Technical Appendix to:
Contracting for Infrequent Restoration and Recovery of Mission-Critical Systems

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A Tables and Figures

λ	1	2	3	4	5	6	7	8	9	10
$\Delta(\lambda)$	0.767	0.577	0.433	0.330	0.258	0.208	0.172	0.147	0.128	0.113
$e^{-\lambda}/\Delta(\lambda)$	0.480	0.235	0.115	0.056	0.026	0.012	0.005	0.002	0.001	4×10^{-4}

Table 1: Numerical values of $\Delta(\lambda)$ and $e^{-\lambda}/\Delta(\lambda)$ for $\lambda = 1, ..., 10$.

B Solution Behavior in the $\lambda \to 0$ Limit

In the $\lambda \to 0$ limit, STC binds at optimum because the condition (2) is trivially satisfied. Both p^{CC} and p^{AC} approach infinity in this limit, as stated in part (i) of Proposition 2; when equipment failures are extremely unlikely the supplier has little incentive to invest in capacity because the chance of his being penalized for poor service time realization is very small. To convince the supplier otherwise and to induce the target capacity μ_I, the customer must threaten him with a very high penalty rate. However, risk premiums under the two contracts converge to the same *finite* number $\frac{c\mu_I}{2}\left(\frac{1+v(\mu_I)^2}{1+\theta(\mu_I)}\right)$ in the $\lambda \to 0$ limit. In fact, this counterintuitive result is more general than what our model allows for, i.e., it continues to hold even if the failure process is not Poisson, as proved in the following proposition.

Proposition B.1 *Let* $\{Y_i\}$ *be arbitrary but i.i.d. random variables representing the failure interarrival time and* $F(\cdot \mid \lambda)$ *be their cdf when the arrival rate is* λ. *Suppose that* $F(\cdot \mid \lambda)$ *satisfies* $\lim_{\lambda \to 0} F(\cdot \mid \lambda) = 0$. *Let* ψ_Y^j, $j \in \{CC, AC\}$ *be the risk premium under either* CC *or* AC. *Then* $\lim_{\lambda \to 0} \psi_Y^{CC} = \lim_{\lambda \to 0} \psi_Y^{AC} = \frac{c\mu_I}{2}\left(\frac{1+v(\mu_I)^2}{1+\theta(\mu_I)}\right)$.

The intuition behind Proposition B.1 is as follows. In the vicinity of $\lambda = 0$, that is, when it is highly unlikely that an equipment failure occurs within the contracting period, the customer faces the following situation. Since the chance is high that the supplier's service is not required, even a fairly large penalty rate will not convince the supplier to invest in capacity. Therefore, the customer has to provide a very high contractual incentive (large p) in order to ensure that the supplier reserves the target capacity μ_I, as we showed above for Poisson failures. At the same time, however, uncertainty in the supplier's performance $\mathrm{Var}[X \mid \lambda, \mu_I]$, where $X = \sum_{i=1}^{N} S_i$ for CC and $X = \widehat{S}\mathbf{1}(N > 0)$ for AC, approaches zero since the supplier does not get a chance to reveal his ability to perform if there is no equipment failure. Since the risk premium combines these two effect, i.e., $\psi = \eta p^2 \mathrm{Var}[X \mid \lambda, \mu_I]$, a tension exists between p that goes to infinity and $\mathrm{Var}[X \mid \lambda, \mu_I]$ that goes to zero. Remarkably, Proposition B.1 states that a middle ground is chosen between these two opposing forces in the $\lambda \to 0$ limit, and that this asymptote depends neither on the supplier's risk aversion coefficient η or the equipment failure process.

To put this last result into a perspective, we compare it to the analysis in Abreu et al. (1991). One of the implications of the analysis in Abreu et al. (1991) is that it becomes infinitely expensive in the $\lambda \to 0$ limit to implement an incentive-compatible fixed-price contract in a repeated setting, which is known to allow for achieving the first-best solution if the discount rate is close to one.[1] Although a side-by-side comparison between our model and that of Abreu et al. (1991) is not possible because our model assumes a single interaction between the customer and the supplier, we find evidence from Proposition B.1 that PBC offers a unique advantage of containing cost even in extreme situations, as an upper bound on ψ exists. Given that repeated interactions can only improve the efficiency of a contract, as is well known in the contracting literature, this advantage over

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[1] Rather than showing that supply chain cost approaches infinity, Abreu et al. (1991) show that an incentive-compatible fixed-price contract that satisfies a budget constraint does not exist if the agent's action is evaluated too frequently. Note that frequent action evaluation (i.e., short period length) in their model is equivalent to infrequent product failures in our model, in that they assume that the signal frequency is fixed, whereas we assume that it is the period length that is fixed.

a fixed-price contract would become even more pronounced in a setting with repeated interactions. Thus, this result advocates the use of a performance-based contract over a fixed-price contract (i.e., the contract that is independent of the performance outcome) in high-reliability environments, if outsourcing is required.

