Technical Appendix to Product Line Design and Production Technology

Appendix A: Uniformly Distributed Consumer Types

The purpose of this appendix is to characterize the extent to which our results extend when consumer types θ are distributed uniformly on $[A, B]$. When there is a continuum of consumer types, the number of products is unbounded, which makes characterizing the optimal product line difficult. For example, as the production cost parameters Z go to zero, the number of products offered under full information goes to infinity. With an unbounded number of products, it is difficult to compare the results with our previous results. To facilitate a comparison, we focus on the case where the firm offers at most two products; this restriction is without loss of generality when the production cost parameters Z are sufficiently large, because then it is optimal to offer at most two products. This restriction facilitates analytical results for the special case $A = 0$ and the development of large-scale numerical results for $A > 0$. It is optimal to segment the market so that consumers of type $[\theta_H, B]$ purchase a higher quality product and consumers of type $[\theta_L, \theta_H)$ purchase a lower quality product, where $\theta_L \leq \theta_H$. We begin by characterizing the optimal product line when $A = 0$, starting with the full information case.

Let θ^{FI} denote the smallest root greater than $3Z^{2/3}B^{1/3}/2\lambda^{1/3}$ of

$$[5(\theta^{FI})^2 - 18B\theta^{FI} + 9B^2]\sqrt{\lambda(1 - \theta^{FI}/B)} + 6Z(9\theta^{FI} - 3B - 4\sqrt{6\theta^{FI}(B - \theta^{FI})} = 0. \tag{1}$$

Proposition 1a *Suppose that the firm offers at most two products and $A = 0$. Under full information the optimal segmentation, product line, batch sizes, and prices are as follows:*
(i) If $Z < 2\sqrt{2\lambda}B/27$, then the firm offers two products, $\theta^{FI} \in (B/3, 3B/5)$ and

$$\theta_H^* = \theta^{FI},\ \theta_L^* = \frac{\theta^{FI}}{3} \tag{2}$$

$$q_H^* = \frac{B + \theta_H^* - 2Z\sqrt{B/\lambda(B - \theta_H^*)}}{4a},\ q_L^* = \frac{4\theta_H^* - 3Z\sqrt{6B/\lambda\theta_H^*}}{12a}, \tag{3}$$

$$Q_H^* = \sqrt{\frac{2K\lambda(B - \theta_H^*)}{Bia}\frac{1}{q_H^*}},\ Q_L^* = \sqrt{\frac{4K\lambda\theta_H^*}{3Bia}\frac{1}{q_L^*}}, \tag{4}$$

$$p^*(\theta) = \begin{cases} \theta q_L^* \text{ for } \theta \in [\theta_L^*, \theta_H^*) \\ \theta q_H^* \text{ for } \theta \in [\theta_H^*, B]. \end{cases} \tag{5}$$

(ii) If $Z \in [2\sqrt{2\lambda}B/27, 2\sqrt{2\lambda}B/3\sqrt{3})$, *then* $\theta_L^* = \theta_H^*$, $q_L^* = Q_L^* = 0$, *the firm offers one product and*

$$\theta_H^* = \frac{B}{3} \tag{6}$$

$$q_H^* = \frac{4B - 3Z\sqrt{6/\lambda}}{12a}, \tag{7}$$

$$Q_H^* = \sqrt{\frac{4K\lambda}{3ia}\frac{1}{q_H^*}}, \tag{8}$$

$$p^*(\theta) = \theta q_H^* \text{ for } \theta \in [\theta_H^*, B]. \tag{9}$$

Otherwise, no product is offered.

Proof: It is optimal to segment the market so that consumers of type $[\theta_H, B]$ purchase a product of quality q_H and consumers of type $[\theta_L, \theta_H)$ purchase a product of quality q_L. It is optimal to price so that a consumer of type θ pays θq when the consumer purchases a product of quality q. Thus, the firm's problem is to choose θ_H, θ_L, q_H, q_L, Q_H and Q_L to maximize

$$\lambda\int_{\theta_L}^{\theta_H} \frac{\theta q_L - aq_L^2}{B}d\theta + \lambda\int_{\theta_H}^{B} \frac{\theta q_H - aq_H^2}{B}d\theta - \frac{\lambda(\theta_H - \theta_L)K}{BQ_L} - \frac{aiq_L^2 Q_L}{2} - \frac{\lambda(B-\theta_H)K}{BQ_H} - \frac{aiq_H^2 Q_H}{2},$$

where $B \geq \theta_H \geq \theta_L \geq 0$. The optimal batch sizes are $Q_L = \sqrt{2\lambda(\theta_H - \theta_L)K/(Baiq_L^2)}$ and $Q_H = \sqrt{2\lambda(B - \theta_H)K/(Baiq_H^2)}$, and the resulting profit is

$$\lambda\int_{\theta_L}^{\theta_H} \frac{\theta q_L - aq_L^2}{B}d\theta + \lambda\int_{\theta_H}^{B} \frac{\theta q_H - aq_H^2}{B}d\theta - Z\sqrt{\lambda(\theta_H - \theta_L)/B}q_L - Z\sqrt{\lambda(1 - \theta_H/B)}q_H.$$

With $Y = Z\sqrt{B/\lambda}$, the problem is equivalent to choosing θ_H, θ_L, q_H and q_L to maximize

$$\Pi^{FI}(\theta_H, \theta_L, q_H, q_L) = \int_{\theta_L}^{\theta_H}(\theta q_L - aq_L^2)d\theta + \int_{\theta_H}^{B}(\theta q_H - aq_H^2)d\theta - Y\sqrt{\theta_H - \theta_L}q_L - Y\sqrt{B - \theta_H}q_H.$$

