + \left( v - \frac{p}{1 - \alpha} \right) g(\hat{w}) \int_0^b \left[ E[N] \ell'(qG(\hat{w})) - E[N] \ell'(b) \right] qf(q) dq \geq 0,
\]

which is equivalent to

\[
\left( v - \frac{p}{1 - \alpha} \right) g(\hat{w}) \int_0^b \left[ \ell'(b) - \ell'(qG(\hat{w})) \right] qf(q) dq \leq \frac{E[(N - k)^+]^2}{E[N]}.
\]
By Definition 2 part (iii), we know that \( \ell'(b) - \ell'(qG(\hat{w})) < 1 - \alpha \) for \( q \leq \frac{b}{\alpha G(\hat{w})} \). Then, the left-hand side of (56) is
\[
\left(v - \frac{p}{1-\alpha}\right) g(\hat{w}) \int_0^{\frac{b}{\alpha G(\hat{w})}} [\ell'(b) - \ell'(qG(\hat{w}))] qf(q) dq
\]

\[
< \left(v - \frac{p}{1-\alpha}\right) g(\hat{w}) \int_0^{\frac{b}{\alpha G(\hat{w})}} [1-\alpha] qf(q) dq
\]

\[
= \left(v - \frac{p}{1-\alpha}\right) \frac{g(\hat{w})}{G(\hat{w})} [1-\alpha] \int_0^{\frac{b}{\alpha G(\hat{w})}} qG(\hat{w}) f(q) dq
\]

\[
< \left(v - \frac{p}{1-\alpha}\right) \frac{g(\hat{w})}{G(\hat{w})} [1-\alpha] \left[ \int_0^{\frac{b}{\alpha G(\hat{w})}} qG(\hat{w}) f(q) dq + \int_{\frac{b}{\alpha G(\hat{w})}}^{\infty} bf(q) dq \right]
\]

\[
= \left(v - \frac{p}{1-\alpha}\right) \frac{g(\hat{w})}{G(\hat{w})} [1-\alpha] E[S]
\]

\[
= \left(v - \frac{p}{1-\alpha}\right) \frac{g(\hat{w})}{G(\hat{w})} E[N].
\]

By (57), a sufficient condition to satisfy (56) is \( \left(v - \frac{p}{1-\alpha}\right) \frac{g(\hat{w})}{G(\hat{w})} E[N] \leq \frac{E[(N-k)^+]^2}{E[N]} \). Recall that \( \beta = \frac{E[(N-k)^+]}{E[N]} \). Therefore, the sufficient condition for (56) to hold is \( \beta(b, \hat{w}) \geq \sqrt{v - \frac{p}{1-\alpha}} \frac{g(\hat{w})}{\alpha G(\hat{w})} \).

**Part (ii).** As a first step, we will show that \( \frac{\partial E[\Pi]}{\partial \hat{w}} > 0 \) for all policies as described at the beginning of the proof that are profit-making. Rearranging (52), we express the equilibrium \( \beta \) in terms of \( \hat{w} \) as
\[
\beta = \frac{v - p/(1-\alpha)}{\hat{w} - c}.
\]

Since \( p < (1-\alpha)v \) and \( \beta > 0 \), we must have \( \hat{w} > c \). Substituting the expression for \( \beta \) in (58) into (13) and differentiating with respect to \( \hat{w} \), we have
\[
\frac{\partial E[\Pi]}{\partial \hat{w}} = \left[ (c + r) \frac{(1-\alpha)v - p}{(\hat{w} - c)^2} \right] \cdot E[\min\{b, QG(\hat{w})\}]
\]

\[
+ \left[ p - (c + r) \frac{(1-\alpha)v - p}{\hat{w} - c} \right] \cdot g(\hat{w}) \cdot P\{QG(\hat{w}) < b\} E[Q \mid QG(\hat{w}) < b]
\]

\[
= \left[ \frac{(1-\alpha)(c + r)\beta}{\hat{w} - c} \right] \cdot E[\min\{b, QG(\hat{w})\}]
\]

\[
+ \left[ p - (1-\alpha)(c + r)\beta \right] g(\hat{w}) \cdot P\{QG(\hat{w}) < b\} E[Q \mid QG(\hat{w}) < b].
\]

The first term of (59) is positive. Since the equilibrium is profit-making, by Definition 3, we must have \( p - (1-\alpha)(c + r)\beta > 0 \) as well. Therefore, the second term of (59) is positive. Hence, we know that \( \frac{\partial E[\Pi]}{\partial \hat{w}} > 0 \).