C Comparing Efficiencies of CC and AC When $v(\mu)$ Is Constant

What drives AC to be more efficient when the service time S does not vary too much? Why is CC more efficient in some cases? The key to answering these questions lies in examining in detail $V^{CC}(\lambda)$ and $V^{AC}(\lambda)$, which determine relative magnitudes of risk premiums (see Proposition 1 for expressions of the two quantities). These two quantities are in fact squares of the coefficients of variations (CV) of the two performance measures $\sum_{i=1}^{N} S_i$ and $\widehat{S}\mathbf{1}(N>0)$ (as opposed to v, which is the CV of S). It is instructive to write them in the following way:

$$V^{CC}(\lambda) = \frac{1}{\lambda} + \frac{1}{\lambda}v^2 \qquad \text{and} \qquad V^{AC}(\lambda) = \frac{e^{-\lambda}}{1-e^{-\lambda}} + \frac{\Delta(\lambda)}{1-e^{-\lambda}}v^2. \tag{C1}$$

As we can see from these expressions, each $V^j(\lambda)$ is separated into terms that are either independent or dependent of v^2. The identities of the independent (first) terms are revealed by rewriting them as

$$\frac{1}{\lambda} = \frac{\lambda}{\lambda^2} = \frac{\text{Var}[N]}{(E[N])^2} \qquad \text{and} \qquad \frac{e^{-\lambda}}{1-e^{-\lambda}} = \frac{e^{-\lambda}(1-e^{-\lambda})}{(1-e^{-\lambda})^2} = \frac{\text{Var}[\mathbf{1}(N>0)]}{(E[\mathbf{1}(N>0)])^2}.$$

In other words, they are the squares of the CVs that originate from uncertainty in the number of equipment failures N. However, they manifest themselves in different forms for the two contracts: while it is the CV of N that enters into $V^{CC}(\lambda)$, it is the CV of $\mathbf{1}(N>0)$ that enters into $V^{AC}(\lambda)$. The latter comes from the no-failure effect of AC. Var$[N]$ is present in CC because uncertainty in N is one of the two components of the total variance in cumulative downtime (see (D2) in Appendix D), whereas under AC, Var$[N]$ is eliminated through division of $\sum_{i=1}^{N} S_i$ by N, leaving the no-failure effect as the only residual of uncertainty from N. As intuition suggests, the variability of CC turns out to be greater than that of AC when only the first terms of $V^{CC}(\lambda)$ and $V^{AC}(\lambda)$ in (C1) are compared: it can be shown that $1/\lambda - e^{-\lambda}/(1-e^{-\lambda}) > 0$.

This, however, does not tell the entire story because we have not taken into account the interactions of N with S that are present in the second terms of $V^{CC}(\lambda)$ and $V^{AC}(\lambda)$ in (C1). The interactions occur because variabilities in performance measures are also impacted by how many samples are collected, i.e., by N. It turns out that there is more variability in AC with regard to these interactions because division of $\sum_{i=1}^{N} S_i$ by a random variable N introduces more noise than when it is not divided by N, as is the case under CC. We confirm this insight by showing that the difference of the second terms in $V^{CC}(\lambda)$ and $V^{AC}(\lambda)$ is negative: $1/\lambda - \Delta(\lambda)/(1-e^{-\lambda}) < 0$, which follows from the property (iv) of $\Delta(\lambda)$ in Lemma D.1.

Combined, the sign of $V^{CC}(\lambda) - V^{AC}(\lambda)$ is ambiguous. However, we can infer from (C1) and the preceding arguments that $V^{CC}(\lambda) > V^{AC}(\lambda)$ if v is sufficiently small but $V^{CC}(\lambda) < V^{AC}(\lambda)$ otherwise. This result answers the questions that we posed above, as the risk premium ψ^j, and hence the supply chain efficiency, is completely determined by $V^j(\lambda)$ in the constant $v(\mu)$ case: AC is more efficient when v is relatively small but the reverse is true if v is large. See Figure 3 that divides the (λ, v) space in terms of relative efficiency of the two contracts. A similar argument can be made for the case where $v(\mu)$ is allowed to vary, although it is more complicated than what we have presented here. The basic insight, however, remains the same.

D Proofs and Auxiliary Results

Proof of Lemma 1. The mean and the variance of a compound Poisson variable is evaluated as (see Ross 1996, pp. 82-89)

$$E[\textstyle\sum_{i=1}^{N} S_i \,|\, \lambda, \mu] = E[N \,|\, \lambda] E[S \,|\, \mu] = \lambda/\mu, \tag{D1}$$

$$\mathrm{Var}[\textstyle\sum_{i=1}^{N} S_i \,|\, \lambda, \mu] = \mathrm{Var}[N \,|\, \lambda](E[S \,|\, \mu])^2 + E[N \,|\, \lambda]\mathrm{Var}[S \,|\, \mu] = \lambda \left(1 + v(\mu)^2\right)/\mu^2. \tag{D2}$$

The supplier utility under CC (with $T = w - p\sum_{i=1}^{N} S_i$) is $u(\mu) = w - p\lambda/\mu - \eta p^2 \lambda \left(1 + v(\mu)^2\right)/\mu^2 - c(\mu - \underline{\mu})$. Differentiating,

$$u'(\mu) = p\lambda/\mu^2 + 2\eta p^2 \lambda \left(1 + \theta(\mu)\right)/\mu^3 - c.$$

