Technical Online Appendix

REVENUE MANAGEMENT THROUGH DYNAMIC CROSS-SELLING IN E-COMMERCE RETAILING

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Proof of Proposition 2

a) Due to the symmetry of the problem, we need to consider only the case of $i = 1$. Using induction, we assume that $G_{n+1}^1(I_1)$ is a nondecreasing concave function of I_1 for some $n = 1, ..., N$, i.e., for $I_1 \geq 1$,

$$G_{n+1}^1(I_1) \geq G_{n+1}^1(I_1 - 1), \tag{52}$$

and

$$G_{n+1}^1(I_1 + 1) + G_{n+1}^1(I_1 - 1) - 2G_{n+1}^1(I_1) \leq 0. \tag{53}$$

Next we show that $G_n^1(I_1)$ is a nondecreasing concave function of I_1. We need to prove that $G_n^1(I_1) \geq G_n^1(I_1-1)$ and $G_n^1(I_1 + 1) + G_n^1(I_1 - 1) - 2G_n^1(I_1) \leq 0$. The proof is carried for two separate cases, $I_1 \geq 2$ and $I_1 = 1$.

Case 1: $I_1 \geq 2$.

Using (9) and the definition of $g_{j1}(\cdot)$ in Lemma A1, we have for $I_1 \geq 1$,

$$G_n^1(I_1) = \lambda_1 \left(p_1 + G_{n+1}^1(I_1 - 1)\right) + (1 - \lambda_1)G_{n+1}^1(I_1) + \sum_{j \in E(1)} \lambda_j g_{j1} \left(p_j + G_{n+1}^1(I_1) - G_{n+1}^1(I_1 - 1)\right). \tag{54}$$

The right-hand side of (54), by the induction assumption, is a nondecreasing function of inventory. Thus, so is $G_n^1(\cdot)$.

Further, denote $x_1 = p_j + G_{n+1}^1(I_1 + 1) - G_{n+1}^1(I_1)$, $x_2 = p_j + G_{n+1}^1(I_1) - G_{n+1}^1(I_1 - 1)$, and $x_3 = p_j + G_{n+1}^1(I_1 - 1) - G_{n+1}^1(I_1 - 2)$. From (53), $x_1 \leq x_2 \leq x_3$. Thus, using (35), we have

$$\begin{aligned} g_{j1}(x_1) + g_{j1}(x_3) - 2g_{j1}(x_2) &\leq x_2 - x_1 \\ &= \left(2G_{n+1}^1(I_1) - G_{n+1}^1(I_1 + 1) - G_{n+1}^1(I_1 - 1)\right). \end{aligned} \tag{55}$$

From (54), for $I_1 \geq 2$ we obtain

$$\begin{aligned} & G_n^1(I_1 + 1) + G_n^1(I_1 - 1) - 2G_n^1(I_1) \\ = \; & \lambda_1 \left(G_{n+1}^1(I_1) + G_{n+1}^1(I_1 - 2) - 2G_{n+1}^1(I_1 - 1)\right) \\ & +(1 - \lambda_1) \left(G_{n+1}^1(I_1 + 1) + G_{n+1}^1(I_1 - 1) - 2G_{n+1}^1(I_1)\right) \end{aligned}$$

$$
\begin{aligned}
& + \sum_{j\in E(1)} \lambda_j \left(g_{j1}(x_1) + g_{j1}(x_3) - 2g_{j1}(x_2)\right) \\
\leq\ & \lambda_1 \left(G^1_{n+1}(I_1) + G^1_{n+1}(I_1 - 2) - 2G^1_{n+1}(I_1 - 1)\right) \\
& +(1-\lambda_1) \left(G^1_{n+1}(I_1 + 1) + G^1_{n+1}(I_1 - 1) - 2G^1_{n+1}(I_1)\right) \\
& - \sum_{j\in E(1)} \lambda_j \left(G^1_{n+1}(I_1 + 1) + G^1_{n+1}(I_1 - 1) - 2G^1_{n+1}(I_1)\right) \quad \text{(from (55))} \\
\leq\ & (1 - \lambda_1 - \sum_{j\in E(1)} \lambda_j) \left(G^1_{n+1}(I_1 + 1) + G^1_{n+1}(I_1 - 1) - 2G^1_{n+1}(I_1)\right) \quad \text{(from (53))} \\
\leq\ & 0.
\end{aligned} \tag{56}
$$