Now note that any optimal \( b \) satisfies \( \frac{d\Pi}{db} = \frac{\partial E[\Pi]}{\partial \hat{w}} \cdot \frac{\partial \hat{w}}{db} = 0 \). Since \( \frac{\partial E[\Pi]}{\partial \hat{w}} > 0 \), \( \frac{\partial \hat{w}}{db} > 0 \) when \( \frac{\partial \hat{w}}{db} < 0 \). The optimal myopic booking limit, \( b^*_m \), satisfies \( \frac{d\Pi}{db} |_{b=b^*_m} = \frac{\partial E[\Pi]}{\partial \hat{w}} \cdot \frac{\partial \hat{w}}{db} |_{b=b^*_m} + \frac{\partial E[\Pi]}{\partial \hat{w}} \cdot \frac{\partial \hat{w}}{db} |_{b=b^*_m} = 0 + \frac{\partial E[\Pi]}{\partial \hat{w}} \cdot \frac{\partial \hat{w}}{db} |_{b=b^*_m} = 0 \). Hence, \( \frac{d\Pi}{db} = 0 \) only when \( \frac{\partial E[\Pi]}{\partial \hat{w}} > 0 \). By Proposition 1 part (iii), \( \frac{\partial E[\Pi]}{\partial \hat{w}} > 0 \) for all \( b < b^*_m \), and \( \frac{\partial E[\Pi]}{\partial \hat{w}} > 0 \) for all \( b > b^*_m \), so the optimal strategic booking limit, \( b^*_s \), is smaller than \( b^*_m \).
Proposition 4. (Optimal Myopic Overbooking Policy)
A myopic airline sets $p^*_m = (1 - \alpha)v$ and $c^*_m = 0$. When $v < r$, it selects a finite optimal booking limit $b^*_m = \max \left\{ \ell^{-1} \left( \frac{(1 - \alpha)\ell}{r} \right), k \right\}$. Otherwise, $b^*_m$ is infinite.

Proof. When the airline takes $\tilde{w}$ as an exogenous quantity, it believes that neither the price nor the fixed compensation affects the demand. Therefore, it charges the highest price, $p = (1 - \alpha)v$, and offers the lowest compensation, $c = 0$.

When $p = (1 - \alpha)v$ and $v \geq r$, by Definition 2 part (iii), $p - \ell(b)r > p - (1 - \alpha)r \geq 0$. Therefore the marginal change in profit is always positive and the airline is incentivized to overbook as much as possible.

When $v < r$, $p - (1 - \alpha)r = (1 - \alpha)v - (1 - \alpha)r < 0$ and either $p - \ell(k)r \leq 0$ or $p - \ell(k)r > 0$. If $p - \ell(k)r > 0$ or $\ell^{-1} \left( \frac{(1 - \alpha)v}{r} \right) > k$, then (19) has a solution $b^*_m = \ell^{-1} \left( \frac{(1 - \alpha)v}{r} \right)$ by the Intermediate Value Theorem. Furthermore, by Definition 2 part (i), $\ell(b)$ increases monotonically in $b$ for $b \geq k$; hence the solution is unique. When $p - \ell(k)r \leq 0$ or $\ell^{-1} \left( \frac{(1 - \alpha)v}{r} \right) \leq k$, the monotonicity of $\ell(b)$ ensures that the marginal increase in profit is always negative and hence the airline does not overbook, i.e., $b^*_m = k$.

Lemma 4. (Boundary Equilibria Not Optimal)
Any optimal strategic overbooking policy induces a customer equilibrium with $U(\beta, \tilde{w}) = 0$.

Proof. The only equilibria for which $U(\beta(\tilde{w}), \tilde{w}) \neq 0$ are $U(\beta(\tilde{w}), \tilde{w}) \geq 0$ and $U(\beta(\tilde{w}), \bar{w}) \leq 0$, and it suffices to show that any strategic overbooking policy that leads to either $U(\beta(\tilde{w}), \tilde{w}) < 0$ or $U(\beta(\tilde{w}), \bar{w}) > 0$ is not optimal.

First consider the policy that induces $U(\beta(\tilde{w}), \tilde{w}) < 0$. In this case, $U(\beta(\tilde{w}), \tilde{w}) < 0$ for all $w$ because $U(\beta(\tilde{w}), w)$ decreases strictly in $w$. Therefore, the equilibrium marginal customer’s response is $\tilde{w} = \tilde{w}$, and the airline thus obtains zero demand and zero profit. The overbooking strategy that results in $U(\beta(\tilde{w}), \tilde{w}) < 0$ is strictly dominated by any policy that charges a positive price and does not overbook.