Observe that

$$\theta'(\mu) = v(\mu)v'(\mu) - \mu[v'(\mu)]^2 - \mu v(\mu) v''(\mu) \leq 0.$$

Hence, $u''(\mu) < 0$, i.e., the supplier's utility maximization problem with CC is concave. The sensitivity analysis results can be found from implicit differentiations, which we omit. The same results for $u(\mu)$ can be shown analogously. ■

Proof of Proposition 1. We show the solution of the CC case only. The solution of the AC case is obtained similarly. The customer's contract design problem is reduced to the cost minimization problem

$$\min_{p} \;\; \Psi(p) \equiv r\lambda/\mu^* + c(\mu^* - \underline{\mu}) + \eta p^2 \lambda \left(1 + v(\mu^*)^2\right)/(\mu^*)^2 \;\; \text{subject to } \mu^* \geq \mu_I$$

as the IR constraint $u(\mu^*) \geq 0$ binds at optimum, by an appropriate selection of w. We can invert μ^* found from the first-order condition in Lemma 1 with respect to p, using the monotonicity relation $\partial\mu^*/\partial p > 0$. Thus the optimal penalty rate that induces the supplier to choose μ is

$$p(\mu) = \frac{-1 + \sqrt{1 + 8\eta c\mu\left(1 + \theta(\mu)\right)/\lambda}}{4\eta\left(1 + \theta(\mu)\right)/\mu} = \frac{2c\mu^2}{\lambda\left(1 + \sqrt{1 + 8\eta c\mu\left(1 + \theta(\mu)\right)/\lambda}\right)}.$$

For notational convenience, let us suppress the argument μ in $p(\mu)$, $\theta(\mu)$, and $v(\mu)$. Observe that

$$\begin{aligned} \frac{\partial}{\partial\mu}\left(\frac{1+v^2}{(1+\theta)^2}\right) &= \frac{2vv'\left(1+\theta\right) - 2(1+v^2)\theta'}{(1+\theta)^3} = \frac{2vv'\left(1 + v^2 - \mu vv'\right) - 2(1+v^2)(vv' - \mu(v')^2 - \mu vv'')}{(1+\theta)^3} \\ &= \frac{2\mu(v')^2 + 2\mu v(1+v^2)v''}{(1+\theta)^3} \geq 0. \end{aligned}$$

Combining this result with (5), we find that the risk premium $\psi = \eta p^2\lambda\left(1+v^2\right)/\mu^2$ (from the expression of $\Psi(p)$ above) is increasing in μ for $\mu \geq \underline{\mu}$:

$$\begin{aligned} \frac{16\eta}{\lambda}\psi'(\mu) &= 16\eta^2\frac{d}{d\mu}\left((p/\mu)^2\left(1+v^2\right)\right) = \frac{d}{d\mu}\left(\left(-1 + \sqrt{1 + 8\eta c\mu\left(1+\theta\right)/\lambda}\right)^2 \frac{1+v^2}{(1+\theta)^2}\right) \\ &= \frac{8\eta c}{\lambda}\left(1 + \theta + \mu\theta'\right)\left(1 - \frac{1}{\sqrt{1 + 8\eta c\mu\left(1+\theta\right)/\lambda}}\right)\frac{1+v^2}{(1+\theta)^2} \\ &\quad + \left(-1 + \sqrt{1 + 8\eta c\mu\left(1+\theta\right)/\lambda}\right)^2 \frac{d}{d\mu}\left(\frac{1+v^2}{(1+\theta)^2}\right) \geq 0\;. \end{aligned}$$

In addition, $r\lambda/\mu + c(\mu - \underline{\mu})$ increases in $\mu \geq \mu_I$ when (2) is satisfied. Thus, the customer cost $\Psi = r\lambda/\mu + c(\mu - \underline{\mu}) + \psi$ increases in μ in the feasible region. Since our goal is to find the minimum of Ψ, which is increasing in μ, in the feasible region $[\mu_I, \infty)$ on the μ-domain, the cost minimizer is found at the left-most boundary, i.e., $\mu = \mu_I$. Hence, the service time constraint binds at the optimal solution. The equilibrium solutions p^{CC} and ψ^{CC} are found by substituting $\mu = \mu_I$ in $p(\mu)$ and $\psi(\mu)$. $\blacksquare$

In the following auxiliary lemma, we evaluate the mean and the variance of the performance measure $\widehat{S}\mathbf{1}(N > 0)$ that are used to prove Lemma 2.

Lemma D.1

$$\begin{aligned} E[\widehat{S}\mathbf{1}(N > 0) \,|\, \lambda, \mu] &= \Pr(N > 0) E[S \,|\, \mu] = (1 - e^{-\lambda})/\mu \quad \textit{and} & \textit{(D3)} \\ \mathit{Var}[\widehat{S}\mathbf{1}(N > 0) \,|\, \lambda, \mu] &= \Pr(N > 0) \left(\Pr(N = 0) (E[S \,|\, \mu])^2 + \Delta(\lambda)\, \mathit{Var}[S \,|\, \mu] \right) \\ &= (1 - e^{-\lambda})[e^{-\lambda} + \Delta(\lambda) v(\mu)^2]/\mu^2, & \textit{(D4)} \end{aligned}$$

where $\Delta(\lambda) \equiv \frac{1}{e^{\lambda}-1} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{1}{n}$ *has the following properties: (i)* $\Delta'(\lambda) < 0$*, (ii)* $\lim_{\lambda \to 0} \Delta(\lambda) = 1$*, (iii)* $\lim_{\lambda \to \infty} \Delta(\lambda) = 0$*, (iv)* $\Delta(\lambda) > (1 - e^{-\lambda})/\lambda$*, and (v)* $\frac{d}{d\lambda} \left(e^{-\lambda}/\Delta(\lambda) \right) < 0$.