Case 2: $I_1 = 1$. This case can be proved by using a similar argument as in Case 1.

b) As (9) indicates, $p^*_{i,j(i)}(\mathbf{I}, n)$ is the maximizer of $g_{i,j(i)}\left(p_i + G^{j(i)}_{n+1}(I_{j(i)}) - G^{j(i)}_{n+1}(I_{j(i)} - 1)\right)$, where $g_{ij}(\cdot)$ is defined in Lemma A1. Clearly, this maximizer is independent of I_k, for $k \neq j(i)$. From the proof in part (a) we know that $G^{j(i)}_{n+1}(I_{j(i)} + 1) - G^{j(i)}_{n+1}(I_{j(i)}) \leq G^{j(i)}_{n+1}(I_{j(i)}) - G^{j(i)}_{n+1}(I_{j(i)} - 1)$, which together with the statement of Lemma A1b indicates that $p^*_{i,j(i)}(\mathbf{I} + \mathbf{e}_{j(i)}, n) \leq p^*_{i,j(i)}(\mathbf{I}, n)$.

Next, we prove that $p^*_{i,j(i)}(\mathbf{I}, n) \geq p^*_{i,j(i)}(\mathbf{I}, n+1)$ for $n = 1, ..., N$. Using the result Lemma A1b, it suffices to prove that $G^{j(i)}_{n}(I_{j(i)}) - G^{j(i)}_{n}(I_{j(i)} - 1) \geq G^{j(i)}_{n+1}(I_{j(i)}) - G^{j(i)}_{n+1}(I_{j(i)} - 1)$ for $I_{j(i)} \geq 1$ and $n = 1, ..., N$. For notational simplicity, we replace $j(i)$ with j. This is clearly true when $n = N$. Using induction, suppose for $I_j \geq 1$, $n = 1, ..., N-1$

$$
G^j_{n+1}(I_j) - G^j_{n+1}(I_j - 1) \geq G^j_{n+2}(I_j) - G^j_{n+2}(I_j - 1). \tag{57}
$$

The case with $I_j = 1$ is trivial. Next we consider $I_j \geq 2$:

$$
\begin{aligned}
& \left(G^j_n(I_j) - G^j_n(I_j - 1)\right) - \left(G^j_{n+1}(I_j) - G^j_{n+1}(I_j - 1)\right) \\
=\ & \lambda_j \left(\left(G^j_{n+1}(I_j - 1) - G^j_{n+1}(I_j - 2)\right) - \left(G^j_{n+2}(I_j - 1) - G^j_{n+2}(I_j - 2)\right)\right) \\
& +(1-\lambda_j) \left(\left(G^j_{n+1}(I_j) - G^j_{n+1}(I_j - 1)\right) - \left(G^j_{n+2}(I_j) - G^j_{n+2}(I_j - 1)\right)\right) \\
& + \sum_{k\in E(j)} \lambda_k \left(g_{kj}\left(p_k + G^j_{n+1}(I_j) - G^j_{n+1}(I_j - 1)\right) - g_{kj}\left(p_k + G^j_{n+1}(I_j - 1) - G^j_{n+1}(I_j - 2)\right)\right) \\
& - \sum_{k\in E(j)} \lambda_k \left(g_{kj}\left(p_k + G^j_{n+2}(I_j) - G^j_{n+2}(I_j - 1)\right) - g_{kj}\left(p_k + G^j_{n+2}(I_j - 1) - G^j_{n+2}(I_j - 2)\right)\right) \\
\geq\ & (1-\lambda_j) \left(\left(G^j_{n+1}(I_j) - G^j_{n+1}(I_j - 1)\right) - \left(G^j_{n+2}(I_j) - G^j_{n+2}(I_j - 1)\right)\right) \\
& + \sum_{k\in E(j)} \lambda_k \left(\begin{array}{c} g_{kj}\left(p_k + G^j_{n+1}(I_j) - G^j_{n+1}(I_j - 1)\right) \\ -g_{kj}\left(p_k + G^j_{n+2}(I_j) - G^j_{n+2}(I_j - 1)\right) \end{array}\right) \quad \text{(from (57,35))}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sum_{k\in E(j)} \lambda_k \left(\left(G^j_{n+1}(I_j) - G^j_{n+1}(I_j-1) \right) - \left(G^j_{n+2}(I_j) - G^j_{n+2}(I_j-1) \right) \right) \\
&\quad + \sum_{k\in E(j)} \lambda_k \begin{pmatrix} g_{kj}\left(p_k + G^j_{n+1}(I_j) - G^j_{n+1}(I_j-1)\right) \\ -g_{kj}\left(p_k + G^j_{n+2}(I_j) - G^j_{n+2}(I_j-1)\right) \end{pmatrix} \quad \text{(from (57))} \\
&\geq 0 \text{ (from (57,35))}. \qquad (58)
\end{aligned}
$$