Next consider the policy that induces $U(\beta(\tilde{w}), \bar{w}) > 0$. Here we know that $U(\beta(\tilde{w}), w) > 0$ for all $w$ and $\tilde{w} = \bar{w}$. Since any admissible policy has $c \leq \bar{w}$, in this case we must have $p < (1 - \alpha)v$. Therefore, the airline can raise the price without affecting demand and ultimately increase profits. Thus, the original policy with $p < (1 - \alpha)v$ cannot be optimal.

Lemma 5. (Multiple Equivalent Policies)
For any admissible policy $(p, b, c)$ for which $\beta > 0$ and $U(\beta, \tilde{w}) = 0$, there exists an infinite set of alternative policies with the same booking limit, $b' \equiv b$, and alternative price and bumping compensation,

$$p' \in \left\{ \max \left\{ 0, (1 - \alpha)(v - \tilde{w}\beta) \right\}, (1 - \alpha)v \right\}, \quad c' = \left( \tilde{w} - \frac{v}{\beta} \right) + \left( \frac{p'}{(1 - \alpha)\beta} \right),$$

with the same equilibrium $(\beta, \tilde{w})$ and expected profits $E[\Pi(p, b, c)] = E[\Pi(p', c', b')] = E[\Pi((1 - \alpha)v, b, \tilde{w})]$.

Proof. We complete the proof in three steps. First, recall that the original policy $(p, b, c)$ induces the equilibrium $(\beta, \tilde{w})$ and that, given fixed $b$ and $\tilde{w}$, the expression for $\beta$ in (46) is independent of $p$ and $c$. As long as $b$ does not change, we need only show $\beta$ to be consistent with $p$, $c$, and $\tilde{w}$ through the equilibrium equation (2).
Second, we use (2) to find consistent \((p', c')\) pairs. Specifically, from (2) we have \(U(\beta, \tilde{w}) = -p' + (1 - \alpha)v + (1 - \alpha)\beta(c' - \tilde{w}) = 0\). Given \(b' \equiv b, \beta > 0,\) and some admissible \(p',\) we can solve (2) for \(c',\) to derive the definition of \(c'\) in (25). Equation (25)'s bounds on \(p'\) then follow from the definition of an admissible policy. Given \(c' \leq \overline{w},\) we can set \(c' = \overline{w}\) and solve (2) for \(p'\) to see that \(p' \leq (1 - \alpha)v + (1 - \alpha)\beta(\overline{w} - \tilde{w}),\) which is looser than the direct bound, \(p' \leq (1 - \alpha)v.\) Similarly, given \(c' \geq 0,\) we can set \(c' = 0\) and solve (2) for \(p'\) to see that \(p' \geq (1 - \alpha)v + (1 - \alpha)\beta(0 - \tilde{w}) = (1 - \alpha)(v - \beta \tilde{w}).\)

While the lower bound may be larger or smaller than the direct bound, \(0 \leq p',\) it is always (weakly) lower than the upper bound, \(p' \leq (1 - \alpha)v.\) Thus, we have \(\max\{0, (1 - \alpha)(v - \beta \tilde{w})\} \leq p' \leq (1 - \alpha)v.\)

Third we show that, for any \((p', c', b')\) that is consistent with \((p, b, c),\) the airline earns the same expected profits. To demonstrate this fact, we recall the definition of expected profits in (13), \(E[\Pi(p', b', c')] = [p' - (1 - \alpha)\beta(c' + r)]E[S].\) From (3) we know that, given \(b\) and \(\tilde{w},\) \(E[S]\) is independent of \(p'\) and \(c'.\) Furthermore, we can use the definition of \(c'\) in (25) to substitute out \(c'\) in the definition of margin per customer to show that

\[
[p' - (1 - \alpha)\beta(c' + r)] = [(1 - \alpha)v - (1 - \alpha)\beta(\tilde{w} + r)].
\]

Thus any \((p', c', b')\) that is consistent with \((p,b,c)\) yields the same expected profit as well. \(\square\)

**Proposition 5.** (Problem Reduction)

If there exists an optimal strategic overbooking policy \(\xi \in \Xi,\) then there exists an optimal strategic policy that sets \(p^*_s = (1 - \alpha)v,\) induces an interior equilibrium \(U(\beta, \tilde{w}),\) and optimizes (10).