Proof of Lemma D.1. For notational convenience, let us suppress the conditional arguments (λ, μ). First, we prove the following intermediate results.

Lemma D.2

$$\begin{aligned} E[\widehat{S}] &= E[S] = 1/\mu, & \textit{(D5)} \\ \mathit{Var}[\widehat{S}] &= \Delta(\lambda)\, \mathit{Var}[S] = \Delta(\lambda) v(\mu)^2/\mu^2, & \textit{(D6)} \end{aligned}$$

Proof. Let $M(t) \equiv E[e^{t\widehat{S}}]$ be the moment generating function for $\widehat{S}$. Then

$$\begin{aligned} M(t) &= E[e^{t(\sum_{i=1}^{N} S_i)/N} \,|\, N > 0] = \frac{1}{\Pr(N > 0)} \sum_{n=1}^{\infty} E[e^{t(\sum_{i=1}^{N} S_i)/N} \,|\, N = n] \Pr(N = n) \\ &= \frac{1}{1 - e^{-\lambda}} \sum_{n=1}^{\infty} E\left[e^{t(\sum_{i=1}^{n} S_i)/n} \right] \frac{\lambda^n e^{-\lambda}}{n!} = \frac{1}{1 - e^{-\lambda}} \sum_{n=1}^{\infty} \left(E[e^{tS_i/n}] \right)^n \frac{\lambda^n e^{-\lambda}}{n!}, \end{aligned}$$

where the last equality follows from independence of $\{S_i\}$. Differentiating,

$$\begin{aligned}
M'(t) &= \frac{1}{1-e^{-\lambda}}\sum_{n=1}^{\infty} n\left(E[e^{tS_i/n}]\right)^{n-1} E\left[\frac{S_i}{n}e^{tS_i/n}\right]\frac{\lambda^n e^{-\lambda}}{n!}, \\
M''(t) &= \frac{1}{1-e^{-\lambda}}\sum_{n=1}^{\infty} n(n-1)\left(E[e^{tS_i/n}]\right)^{n-2}\left(E\left[\frac{S_i}{n}e^{tS_i/n}\right]\right)^2\frac{\lambda^n e^{-\lambda}}{n!} \\
&\quad +\frac{1}{1-e^{-\lambda}}\sum_{n=1}^{\infty} n\left(E[e^{tS_i/n}]\right)^{n-1} E\left[\frac{S_i^2}{n^2}e^{tS_i/n}\right]\frac{\lambda^n e^{-\lambda}}{n!}
\end{aligned}$$

The first and second moments are

$$\begin{aligned}
E[\widehat{S}] &= M'(0) = \frac{E[S]}{1-e^{-\lambda}}\sum_{n=1}^{\infty}\frac{\lambda^n e^{-\lambda}}{n!} = E[S], \\
E[\widehat{S}^2] &= M''(0) = \frac{(E[S])^2}{e^{\lambda}-1}\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)\frac{\lambda^n}{n!} + \frac{E[S^2]}{e^{\lambda}-1}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{1}{n} = (E[S])^2 + \frac{\mathrm{Var}[S]}{e^{\lambda}-1}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{1}{n},
\end{aligned}$$

which together yield $\mathrm{Var}[\widehat{S}] = E[\widehat{S}^2] - (E[\widehat{S}])^2 = \left(\frac{1}{e^{\lambda}-1}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{1}{n}\right)\mathrm{Var}[S]$. ■

Next, we prove (D3) and (D4). Let $I \equiv \mathbf{1}(N > 0)$. Note that, using (D5) and (D6), $E[\widehat{S}I \mid I = 0] = 0$, $E[\widehat{S}I \mid I = 1] = E[\widehat{S}] = E[S]$, $\mathrm{Var}[\widehat{S}I \mid I = 0] = 0$, and $\mathrm{Var}[\widehat{S}I \mid I = 1] = \mathrm{Var}[\widehat{S}] = \Delta(\lambda)\mathrm{Var}[S]$. The mean is

$$E[\widehat{S}I] = E[E[\widehat{S}I \mid I]] = \Pr(I = 1)E[S].$$

To compute the variance, first observe that

$$\begin{aligned}
\mathrm{Var}[E[\widehat{S}I \mid I]] &= E[E[\widehat{S}I \mid I]^2] - (E[E[\widehat{S}I \mid I]])^2 = E[E[\widehat{S}I \mid I]^2] - (E[\widehat{S}I])^2 \\
&= \Pr(I = 1)(E[\widehat{S}I \mid I = 1])^2 + \Pr(I = 0)(E[\widehat{S}I \mid I = 0])^2 - (\Pr(I = 1)E[S])^2 \\
&= \Pr(I = 1)(E[S])^2 - (\Pr(I = 1))^2(E[S])^2 = \Pr(I = 0)\Pr(I = 1)(E[S])^2,
\end{aligned}$$

where we have used the results obtained above. Therefore,

$$\begin{aligned}\mathrm{Var}[\widehat{S}I] &= \mathrm{Var}[E[\widehat{S}I \mid I]] + E[\mathrm{Var}[\widehat{S}I \mid I]] \\ &= \Pr(I=0)\Pr(I=1)(E[S])^2 + \Pr(I=1)\Delta(\lambda)\mathrm{Var}[S] \\ &= \Pr(I=1)\left(\Pr(I=0)(E[S])^2 + \Delta(\lambda)\mathrm{Var}[S]\right).\end{aligned}$$