Proof of Proposition 5

a) Note that for $I_1, I_2 \geq 1$, (7) can be rewritten as

$$
\begin{aligned}
V_n(I_1,I_2) &= \lambda_1 \left(p_1 + V_{n+1}(I_1-1,I_2) + g_{12}\left(p_1 + V_{n+1}(I_1-1,I_2) - V_{n+1}(I_1-1,I_2-1)\right)\right) \\
&\quad + \lambda_2 \left(p_2 + V_{n+1}(I_1,I_2-1) + g_{21}\left(p_2 + V_{n+1}(I_1,I_2-1) - V_{n+1}(I_1-1,I_2-1)\right)\right) \\
&\quad + (1-\lambda_1-\lambda_2) V_{n+1}(I_1,I_2), \qquad (59)
\end{aligned}
$$

where $g_{12}(\cdot)$ and $g_{21}(\cdot)$ are defined as in Lemma A1.

The statement of part a is clearly true for $n = N+1$. By induction, suppose that for some $n = 1,...,N$

$$
V_{n+1}(I_1+1,I_2) - V_{n+1}(I_1,I_2) \geq 0 \quad \text{and} \quad V_{n+1}(I_1,I_2+1) - V_{n+1}(I_1,I_2) \geq 0. \qquad (60)
$$

By symmetry, we need to prove only that $V_n(I_1,I_2)$ is nondecreasing in I_1 for fixed I_2. We consider four separate cases.

Case 1: $I_1 \geq 1$ and $I_2 \geq 1$.

Denote $x_1 = V_{n+1}(I_1,I_2) - V_{n+1}(I_1-1,I_2)$, $x_2 = V_{n+1}(I_1,I_2) - V_{n+1}(I_1,I_2-1)$, $x_3 = V_{n+1}(I_1-1,I_2) - V_{n+1}(I_1-1,I_2-1)$, $x_4 = V_{n+1}(I_1+1,I_2-1) - V_{n+1}(I_1,I_2-1)$, $x_5 = V_{n+1}(I_1,I_2-1) - V_{n+1}(I_1-1,I_2-1)$, and $x_6 = V_{n+1}(I_1+1,I_2) - V_{n+1}(I_1,I_2)$. From (60), we have

$$
x_i \geq 0, i = 1,...,6. \qquad (61)
$$

Furthermore, from (60), it is easy to check that

$$
x_1 - x_2 + x_3 \geq 0. \qquad (62)
$$

Using (35) and (59-62), we obtain

$$
\begin{aligned}
& V_n(I_1+1,I_2) - V_n(I_1,I_2) \\
= \;& \lambda_1 \left(x_1 + g_{12}(p_1+x_2) - g_{12}(p_1+x_3)\right) + \lambda_2 \left(x_4 + g_{21}(p_2+x_4) - g_{21}(p_2+x_5)\right) + (1-\lambda_1-\lambda_2)x_6 \\
\geq \;& \lambda_1 \left(x_1 + \min(0, x_3 - x_2)\right) + \lambda_2 \left(x_4 + \min(0, x_5 - x_4)\right) \\
\geq \;& \lambda_1 \min(x_1, x_1 - x_2 + x_3) + \lambda_2 \min(x_4, x_5) \geq 0. \qquad (63)
\end{aligned}
$$