**Proof.** For any optimal policy \((p^*_s, c^*_s, b^*_s)\) with \(\beta > 0,\) Lemma 5 implies that there exists a policy with \(p = (1 - \alpha)v\) that generates the same equilibrium and expected profit, and the result follows. Now suppose there exists an optimal policy with \(\beta = 0.\) Then from (2) and Lemma 4, we see that \(U(\beta, \tilde{w}) = -p + (1 - \alpha)v + 0(c - \tilde{w}) = 0.\) Thus, \(p = (1 - \alpha)v\) here as well. \(\square\)

**Proposition 6.** (Booking Limit for Optimal Strategic Overbooking Policy)

(i) If \(c^*_s \leq v - r,\) then \(b^*_s = \infty.\)

(ii) If \(v - r < c^*_s < \frac{(1-\alpha)v}{\ell(k)} - r,\) then \(b^*_s = \ell^{-1}\left(\frac{(1-\alpha)v}{c^*_s + r}\right).\)

(iii) If \(c^*_s \geq \frac{(1-\alpha)v}{\ell(k)} - r,\) then \(b^*_s = k.\)

**Proof.** The proof of the proposition can be found in the main text. \(\square\)

**Proposition 7.** (Properties of the Auction with No Cap)

Suppose that, when \(n > k\) customers show up for a flight, the airline runs a reverse, uniform price, multi-unit auction. Then we have the following.

(i) Customers’ optimal bids match their underlying hassle costs: \(\{\omega_{1:n} = w_{1:n}, \ldots, \omega_{n:n} = w_{n:n}\}.\)

(ii) All customers are willing to purchase tickets, irrespective of their hassle cost \(w \in [\underline{w}, \overline{w}].\)

(iii) The airline’s optimal price is \(p^*_a = (1 - \alpha)v.\)

**Proof.**

Part (i). We will show that, in the \(k^{th}\)-price reverse auction problem outlined above, bidding the true hassle cost is a dominant strategy for customers. Consider a customer with hassle cost \(w.\) Assume that other customers bid in some arbitrary way.
1. Suppose the customer can board if she bids her true hassle cost \( w \), i.e., \( w \geq w_{n-k+1:n} \). Then bidding higher than \( w \) still allows her to board whereas bidding lower than \( w \) may result in bumping with compensation \( w_{n-k:n} \). Since \( w_{n-k:n} < w \), bidding lower than \( w \) is dominated by bidding \( w \).

2. Suppose the customer is bumped if she bids her true hassle cost \( w \), i.e., \( w < w_{n-k+1:n} \). In this case, she gets positive utility \( w_{n-k+1:n} - w \). If she were to bid lower than \( w \), she would still be bumped and receive \( w_{n-k+1:n} \), and this does not improve her utility. If she bids \( w' > w \), then one of the following two cases holds. If \( w < w' \leq w_{n-k+1:n} \), then she is still bumped and receive \( w_{n-k+1:n} \). If she bids \( w' > w_{n-k+1:n} \), then she isn’t bumped. In this case she ends up receiving no compensation and is strictly worse off because \( 0 < w_{n-k+1:n} - w \). Therefore, bidding higher than \( w \) is dominated by bidding \( w \).

Considering both cases, we see that bidding \( w \) is a dominant strategy for the customer.

**Part (ii).** By construction of the \( k^{th} \)-price auction, we know that every bumped customer is more than fairly compensated: \( w_{n-k+1:n} \geq w_{i:n} \) for \( i \in \{1, 2, ..., n-k\} \). Therefore, all customers receive non-negative utility from buying a ticket under any admissible policy, and \( \hat{w} = \overline{w} \).

**Part (iii).** From part (ii), we know that demand is not thinned by the airline’s profit-maximizing overbooking strategy since \( G(\hat{w}) = G(\overline{w}) = 1 \). Therefore, the airline should maximize profit per customer and set \( p^*_a = (1 - \alpha)v \).

**Proposition 8.** (Properties of the Auction with a Cap)
Suppose that the airline sets the price \( p = (1 - \alpha)v \) and \( b > k \). When \( n > k \) customers show up for a flight, it runs a reverse, uniform price, multi-unit auction with compensation cap \( c_a \leq \overline{w} \). Then we have the following.
(i) Customers are willing to purchase tickets, if and only if their hassle costs are \( w \leq c_a \).
(ii) Customers’ optimal bids match their underlying hassle costs: \( \{\overline{w}_{1:n} = w_{1:n}, ..., \overline{w}_{n:n} = w_{n:n}\} \).