Finally, we show the properties of $\Delta(\lambda)$. The following facts are useful:

$$\begin{aligned}\Delta'(\lambda) &= -\frac{e^{\lambda}}{(e^{\lambda}-1)^2}\sum_{n=1}^{\infty}\left(\frac{\lambda^n}{n!}\frac{1}{n}\right) + \frac{1}{e^{\lambda}-1}\sum_{n=1}^{\infty}\left(\frac{\lambda^{n-1}}{n!}\right) \\ &= -\frac{e^{\lambda}}{(e^{\lambda}-1)^2}\sum_{n=1}^{\infty}\left(\frac{\lambda^n}{n!}\frac{1}{n}\right) + \frac{1}{e^{\lambda}-1}\frac{1}{\lambda}\sum_{n=1}^{\infty}\left(\frac{\lambda^n}{n!}\right) = -\frac{e^{\lambda}}{(e^{\lambda}-1)^2}\sum_{n=1}^{\infty}\left(\frac{\lambda^n}{n!}\frac{1}{n}\right) + \frac{1}{\lambda} \\ &= -\frac{\Delta(\lambda)}{1-e^{-\lambda}} + \frac{1}{\lambda}, \qquad \text{(D7)}\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{1}{n} &= \lambda + \sum_{n=2}^{\infty}\frac{\lambda^n}{n!}\frac{1}{n} \\ &> \lambda + \sum_{n=2}^{\infty}\frac{\lambda^n}{(n+1)!} = \lambda + \frac{1}{\lambda}\sum_{n=3}^{\infty}\frac{\lambda^n}{n!} \\ &= \lambda + \frac{1}{\lambda}\left(e^{\lambda} - 1 - \lambda - \frac{\lambda^2}{2}\right) = \frac{1}{\lambda}\left(e^{\lambda} - 1 - \lambda + \frac{\lambda^2}{2}\right), \qquad \text{(D8)}\end{aligned}$$

and

$$e^{-\lambda} \leq 1 - \lambda + \lambda^2/2, \qquad \text{(D9)}$$

which can be shown as follows. Let $\xi(\lambda) \equiv 1 - \lambda + \lambda^2/2 - e^{-\lambda}$. Then $\xi'(\lambda) = -1 + \lambda + e^{-\lambda}$ and $\xi''(\lambda) = 1 - e^{-\lambda}$. Since $\xi'(0) = 0$ and $\xi''(\lambda) \geq 0$, we have $\xi'(\lambda) \geq 0$. But this implies $\xi(\lambda) \geq 0$ since $\xi'(0) = 0$.

(i) Differentiating $\Delta(\lambda)$, we obtain

$$\begin{aligned}
\Delta'(\lambda) &= -\frac{e^{\lambda}}{(e^{\lambda}-1)^2}\sum_{n=1}^{\infty}\left(\frac{\lambda^n}{n!}\frac{1}{n}\right)+\frac{1}{\lambda} \\
&< -\frac{e^{2\lambda}-e^{\lambda}-\lambda e^{\lambda}+\lambda^2 e^{\lambda}/2}{\lambda(e^{\lambda}-1)^2}+\frac{1}{\lambda} = -\frac{e^{2\lambda}-e^{\lambda}-\lambda e^{\lambda}+\lambda^2 e^{\lambda}/2-e^{2\lambda}+2e^{\lambda}-1}{\lambda(e^{\lambda}-1)^2} \\
&= -\frac{e^{\lambda}-\lambda e^{\lambda}+\lambda^2 e^{\lambda}/2-1}{\lambda(e^{\lambda}-1)^2} = -\frac{e^{\lambda}(1-\lambda+\lambda^2/2-e^{-\lambda})}{\lambda(e^{\lambda}-1)^2} \leq 0,
\end{aligned}$$

where the first and second inequalities follow from (D8) and (D9), respectively.

(ii) By l'Hopital's rule,

$$\lim_{\lambda\to 0}\Delta(\lambda)=\lim_{\lambda\to 0}\frac{\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{1}{n}}{e^{\lambda}-1}=\lim_{\lambda\to 0}\frac{\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{n!}}{e^{\lambda}}=\lim_{\lambda\to 0}\frac{1+\sum_{n=2}^{\infty}\frac{\lambda^{n-1}}{n!}}{e^{\lambda}}=1.$$

(iii) Applying l'Hopital's rule n times,

$$\lim_{\lambda\to\infty}\Delta(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{n}\left(\lim_{\lambda\to\infty}\frac{\lambda^n}{e^{\lambda}-1}\right)=\sum_{n=1}^{\infty}\frac{1}{n!}\frac{1}{n}\left(\lim_{\lambda\to\infty}\frac{n!}{e^{\lambda}}\right)=\sum_{n=1}^{\infty}\frac{1}{n}\left(\lim_{\lambda\to\infty}\frac{1}{e^{\lambda}}\right)=0.$$

(iv) The lower bound of $\Delta(\lambda)$ follows from (D7) and part (i).