Case 2**:** $I_1 \geq 1$ and $I_2 = 0$.

Using (23) and (60), we obtain

$$\begin{aligned} & V_n(I_1+1,0)-V_n(I_1,0) \\ = \; & \lambda_1[V_{n+1}(I_1,0)-V_{n+1}(I_1-1,0)]+(1-\lambda_1)[V_{n+1}(I_1+1,0)-V_{n+1}(I_1,0)] \geq 0. \end{aligned} \tag{64}$$

Case 3: $I_1=0$ and $I_2 \geq 1$.

Using (59), (60) and (24)

$$\begin{aligned} V_n(1,I_2) & \geq \lambda_1 V_{n+1}(0,I_2)+\lambda_2[p_2+V_{n+1}(1,I_2-1)]+(1-\lambda_1-\lambda_2)V_{n+1}(1,I_2) \\ & \geq \lambda_2[p_2+V_{n+1}(0,I_2-1)]+(1-\lambda_2)V_{n+1}(0,I_2)=V_n(0,I_2). \end{aligned} \tag{65}$$

Case 4: $I_1=0$ and $I_2=0$. The result trivially holds in this case.

b) The statement clearly holds for $n=N+1$. Suppose that for some $n=1,...,N$

$$V_{n+1}(I_1+1,I_2+1)-V_{n+1}(I_1,I_2+1) \geq V_{n+1}(I_1+1,I_2)-V_{n+1}(I_1,I_2). \tag{66}$$

Again, we look at four cases.

Case 1: $I_1 \geq 1$ and $I_2 \geq 1$.

We reuse the notation x_i ($i=1,2,\dots,6$) defined in the proof of part a. In addition, denote $x_1' = V_{n+1}(I_1,I_2+1)-V_{n+1}(I_1-1,I_2+1)$, $x_2' = V_{n+1}(I_1,I_2+1)-V_{n+1}(I_1,I_2)$, $x_3' = V_{n+1}(I_1-1,I_2+1)-V_{n+1}(I_1-1,I_2)$, $x_4' = V_{n+1}(I_1+1,I_2)-V_{n+1}(I_1,I_2)$, $x_5' = V_{n+1}(I_1,I_2)-V_{n+1}(I_1-1,I_2)$, and $x_6' = V_{n+1}(I_1+1,I_2+1)-V_{n+1}(I_1,I_2+1)$. From (60) and (66) we have for $i=1,2,\dots,6$,

$$x_i' \geq x_i \geq 0. \tag{67}$$

Furthermore, from (66), we have

$$\begin{aligned} x_2-x_3 & \geq 0, \\ x_2'-x_3' & \geq 0. \end{aligned} \tag{68}$$

It is also easy to check that

$$x_1'-x_2'+x_3'=x_1. \tag{69}$$

Using (59),

$$\begin{aligned} & V_n(I_1+1,I_2+1)-V_n(I_1,I_2+1) \\ = \; & \lambda_1\left(x_1'+g_{12}(p_1+x_2')-g_{12}(p_1+x_3')\right)+\lambda_2\left(x_4'+g_{21}(p_2+x_4')-g_{21}(p_2+x_5')\right)+(1-\lambda_1-\lambda_2)x_6' \end{aligned}$$