**Proof.**
**Part (i).** It is easy to see that customers with hassle costs \( w \leq c_a \) are always more than fairly compensated if bumped. Since \( p = (1 - \alpha)v \), buying a ticket always gives them non-negative utility. For a customer with hassle cost \( w > c_a \), her net value of being bumped is \( c_a - w < 0 \). As long as her probability of being bumped is positive, her expected value from purchasing a ticket is negative. Therefore, buying a ticket is dominated by taking the outside option, which gives her zero utility.

**Part (ii).** Since \( c_a \leq \overline{w} \), we have \( \overline{w} = c_a = \hat{w} \) and can apply the proof of Proposition 7 part (i).

**Proposition 9.** (Auction with Cap Dominates Fixed Compensation)
Given any fixed-compensation policy with \( p = (1 - \alpha)v \), \( b > k \), \( \underline{w} < c \leq \overline{w} \), and equilibrium \( \beta > 0 \), an auction-based policy with the same price, \( p = (1 - \alpha)v \), the same booking limit, \( b \), and an analogous cap, \( c_a = c \), earns strictly higher expected profits: \( E[\Pi_a((1 - \alpha)v, b, c)] > E[\Pi((1 - \alpha)v, b, c)] \).

**Proof.** Consider a fixed compensation scheme with price \( p = (1 - \alpha)v \), booking limit \( b > k \), and compensation \( \underline{w} \leq c \leq \overline{w} \) that induces expected profit \( E[\Pi(p, b, c)] \). We define the analogous capped auction with the same price and booking limit, the analogous cap \( c_a = c \), and expected profit \( E[\Pi_a(p, b, c)] \).
Observe that demand $S = \min\{b, QG(c)\}$ is the same in both cases. In the fixed compensation scheme with $p = (1 - \alpha)v$, (25) shows that $\hat{w} \equiv c$. In the capped auction scheme, Proposition 8 part (i) similarly implies that the effective $\hat{w} \equiv c_a$. Because both schemes also have identical price, their expected revenues are the same.

Similarly, both schemes have identical numbers of bumped customers, $(N(s, \alpha) - k)^+$, and rerouting costs per customer. Thus, expected rerouting costs are also the same in both cases.

Finally, while the number of bumped passengers is, again, the same in both cases, we can show that the expected per-customer bumping compensation is strictly lower in the auction scheme. In particular, given continuous $W$, $E[w_{n-k:n}] < c$ with probability 1. Hence, $E[\Pi_a(p, b, c)] > E[\Pi(p, b, c)]$.

**Lemma 6.** (Convexity of Auction-Based Expected Bumping Cost)

Suppose $N(s, \alpha)$ is SICX in $s$.

(i) If $\partial E[w(c_a)_{n-k+1:n}] / \partial n \geq 0$, then $C'(s) > 0$.

(ii) If in addition $\partial^2 E[w(c_a)_{n-k+1:n}] / \partial n^2 > 0$, then $C''(s) > 0$.

Proof. For $s \geq k$, $C(s, c_a) = \int_k^s (n-k) E[w(c_a)_{n-k+1:n}] p_N(n|s) \, dn$. Let $\psi(x) = (x-k)^+ E[w(c_a)_{x-k+1:x}]$.

Then $C(s, c_a) = E[\psi(N(s, \alpha))]$. Since $N(s, \alpha)$ is SICX in $s$, to show that $C(s, c_a)$ is increasing convex in $s$, it suffices to show that $\psi(x)$ is increasing convex in $x$.

**Part (i).** Since $(x-k)^+$ is strictly increasing in $x$ for $s \geq k$, if $E[w(c_a)_{x-k+1:x}]$ is increasing in $x$, i.e., $\partial E[w(c_a)_{x-k+1:x}] / \partial x \geq 0$, $\psi(x)$ must be increasing in $x$.

**Part (ii).** For $s \geq k$, $\psi'(x) = (x-k) \frac{\partial E[w(c_a)_{x-k+1:x}]}{\partial x} + E[w(c_a)_{x-k+1:x}]$ and $\psi''(x) = (x-k) \frac{\partial^2 E[w(c_a)_{x-k+1:x}]}{\partial x^2}$.

Therefore, one sufficient condition for $\psi''(x) > 0$ is that $\frac{\partial^2 E[w(c_a)_{x-k+1:x}]}{\partial x^2} > 0$.

**Lemma 7.** (Convexity of the Approximation $\tilde{G}$)

Suppose $N(s, \alpha)$ is SICX in $s$ and we use the specific approximation $E[w(c_a)_{n-k+1:n}] \approx \tilde{G}(c_a, n)$.