(v) Since

$$\frac{\Delta(\lambda)}{e^{-\lambda}}=\frac{1}{1-e^{-\lambda}}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{1}{n},$$

we see that

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\left(\frac{\Delta(\lambda)}{e^{-\lambda}}\right) &= \frac{1}{(1-e^{-\lambda})^2}\left((1-e^{-\lambda})\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{n!}-e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{1}{n}\right) \\
&\geq \frac{1}{(1-e^{-\lambda})^2}\left((1-e^{-\lambda})\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{n!}-e^{-\lambda}\sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\right) \\
&= \frac{1}{(1-e^{-\lambda})}\left(\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{n!}-1\right)=\frac{\left(e^{\lambda}-1\right)/\lambda-1}{1-e^{-\lambda}}>0,
\end{aligned}$$

where the last inequality comes from $e^{\lambda}-1>\lambda$ for $\lambda>0$.

■

Proof of Lemma 2. The proofs for all results in the lemma are analogous to those of Lemma 1, except for the last result concerning the sign of $\partial\mu^*/\partial\lambda$. Suppose that λ is close to zero such that terms of order λ^2 and above can be dropped. Then the first-order condition is approximated as

$$c = \frac{p}{\mu^2}(1-e^{-\lambda}) + \frac{2\eta p^2}{\mu^3}\left(e^{-\lambda} + \Delta(\lambda)\theta(\mu)\right)(1-e^{-\lambda}) \approx \frac{p}{\mu^2}\lambda + \frac{2\eta p^2}{\mu^3}\left(1+\theta(\mu)\right)\lambda,$$

where we have used (ii) of Lemma D.1. This result identical to the first-order condition in Lemma 1 for CC. Hence, $\partial\mu^*/\partial\lambda > 0$ for small λ. On the other hand, if λ is sufficiently large so that $e^{-\lambda}/\Delta(\lambda) \approx 0$ (see (v) of Lemma D.1 and Table 1), $1-e^{-\lambda} \approx 1$ and the first-order condition becomes

$$c = \frac{p}{\mu^2}(1-e^{-\lambda}) + \frac{2\eta p^2}{\mu^3}\left(e^{-\lambda} + \Delta(\lambda)\theta(\mu)\right)(1-e^{-\lambda}) \approx \frac{p}{\mu^2} + \frac{2\eta p^2}{\mu^3}\Delta(\lambda)\theta(\mu).$$

Since $\Delta'(\lambda) < 0$ by (i) of Lemma D.1, it is clear from this expression that $\partial\mu^*/\partial\lambda < 0$. ■

Proof of Proposition 2.

(i) Rewriting p^{CC} and p^{AC} derived in Proposition 1,

$$p^{CC} = 2c\mu_I^2\left(\lambda + \sqrt{\lambda^2 + 8\eta c\mu_I\left(1+\theta(\mu_I)\right)\lambda}\right)^{-1} \quad \text{and} \tag{D10}$$

$$p^{AC} = 2c\mu_I^2\left((1-e^{-\lambda}) + \sqrt{(1-e^{-\lambda})^2 + 8\eta c\mu_I\left(e^{-\lambda} + \Delta(\lambda)\theta(\mu_I)\right)(1-e^{-\lambda})}\right)^{-1} \tag{D11}$$

$\partial p^{CC}/\partial\lambda < 0$ is clear from (D10). Since performance measures under CC and AC converge near $\lambda = 0$ (see the discussion below Lemma 2), $p^{AC} \to p^{CC}$ as $\lambda \to 0$, so $\lim_{\lambda\to 0}\partial p^{AC}/\partial\lambda < 0$. On the other hand, if λ is sufficiently large so that $e^{-\lambda}/\Delta(\lambda) \approx 0$ (see Table 1), (D11) can be approximated as $p^{AC} \approx 2c\mu_I^2\left(1+\sqrt{1+8\eta c\mu_I\Delta(\lambda)\theta(\mu_I)}\right)^{-1}$, from which we find that $\partial p^{AC}/\partial\lambda > 0$ since $\Delta'(\lambda) < 0$ by (i) of Lemma D.1. To show $\lim_{\lambda\to 0} p^{CC} = \lim_{\lambda\to 0} p^{AC} = \infty$, notice that term-by-term comparison of the denominators of (D10) and (D11) reveals that $p^{CC} < p^{AC}$, since $1-e^{-\lambda} < \lambda$, $e^{-\lambda} < 1$, and $\Delta(\lambda) < 1$. Retaining only the terms up to $O(\lambda)$, we see that $p^{AC} \to p^{CC}$ in the $\lambda \to 0$ limit. Moreover, $\lim_{\lambda\to 0} p^{CC} = \infty$ is clear from (D10).