$$
\begin{aligned}
&\geq \lambda_1\left(x_1' + \min(0, x_3' - x_2')\right) + \lambda_2\left(x_4' + g_{21}(p_2 + x_4') - g_{21}(p_2 + x_5')\right) + (1-\lambda_1-\lambda_2)x_6' \quad \text{(from (35))} \\
&\geq \lambda_1(x_1' - x_2' + x_3') + \lambda_2\left(x_4' + g_{21}(p_2 + x_4') - g_{21}(p_2 + x_5')\right) + (1-\lambda_1-\lambda_2)x_6' \quad \text{(from (68))} \\
&\geq \lambda_1 x_1 + \lambda_2\left(x_4' + g_{21}(p_2 + x_4') - g_{21}(p_2 + x_5)\right) + (1-\lambda_1-\lambda_2)x_6 \quad \text{(from (35), (67) and (69))} \\
&\geq \lambda_1 x_1 + \lambda_2\left(x_4 + g_{21}(p_2 + x_4) - g_{21}(p_2 + x_5)\right) \\
&\quad +(1-\lambda_1-\lambda_2)x_6. \quad \text{(by (35) and (67))} \qquad (70)
\end{aligned}
$$

On the other hand, from (59), we have

$$
\begin{aligned}
&V_n(I_1+1, I_2) - V_n(I_1, I_2) \\
&= \lambda_1\left(x_1 + g_{12}(p_1 + x_2) - g_{12}(p_1 + x_3)\right) + \lambda_2\left(x_4 + g_{21}(p_2 + x_4) - g_{21}(p_2 + x_5)\right) + (1-\lambda_1-\lambda_2)x_6 \\
&\leq \lambda_1\left(x_1 + \max(0, x_3 - x_2)\right) + \lambda_2\left(x_4 + g_{21}(p_2 + x_4) - g_{21}(p_2 + x_5)\right) + (1-\lambda_1-\lambda_2)x_6 \quad \text{(from (35))} \\
&\leq \lambda_1 x_1 + \lambda_2\left(x_4 + g_{21}(p_2 + x_4) - g_{21}(p_2 + x_5)\right) + (1-\lambda_1-\lambda_2)x_6. \quad \text{(from (68))} \qquad (71)
\end{aligned}
$$

Combining (70) and (71), we obtain the desired result.

Case 2): $I_1 \geq 1$ and $I_2 = 0$.

First, note that it is easy to prove by induction that for $I_1 \geq 1$ and $n = 1, ..., N$,

$$V_n(I_1+1, 0) - V_n(I_1, 0) \leq V_n(I_1, 0) - V_n(I_1-1, 0). \qquad (72)$$

Denote $z_1 = V_{n+1}(I_1, 1) - V_{n+1}(I_1-1, 1)$, $z_2 = V_{n+1}(I_1, 1) - V_{n+1}(I_1, 0)$, $z_3 = V_{n+1}(I_1-1, 1) - V_{n+1}(I_1-1, 0)$, $z_4 = V_{n+1}(I_1+1, 0) - V_{n+1}(I_1, 0)$, $z_5 = V_{n+1}(I_1, 0) - V_{n+1}(I_1-1, 0)$, and $z_6 = V_{n+1}(I_1+1, 1) - V_{n+1}(I_1, 1)$. From (72), we have

$$z_4 \leq z_5. \qquad (73)$$

Similar to the proof in Case 1, from (59), we obtain

$$
\begin{aligned}
&V_n(I_1+1, 1) - V_n(I_1, 1) \\
&= \lambda_1[z_1 + g_{12}(p_1 + z_2) - g_{12}(p_1 + z_3)] + \lambda_2[z_4 + g_{21}(p_2 + z_4) - g_{21}(p_2 + z_5)] + (1-\lambda_1-\lambda_2)z_6 \\
&\geq \lambda_1(z_1 - z_2 + z_3) + \lambda_2 z_4 + (1-\lambda_1-\lambda_2)z_6 \quad \text{(from (35) and (73))} \\
&\geq \lambda_1(z_1 - z_2 + z_3) + (1-\lambda_1)z_4 \quad \text{(from (66))} \\
&= V_n(I_1+1, 0) - V_n(I_1, 0). \quad \text{(from (23))} \qquad (74)
\end{aligned}
$$

The proofs for the case of $I_1 = 0$ and $I_2 \geq 1$, and for the case of $I_1 = 0$ and $I_2 = 0$ are similar to the proof of Case 2.

c) (28) follows from (59) and the result of part b.