If for any $c_a \in (\underline{w}, \overline{w})$, $g'(w) < 0$ for all $w \in [G^{-1}(\frac{G(c_a)}{k+1}), c_a]$, then $C''(s) \geq 0$.

Proof. For $s \geq k$, $C(s, c_a) = \int_k^s (n-k) E[w(c_a)_{n-k+1:n}] p_N(n|s) \, dn$. Let $\psi(x) = (x-k)^+ E[w(c_a)_{x-k+1:x}]$.

Then $C(s, c_a) = E[\psi(N(s, \alpha))]$. If $N(s, \alpha)$ is stochastically increasing and linear (e.g. binomial), then to show that $C(s, c_a)$ is convex in $s$, it suffices to show that $\psi(x)$ is convex in $x$.

Let $y(x) = \frac{x-k}{x+1}$. Then, $\tilde{G}(c_a, x) = \tilde{G}(x) = G^{-1}(y(x))$. Here we drop $c_a$ in the argument because we consider a fixed $c_a$. Thus,

$$\tilde{G}'(x) = \frac{k}{g_{c_a}(G^{-1}(y(x)))(x+1)^2}$$

and

$$\tilde{G}''(x) = -\frac{g_{c_a}'(G^{-1}(y(x))))}{g_{c_a}(G^{-1}(y(x)))}(G'(x))^2 \left(\frac{2\tilde{G}'(x)}{x+1}\right).$$

By (60) and (61),

$$\psi''(x) = \frac{g_{c_a}'(G^{-1}(y(x))}{g_{c_a}(G^{-1}(y(x)))}(G'(x))^2 \left(\frac{2(k+1)\tilde{G}'(x)}{x+1}\right).$$

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By (62), a sufficient condition for \( \psi''(x) \geq 0 \) is \( g'_{c_a}(G^{-1}_{c_a}(y(x))) \leq 0 \), which is equivalent to \( g'(G^{-1}_{c_a}(y(x))) \leq 0 \) since \( g_{c_a}(x) = \frac{g(x)}{c_a} \). Note that for \( x \in (k, \infty) \), we have \( y(x) \in (\frac{1}{k+1}, 1) \). Therefore, \( G^{-1}_{c_a}(y(x)) \in (G^{-1}(\frac{1}{k+1}), c_a) \). Clearly, \( G^{-1}_{c_a}(\frac{1}{k+1}) = G^{-1}(\frac{G(c_a)}{k+1}) \). Hence, the sufficient condition becomes \( g'(w) \leq 0 \) for \( w \in [G^{-1}(\frac{G(c_a)}{k+1}), c_a] \).

\[ \square \]

**Proposition 10.** (Optimal Booking Limit for the Auction)

Suppose \( p = (1 - \alpha)v \) and \( C''(s) > 0 \). Then there exists a unique optimal booking limit, \( b^*_a \), with the following properties.

(i) If \( \ell'(k) \geq \frac{(1-\alpha)v}{r} \) then \( b^*_a = k \).
(ii) If \( \ell'(k) < \frac{(1-\alpha)v}{r} \) and \( \exists b \in (k, \infty) \) s.t. \( C'(b) \geq (v - r)(1 - \alpha) \), then \( (36) \) determines \( b^*_a \in (k, \infty) \).

(iii) If \( C'(b) < (v - r)(1 - \alpha) \) for all \( b \geq k \) then \( b^*_a = \infty \).

**Proof.** By the first-order condition \( (36) \), an interior \( b^*_a \) satisfies \( \ell'(b) = \frac{p - C'(b)}{r} \). Recall from Definition 2 part (i) that \( \ell''(b) > 0 \) and from Lemma 6 part (ii) that \( C''(b) > 0 \). Given the convexity of these costs, expected profits are concave in \( b \), and at most one such \( b^*_a \) exists. By the second-order condition \( (37) \), \( \frac{d^2H}{db^2} \big|_{b=b^*_a} = 0 + P\{Q > b\}[-r\ell''(b) - C''(b)] < 0 \). Therefore, an interior \( b^*_a \), if it exists, is a global maximum.

The airline charges price \( p = (1 - \alpha)v \), and we prove the results in the order (i), (iii), (ii).

Part (i). \( \ell'(k) \geq \frac{(1-\alpha)v}{r} \) implies \( \ell'(k) \geq \frac{p}{r} = \frac{p - C'(k)}{r} \). Therefore, \( \frac{d\Pi_a}{db} = p - r\ell'(b) - C'(b) \leq p - r\ell'(k) - C'(k) \leq 0 \) for all \( b \geq k \), which implies \( b^*_a = k \).