(ii) $\partial\psi^{CC}/\partial\lambda < 0$ is clear from the expression of ψ^{CC} in Proposition 1. To show $\partial\psi^{AC}/\partial\lambda < 0$, let $\varphi(\lambda) \equiv \frac{e^{-\lambda}+\Delta(\lambda)v(\mu_I)^2}{e^{-\lambda}+\Delta(\lambda)\theta(\mu_I)}$ be the multiplicative factor that appears in ψ^{AC} (see the same proposition). Define $\chi \equiv -\mu_I v(\mu_I)v'(\mu_I) \geq 0$. Note that

$$\varphi'(\lambda) = \frac{d}{d\lambda}\left(1 + \frac{\chi}{e^{-\lambda}/\Delta(\lambda) + v(\mu_I)^2}\right)^{-1} < 0,$$

by the property (v) of Lemma D.1. Using this and $\Delta'(\lambda) < 0$, $\partial\psi^{AC}/\partial\lambda < 0$ follows. Notice that $\lim_{\lambda\to 0}\varphi(\lambda) = \varphi_c$, where $\varphi_c \equiv \left(1 + v(\mu_I)^2\right) / \left(1 + \theta(\mu_I)\right)$ is the multiplicative factor that appears in ψ^{CC}. Together with $\varphi'(\lambda) < 0$, this implies $\varphi(\lambda) < \varphi_c$. With the latter, the stated condition $V^{CC}(\lambda) \geq V^{AC}(\lambda)$, or $\left(1 + \theta(\mu_I)\right)/\lambda \geq \left(e^{-\lambda} + \Delta(\lambda)\theta(\mu_I)\right)/(1 - e^{-\lambda})$, implies $\psi^{CC} \geq \psi^{AC}$, as can be verified from their respective expressions. The $\lambda \to 0$ limit is immediate from the same expressions.

■

Proof of Lemma 3. For notational convenience, let $u^* \equiv u(\mu^*)$, $u_t^* \equiv u(\mu_t^*)$, and $U_t^* \equiv U_t(\mu_t^*)$.

(i) Under CC or under AC with $\lambda_t \sim 0$, the first-order condition for μ_t corresponding to λ_t is (or is approximated as, in the case of AC, since AC mimics CC if $\lambda_t \sim 0$)

$$\frac{p}{\mu_t^2} + \frac{2\eta p^2}{\mu_t^3}\left(1 + v^2\right) = \frac{c}{\lambda_t},$$

from Lemma 1. For each $t \in \{L, H\}$, the supplier's utility at μ_t^* that solves this optimality condition is $U_t^* = w - p\lambda_t/\mu_t^* - \eta p^2\left(1 + v^2\right)\lambda_t/(\mu_t^*)^2 - c(\mu_t^* - \underline{\mu}) - K\mathbf{1}(t = L)$. Their difference is $U_H^* - U_L^* = -\zeta(p) + K$, where

$$\zeta(p) \equiv p\left(\frac{\lambda_H}{\mu_H^*} - \frac{\lambda_L}{\mu_L^*}\right) + \eta p^2\left(1 + v^2\right)\left(\frac{\lambda_H}{(\mu_H^*)^2} - \frac{\lambda_L}{(\mu_L^*)^2}\right) + c(\mu_H^* - \mu_L^*).$$

Hence, $U_H^* \geq U_L^*$, i.e., the supplier chooses λ_H, if and only if $K \geq \zeta(p)$. Note that, from the first-order condition above for a generic λ,

$$\frac{\lambda}{\mu^*} = c\left(\frac{p}{\mu^*} + \frac{2\eta p^2}{(\mu^*)^2}\left(1 + v^2\right)\right)^{-1} \text{ and } \frac{\lambda}{(\mu^*)^2} = c\left(p + \frac{2\eta p^2}{\mu^*}\left(1 + v^2\right)\right)^{-1},$$

both of which are increasing in μ^* for fixed p. On the other hand, $\mu_L^* < \mu_H^*$ since $\partial\mu^*/\partial\lambda > 0$

for fixed p, according to Lemma 1. Therefore, we have $\lambda_L/\mu_L^* < \lambda_H/\mu_H^*$ and $\lambda_L/(\mu_L^*)^2 < \lambda_H/(\mu_H^*)^2$ and conclude that $\zeta(p) > 0$. Using these results and applying the envelope theorem, we have $\frac{d}{dp}\left(U_H^* - U_L^*\right) = \frac{\partial}{\partial p}\left(U_H^* - U_L^*\right) = -\frac{\partial}{\partial p}\zeta(p) = -\left(\frac{\lambda_H}{\mu_H^*} - \frac{\lambda_L}{\mu_L^*}\right) - 2\eta p\left(1+v^2\right)\left(\frac{\lambda_H}{(\mu_H^*)^2} - \frac{\lambda_L}{(\mu_L^*)^2}\right) < 0$. Because of this monotonicity, $U_H^* - U_L^*$ crosses zero at most once, i.e., $p^\dagger$ that satisfies $\zeta(p^\dagger) = K$ is unique if it exists. Suppose that $K < \zeta(\underline{\underline{\underline{p}}})$. Then $U_H^* - U_L^*$ starts from a negative value at $p = \underline{\underline{\underline{p}}}$ and becomes more negative as p increases. Hence, the supplier always chooses λ_L in this case (the statement in (i) of the lemma is true by setting $p^\dagger = \underline{\underline{\underline{p}}}$). On the other hand, if $K \geq \zeta(\underline{\underline{\underline{p}}})$, there may be a value (which we have shown to be unique) $p^\dagger \geq \underline{\underline{\underline{p}}}$ for which $u_H^* - u_L^*$ crosses zero from positive to negative. In this case, the supplier chooses λ_H if $p \leq p^\dagger$ and λ_L if $p > p^\dagger$.