Proof of Proposition 6

Using induction, we assume that

$$V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i) \geq V_{n+1}^{LS}(\mathbf{I}) \tag{75}$$

for some $n = 1, ..., N$. Next we show that $V_n^{LS}(\mathbf{I}+\mathbf{e}_i) \geq V_n^{LS}(\mathbf{I})$. The proof is carried for two separate cases, $I_i \geq 1$ and $I_i = 0$.

Case 1: $I_i \geq 1$.

Using (30) and the definition of $g_{ij}(\cdot)$ in Lemma A1, we have

$$\begin{aligned}
& V_n^{LS}(\mathbf{I}+\mathbf{e}_i) - V_n^{LS}(\mathbf{I}) \\
= & \sum_{k\in A_n, j(k)\in A_n} \lambda_k \left(\begin{array}{c} V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i-\mathbf{e}_k) + g_{k,j(k)}\left(p_k + V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i-\mathbf{e}_k) - V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i-\mathbf{e}_k-\mathbf{e}_{j(k)})\right) \\ -V_{n+1}^{LS}(\mathbf{I}-\mathbf{e}_k) - g_{k,j(k)}\left(p_k + V_{n+1}^{LS}(\mathbf{I}-\mathbf{e}_k) - V_{n+1}^{LS}(\mathbf{I}-\mathbf{e}_k-\mathbf{e}_{j(k)})\right) \end{array} \right) \\
& + \sum_{k\in A_n, j(k)\notin A_n} \lambda_k \left(V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i-\mathbf{e}_k) - V_{n+1}^{LS}(\mathbf{I}-\mathbf{e}_k)\right) + \left(1 - \sum_{i\in A_n}\lambda_i\right)\left(V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i) - V_{n+1}^{LS}(\mathbf{I})\right) \\
\geqslant & \sum_{k\in A_n, j(k)\in A_n} \lambda_k \min\left(V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i-\mathbf{e}_k) - V_{n+1}^{LS}(\mathbf{I}-\mathbf{e}_k), V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i-\mathbf{e}_k-\mathbf{e}_{j(k)}) - V_{n+1}^{LS}(\mathbf{I}-\mathbf{e}_k-\mathbf{e}_{j(k)})\right) \\
& + \sum_{\substack{k\in A_n, \\ j(k)\notin A_n}} \lambda_k \left(V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i-\mathbf{e}_k) - V_{n+1}^{LS}(\mathbf{I}-\mathbf{e}_k)\right) + \left(1 - \sum_{i\in A_n}\lambda_i\right)\left(V_{n+1}^{LS}(\mathbf{I}+\mathbf{e}_i) - V_{n+1}^{LS}(\mathbf{I})\right) \qquad \text{(from (35))} \\
\geqslant & \ 0. \text{ (from (75))}
\end{aligned}$$

Case 2: $I_i = 0$. This case can be proven by using an argument similar to the one used in the proof of Case 1.

Proof of Proposition 7

a) Using induction, we assume that

$$L_{n+1}(\mathbf{I}) \leq V_{n+1}^{LS}(\mathbf{I}), \tag{76}$$

and

$$V_{n+1}^{LS}(\mathbf{I}) \leq U_{n+1}(\mathbf{I}) \tag{77}$$

for some $n = 1, ..., N$. From (31) and (32), it is easy to verify that

$$L_n(\mathbf{I}) = \sum_{i\in A_n} \lambda_i \left(p_i + L_{n+1}(\mathbf{I}-\mathbf{e}_i)\right) + \left(1 - \sum_{i\in A_n}\lambda_i\right) L_{n+1}(\mathbf{I}), \tag{78}$$

and

$$U_n(\mathbf{I}) = \sum_{i \in A_n} \lambda_i \left(p_i + U_{n+1}(\mathbf{I} - \mathbf{e}_i) + \overline{F}_{i,j(i)}(p^M_{i,j(i)})(p^M_{i,j(i)} - p_i)\right) + \left(1 - \sum_{i \in A_n} \lambda_i\right) U_{n+1}(\mathbf{I}). \tag{79}$$