Part (iii). Together, \( p = (1 - \alpha)v \) and \( C'(b) < (v - r)(1 - \alpha) \) \( \forall b \geq k \) imply that \( \ell'(b) \leq 1 - \alpha < \frac{p - C'(b)}{r} \) and \( \frac{d\Pi_a}{db} > 0 \) \( \forall b \), which in turn means that \( b^*_a = \infty \).

Part (ii). From parts (i) and (iii) and the Intermediate Value Theorem, the necessary and sufficient conditions for the existence of an interior \( b^*_a \) are a) \( \ell'(k) < \frac{(1-\alpha)v}{r} \) and b) \( \exists b > k \) s.t. \( C'(b) \geq (v - r)(1 - \alpha) \).

\[ \square \]

**Lemma 8.** (Bumping Compensation Grows with the Cap)

For \( c_a \in (\underline{w}, \bar{w}) \) and \( s \geq k \), (i) \( \partial C(s, c_a)/\partial c_a > 0 \), and (ii) \( \partial^2 C(s, c_a)/\partial s \partial c_a > 0 \).

**Proof.** Since \( E[w(c_a)] \) increases in \( c_a \), by (33), \( C(s, c_a) \) increases in \( c_a \) for any fixed \( s \). Then from (33) we have

\[
\frac{\partial}{\partial s} C(s, c_a) = (s - k)E[w(c_a)]P_N(s|s) + \int_k^s (n - k)E[w(c_a)]dP_N(n|s)\frac{dP_N(n)}{ds} \, dn. \tag{63}
\]

Since both \( E[w(c_a)] \) and \( E[w(c_a)] \) increase in \( c_a \), by (63), for any fixed \( s \) we have \( \frac{\partial^2}{\partial s \partial c_a} C(s, c_a) > 0 \).

\[ \square \]

**Proposition 11.** (Optimal Auction Parameters)

For fixed \( p \), let \( b^*_a(c_a) \) be the optimal booking limit induced by \( c_a \). Suppose \( p = (1 - \alpha)v, \underline{w} < c_a < \bar{w}, k < b^*_a(c_a) < \infty \) and \( C''(s) > 0 \). Then we have the following.

(i) The optimal booking limit, \( b^*_a \), is decreasing in \( c_a \).
(ii) The resulting expected profit, \( E[\Pi_a] \), is increasing in \( c_a \).
Proof.
Part (i). By Proposition 9 part (i), the equilibrium customers’ response is independent of the booking limit. Therefore, as in (34),
\[
\frac{d}{db} \Pi_a(b) = p \frac{dE[S]}{db} - r \frac{dE[(N - k)^+]}{db} - \frac{dE[C(S)]}{db}
\]
\[
= pP\{Q > b\} - r \ell'(b)P\{Q > b\} - C'(b)P\{Q > b\},
\]
and the optimal booking limit with \(c_a, b^*_a(c_a)\), satisfies
\[
p - r \ell'(b^*_a(c_a)) - C'(b^*_a(c_a)) = 0.
\]
Consider \(c'_a > c_a\). Then by Lemma 8 part (ii), \(C'(s, c'_a)(b^*_a(c_a)) > C'(s, c_a)(b^*_a(c_a))\). Since \(C'' > 0\), \(\ell'' > 0\) and \(b^*_a(c_a)\) is interior, we must have \(b^*_a(c'_a) < b^*_a(c_a)\).

Part (ii). Recall that, for a capped auction, \(\hat{\omega} \equiv c_a\). Suppose the airline always implements the optimal booking limit \(b^*_a(c_a)\) w.r.t. \(c_a\). Then by the Envelope Theorem,
\[
\frac{d}{dc_a} E[\Pi_a(b, \hat{\omega})] \bigg|_{b = b^*_a(c_a)} = \frac{d}{d\hat{\omega}} E[\Pi_a(b, \hat{\omega})] \bigg|_{b = b^*_a(c_a)}
\]
\[
= \frac{\partial}{\partial \hat{\omega}} \Pi_a(b, \hat{\omega})
\]
\[
= p \frac{\partial E[S]}{\partial \hat{\omega}} - r \frac{\partial E[(N - k)^+]}{\partial \hat{\omega}} - \frac{\partial E[C(S)]}{\partial \hat{\omega}}
\]
\[
= g(\hat{\omega}) \int_0^{b^*_a(c_a) \frac{C'(0)}{C'(\hat{\omega})}} [p - r \ell'(qG(\hat{\omega})) - C'(qG(\hat{\omega}))] qf(q) dq.
\]
Since \(p - r \ell'(qG(\hat{\omega})) - C'(qG(\hat{\omega})) = 0\) for \(q = \frac{b^*_a(c_a)}{C'(\hat{\omega})}\), and since \(\ell'(s), C'(s) > 0\), we know that \(p - r \ell'(qG(\hat{\omega})) - C'(qG(\hat{\omega})) \geq 0\) for \(q \in [0, \frac{b^*_a(c_a)}{C'(\hat{\omega})}]\). This shows that \(\frac{d}{dc_a} \Pi_a(b, \hat{\omega}) \bigg|_{b = b^*_a(c_a)} > 0\), which concludes the proof.