(ii) Fix p and λ. Under AC with λ sufficiently large for which $e^{-\lambda}/\Delta(\lambda) \approx 0$, the supplier's utility at μ^*, which satisfies the first-order condition in Lemma 2, is approximated as $u^* \approx w - p/\mu^* - \eta p^2\Delta(\lambda)v^2/(\mu^*)^2 - c(\mu^* - \underline{\mu})$. By the envelope theorem, $du^*/d\lambda = \partial u^*/\partial\lambda \approx -\eta p^2\Delta'(\lambda)v^2/(\mu^*)^2 > 0$, implying that $u_H^* - u_L^*$ increases as the distance between λ_L and λ_H becomes larger. Since $U_t^* = u_t^* - K\mathbf{1}(t = L)$ and $u_H^* - u_L^* \to 0$ as $\lambda_H - \lambda_L \to 0$, $U_H^* - U_L^* \to K \geq 0$ as $\lambda_H - \lambda_L \to 0$ while $U_H^* - U_L^*$ increases as $\lambda_H - \lambda_L$ becomes larger. In other words, $U_H^* - U_L^* > K \geq 0$ for any $\lambda_L < \lambda_H$ regardless of p. Therefore, the supplier always chooses λ_H.

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Proof of Proposition B.1. The CC and AC converge to one another when $\lambda \sim 0$ since their performance measures become indistinguishable, as $\sum_{i=1}^{N} S_i \sim S_1\mathbf{1}(N = 1)$ and $\widehat{S}\mathbf{1}(N > 1) \sim S_1\mathbf{1}(N = 1)$. Let us consider AC with $\lambda \sim 0$, for which $T_a \approx w - pS_1\mathbf{1}(N = 1)$. Since the period length is normalized to one, $\mathbf{1}(N = 1) = \mathbf{1}(Y_1 < 1)$. Thus, $T \approx w - pS_1\mathbf{1}(Y_1 < 1)$. By the law of total variance (proof is similar to that of Lemma D.1), it can be shown that

$$\begin{aligned} E[T \,|\, \lambda, \mu] &\approx w - pF(1 \,|\, \lambda)E[S \,|\, \mu], \\ \operatorname{Var}[T \,|\, \lambda, \mu] &\approx p^2F(1 \,|\, \lambda)\left([1 - F(1 \,|\, \lambda)](E[S \,|\, \mu])^2 + \operatorname{Var}[S \,|\, \mu]\right) \approx p^2F(1 \,|\, \lambda)\left((E[S \,|\, \mu])^2 + \operatorname{Var}[S \,|\, \mu]\right), \end{aligned}$$

where the last approximation is valid since $[F(1 \,|\, \lambda)]^2$ is negligible when $\lambda \sim 0$. Compare these expressions to their Poisson counterparts for λ around zero (where the terms of order only up to $O(\lambda)$ in (D3) and (D4) are retained):

$$\begin{aligned} E[T \,|\, \lambda, \mu] &= w - p(1-e^{-\lambda})E[S \,|\, \mu] \approx w - p\lambda E[S \,|\, \mu], \\ \mathrm{Var}[T \,|\, \lambda, \mu] &= p^2(1-e^{-\lambda})\left(e^{-\lambda}(E[S \,|\, \mu])^2 + \Delta(\lambda)\mathrm{Var}[S \,|\, \mu]\right) \approx p^2\lambda\left((E[S \,|\, \mu])^2 + \mathrm{Var}[S \,|\, \mu]\right). \end{aligned}$$

We see that they have the same forms, the only difference being that $F(1 \,|\, \lambda)$ is substituted by λ. Hence, the analysis of AC with Y is equivalent to that with the Poisson failures. Therefore, when $\lambda \sim 0$,

$$\psi_Y^{AC} \approx \frac{c\mu_I}{2}\left(\frac{1+v(\mu_I)^2}{1+\theta(\mu_I)}\right)\left(1 - 2\left(1 + \sqrt{1 + 8\eta c\mu_I \frac{1+\theta(\mu_I)}{F(1 \,|\, \lambda)}}\right)^{-1}\right),$$

which is obtained from ψ^{AC} in Proposition 1 with $1 - e^{-\lambda} \approx \lambda$ replaced by $F(1 \,|\, \lambda)$ and $e^{-\lambda} \to 1$ and $\Delta(\lambda) \to 1$. $\lim_{\lambda \to 0} \psi_Y^{AC} = \frac{c\mu_I}{2}\left(\frac{1+v(\mu_I)^2}{1+\theta(\mu_I)}\right)$ follows after letting $\lambda \to 0$, since $\lim_{\lambda \to 0} F(\cdot \,|\, \lambda) = 0$ by assumption. ■