We denote the optimal price under the LS model by p^{LS}_{ij}. By the definition of p^{LS}_{ij}, we have for $i = 1, ..., m$, and $n = 1, ..., N$,

$$\overline{F}_{i,j(i)}(p^{LS}_{i,j(i)}) \left(p^{LS}_{i,j(i)} - p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i - \mathbf{e}_{j(i)}) - V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i)\right) \geqslant 0. \tag{80}$$

From (30), we have

$$\begin{aligned}
V^{LS}_n(\mathbf{I}) &= \sum_{i \in A_n, j(i) \in A_n} \lambda_i \left(\begin{array}{c} p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i) + \\ \overline{F}_{i,j(i)}(p^{LS}_{i,j(i)}) \left(p^{LS}_{i,j(i)} - p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i - \mathbf{e}_{j(i)}) - V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i)\right) \end{array} \right) \\
&\quad + \sum_{i \in A_n, j(i) \notin A_n} \lambda_i \left(p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i)\right) + \left(1 - \sum_{i \in A_n} \lambda_i\right) V^{LS}_{n+1}(\mathbf{I}) \\
&\geqslant \sum_{i \in A_n} \lambda_i \left(p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i)\right) + \left(1 - \sum_{i \in A_n} \lambda_i\right) V^{LS}_{n+1}(\mathbf{I}) \ \text{(from (80))} \\
&\geqslant \sum_{i \in A_n} \lambda_i \left(p_i + L_{n+1}(\mathbf{I} - \mathbf{e}_i)\right) + \left(1 - \sum_{i \in A_n} \lambda_i\right) L_{n+1}(\mathbf{I}) \ \text{(from (76))} \\
&= L_n(\mathbf{I}). \ \text{(from (78))}
\end{aligned}$$

Also,

$$\begin{aligned}
& V^{LS}_n(\mathbf{I}) \\
= & \sum_{i \in A_n, j(i) \in A_n} \lambda_i \left(\begin{array}{c} p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i) + \\ \overline{F}_{i,j(i)}(p^{LS}_{i,j(i)}) \left(p^{LS}_{i,j(i)} - p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i - \mathbf{e}_{j(i)}) - V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i)\right) \end{array} \right) \\
& + \sum_{i \in A_n, j(i) \notin A_n} \lambda_i \left(p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i)\right) + \left(1 - \sum_{i \in A_n} \lambda_i\right) V^{LS}_{n+1}(\mathbf{I}) \\
\leq & \sum_{i \in A_n, j(i) \in A_n} \lambda_i \left(p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i) + \overline{F}_{i,j(i)}(p^{LS}_{i,j(i)}) \left(p^{LS}_{i,j(i)} - p_i\right)\right) \\
& + \sum_{i \in A_n, j(i) \notin A_n} \lambda_i \left(p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i)\right) + \left(1 - \sum_{i \in A_n} \lambda_i\right) V^{LS}_{n+1}(\mathbf{I}) \ \text{(from Proposition 6)} \\
\leq & \sum_{i \in A_n} \lambda_i \left(p_i + V^{LS}_{n+1}(\mathbf{I} - \mathbf{e}_i) + \overline{F}_{i,j(i)}(p^M_{i,j(i)}) \left(p^M_{i,j(i)} - p_i\right)\right) \\
& + \left(1 - \sum_{i \in A_n} \lambda_i\right) V^{LS}_{n+1}(\mathbf{I}) \ \text{(from (13))}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i\in A_n} \lambda_i \left(p_i + U_{n+1}\left(\mathbf{I}-\mathbf{e}_i\right) + \overline{F}_{i,j(i)}(p^M_{i,j(i)}) \left(p^M_{i,j(i)} - p_i\right)\right) \\
&\quad + \left(1 - \sum_{i\in A_n} \lambda_i\right) U_{n+1}\left(\mathbf{I}\right) \text{ (from (77))} \\
&= U_n\left(\mathbf{I}\right). \text{ (from (79))}
\end{aligned}
$$