**Proposition 12.** (Optimality of Overbooking Policy)

(i) The optimal overbooking policy uses an auction to determine customers’ bumping compensation.

When \(N(s, \alpha)\) is SICX, \(C''(s) > 0\), and the auction-based overbooking policy sets \(p = (1 - \alpha)v\), we also have the following.

(ii) The optimal cap on bumping compensation is effectively unbounded: \(c_a^* = \overline{\omega}\).

(iii) The optimal booking limit \(b_a^*\) is defined as in Proposition 10.

**Proof.** Part (i) follows immediately from Proposition 9. Part (ii) follows immediately from Proposition 11 part (ii). For Part (iii) See the proof of Proposition 10. □
C Numerical Experiments

The results reported in Table 1 are based on the optimal booking limits found for the following primitive parameters.

\[
p = 400 \quad \text{All other parameters are pegged off the ticket price.}
\]

\[
c \in \{0, 100, 200, 400, 800\} \quad \text{Bumping compensation ranges from 0 to 2 times the ticket price.}
\]

\[
\alpha \in \{0.05, 0.1, 0.2\} \quad \text{No-show probabilities range from low to high.}
\]

\[
k \in \{50, 100, 200, 400\} \quad \text{The plane’s capacity ranges from low to high.}
\]

\[
r \in \{0, 200, 320, 400\} \quad \text{Rerouting costs run from 0 to the ticket price.}
\]

\[
v (1 - \alpha) - 400 \in \{0.01, 1, 4\} \quad \text{Value set to ensure a small amount of consumer surplus.}
\]

\[
F \sim \mathcal{N}(1.2 k, k/3) \quad \text{Support is } [1, 2.4 k]; \text{ distribution renormalized so probabilities sum to one.}
\]

\[
G \sim \mathcal{N}(v, v/3) \quad \text{Support is } [0, 2.2 v]; \text{ distribution renormalized so probabilities sum to one.}
\]

Notes: (1) The range of customers’ expected values of flying \(v (1 - \alpha) - 400 \in \{0.01, 1, 4\}\) is low and reflects the fact that larger \(v\)’s generate enough customer surplus that booking limits become unbounded. This numerical result also suggests that the price should be roughly \(p \approx (1 - \alpha)v\). (2) The demand distribution \(F\) is scaled to offer slightly more demand than capacity available. (3) The hassle-cost distribution \(G\) is scaled to be on the order of the value the customer receives from flying.

The results reported in Table 2 are based on the optimal booking limits and bumping compensation found for the following primitive parameters.

\[
v \in \{200, 400, 500, 600, 800\} \quad \text{Value of flying ranges from low to high.}
\]

\[
\alpha \in \{0.05, 0.1, 0.2\} \quad \text{No-show probabilities range from low to high.}
\]

\[
p = (1 - \alpha)v \quad \text{Optimal prices for both the fixed and uncapped-auction schemes.}
\]

\[
k \in \{50, 100, 200, 400\} \quad \text{The plane’s capacity ranges from low to high.}
\]

\[
r \in \{0, 200, 400, 600, 800, 1000\} \quad \text{Rerouting costs run from 0 to very high.}
\]

\[
F \sim \mathcal{N}(1.2 k, k/3) \quad \text{Support is } [1, 2.4 k]; \text{ distribution renormalized so probabilities sum to one.}
\]

\[
G \sim \mathcal{N}(v, v/3) \quad \text{Support is } [0, 2.2 v]; \text{ distribution renormalized so probabilities sum to one.}
\]

Notes: (1) The demand distribution \(F\) is scaled to offer slightly more demand than capacity available. (2) The hassle-cost distribution is scaled to be on the order of the value the customer receives from flying. (3) Hassles costs are normally distributed, and to evaluate auction-based policies we approximate expected order statistics using results from Harter (1961), which includes correction terms for the fractile approach of Arnold et al. (2008) we describe in 5.4.1.