b) From part a, it suffices to prove that

$$\frac{U_n\left(\mathbf{I}\right) - L_n\left(\mathbf{I}\right)}{L_n\left(\mathbf{I}\right)} \leq \max\left\{\frac{p_{j(i)}}{p_i} | i = 1, ..., m\right\},$$

or equivalently,

$$\frac{U_n\left(\mathbf{I}\right)}{L_n\left(\mathbf{I}\right)} \leq \max\left\{\frac{p_{j(i)} + p_i}{p_i} | i = 1, ..., m\right\}.$$

It suffices to prove

$$\frac{U^i_n\left(\mathbf{I}\right)}{L^i_n\left(\mathbf{I}\right)} \leq \frac{p_{j(i)} + p_i}{p_i}.$$

Let $B = \left(p_{j(i)} + p_i\right)/p_i$. Using induction, we assume that

$$\frac{U^i_{n+1}\left(\mathbf{I}\right)}{L^i_{n+1}\left(\mathbf{I}\right)} \leq B \tag{81}$$

for some $n = 1, ..., N$. From (32),

$$
\begin{aligned}
U^i_n\left(\mathbf{I}\right) &= \lambda_i\left(p_i + U^i_{n+1}\left(\mathbf{I}-\mathbf{e}_i\right) + \overline{F}_{i,j(i)}(p^M_{i,j(i)})(p^M_{i,j(i)} - p_i)\right) + \left(1-\lambda_i\right) U^i_{n+1}\left(\mathbf{I}\right) \\
&\leq \lambda_i\left(p_i + BL^i_{n+1}\left(\mathbf{I}-\mathbf{e}_i\right) + \overline{F}_{i,j(i)}(p^M_{i,j(i)})(p^M_{i,j(i)} - p_i)\right) \\
&\quad + \left(1-\lambda_i\right) BL^i_{n+1}\left(\mathbf{I}\right) \text{ (from (81))} \\
&\leq \lambda_i\left(p_i + BL^i_{n+1}\left(\mathbf{I}-\mathbf{e}_i\right) + p_{j(i)}\right) + \left(1-\lambda_i\right) BL^i_{n+1}\left(\mathbf{I}\right) \\
&= BL^i_n(\mathbf{I}). \text{ (from (31))}
\end{aligned}
$$

Proof of Proposition 8

a) Note that $p^*_{ij}(\mathbf{I}, n)$ is the maximizer of $g_{ij}\left(p_i + \left(V^{LS}_{n+1}(\mathbf{I}-\mathbf{e}_i) - V^{LS}_{n+1}(\mathbf{I}-\mathbf{e}_i-\mathbf{e}_j)\right)\right)$. (13) indicates that p^M_{ij} is the maximizer of $g_{ij}(p_i)$, where $g_{ij}(\cdot)$ is defined in Lemma A1. From the result of Proposition 6, we know that $\left(V^{LS}_{n+1}(\mathbf{I}-\mathbf{e}_i) - V^{LS}_{n+1}(\mathbf{I}-\mathbf{e}_i-\mathbf{e}_j)\right) \geq 0$, which together with the result of Lemma A1b, yields $p^*_{ij}(\mathbf{I}, n) \geq p^M_{ij}$.

b) Denote $\mathbf{I} = (I_1, I_2, ..., I_m)$. It suffices to prove that if $I_k \geq N + 2 - n$ for $k = 1, 2, ..., m$, then $V^{LS}_{n+1}(\mathbf{I}-\mathbf{e}_i) - V^{LS}_{n+1}(\mathbf{I}-\mathbf{e}_i-\mathbf{e}_j) = 0$ for any i and j. From (30), we can prove this statement using the induction method on n.