$\Pi^{FI}(\theta_H, \theta_L, q_H, q_L)$ is jointly concave in (q_H, q_L) and is maximized at

$$q_H^*(\theta_H) = \frac{[B + \theta_H - 2Y/\sqrt{B - \theta_H}]^+}{4a} \text{ and } q_L^*(\theta_H, \theta_L) = \frac{[\theta_H + \theta_L - 2Y/\sqrt{\theta_H - \theta_L}]^+}{4a}.$$

Because

$$(\partial/\partial\theta_L)\Pi^{FI}(\theta_H, \theta_L, q_H^*(\theta_H), q_L^*(\theta_H, \theta_L)) = (\theta_H - 3\theta_L)q_L^*(\theta_H, \theta_L)/4,$$

$\Pi^{FI}(\theta_H, \theta_L, q_H^*(\theta_H), q_L^*(\theta_H, \theta_L))$ is maximized at $\theta_L^*(\theta_H) = \theta_H/3$ and the constraint $\theta_H \geq \theta_L$ is satisfied at the optimal θ_L. Thus, the profit function can be written as a function of a single variable $\Pi^{FI}(\theta_H, \theta_L^*(\theta_H), q_H^*(\theta_H), q_L^*(\theta_H, \theta_L^*(\theta_H)))$, which we write more compactly as $\Pi^{FI}(\theta_H)$. We begin by assuming that $Y \leq (2B/3)^{3/2}$. This implies that there exist $\underline{\theta}$ and $\overline{\theta}$ such that

$$B + \theta_H - 2Y/\sqrt{B - \theta_H} \geq 0 \text{ if and only if } \theta_H \in [\underline{\theta}, \overline{\theta}].$$

Further, $\underline{\theta} \leq B/3 \leq \overline{\theta}$. We begin by restricting attention to $\theta_H \in [\underline{\theta}, \overline{\theta}]$. Let $\tilde{\theta} = 3Y^{2/3}/2$. Note $q_L^*(\theta_H, \theta_L^*(\theta_H)) > 0$ if and only if $\theta > \tilde{\theta}$. It is straightforward to verify that $\tilde{\theta} > \underline{\theta}$ and

$$\Pi^{FI}(\theta_H) = \begin{cases} [4Y^2 - 4Y\sqrt{B-\theta_H}(B+\theta_H) + (B-\theta_H)(B+\theta_H)^2]/16a & \text{if } \theta_H \in [\underline{\theta}, \min(\tilde{\theta}, \overline{\theta})] \\ [216Y^2 - 12Y[9(B+\theta_H)\sqrt{B-\theta_H} + 4\sqrt{6}\theta_H^{3/2}] & \text{if } \theta_H \in [\min(\tilde{\theta}, \overline{\theta}), \overline{\theta}]. \\ \quad +(3B-\theta_H)^2(3B+5\theta_H)]/432a & \end{cases}$$

In the region $\theta_H \in [\underline{\theta}, \min(\tilde{\theta}, \overline{\theta})]$, Π^{FI} is maximized at $\theta_H = \min(B/3, \tilde{\theta})$. Further, $B/3 \leq \tilde{\theta}$ if and only if $Y \geq 2\sqrt{2}B^{3/2}/27$. In the region $\theta_H \in [\min(\tilde{\theta}, \overline{\theta}), \overline{\theta}]$, $(\partial^3/\partial\theta_H^3)\Pi^{FI}(\theta_H) > 0$, so profit is either concave, convex, or concave-convex in θ_H. Further,

$$\Pi^{FI}(\overline{\theta}) \leq \Pi^{FI}(B/3) = \frac{([\sqrt{8}B^{3/2} - 3\sqrt{3}Y]^+)^2}{108a} + \frac{([\sqrt{8}B^{3/2} - 27Y]^+)^2}{2916a}.$$

If $Y < 2\sqrt{2}B^{3/2}/27$, then $\tilde{\theta} < \overline{\theta}$ and the optimal solution is on $[\tilde{\theta}, \overline{\theta}]$ and in this region the first order condition is given by (1); because $(\partial/\partial\theta_H)\Pi^{FI}(B/3) > 0$ and $(\partial/\partial\theta_H)\Pi^{FI}(3B/5) < 0$, $\theta^{FI} \in (B/3, 3B/5)$. Therefore, if $Y < 2\sqrt{2}B^{3/2}/27$, then the optimal solution is given by (2)-(5). If $Y \in [2\sqrt{2}B^{3/2}/27, (2B/3)^{3/2}]$, then an optimal solution is given by (6)-(9), $\theta_L^* = \theta_H^*$, and $q_L^* = Q_L^* = 0$. We conclude by relaxing the restrictions on θ_H and Y imposed above. First, if $\theta_H \notin [\underline{\theta}, \overline{\theta}]$, then $q_H^*(\theta_H) = 0$; the single-product product line in (6)-(9) yields weakly greater profit than any product line with $\theta_H \notin [\underline{\theta}, \overline{\theta}]$. Second, if $Y = (2B/3)^{3/2}$, then the profit under the optimal product line is zero; this continues to be true when $Y > (2B/3)^{3/2}$. ■

The next proposition characterizes the optimal product line under asymmetric information when $A = 0$. Let θ^{AI} denote the smallest root greater than $(2B + 3\ 2^{1/3}Z^{2/3}B^{1/3}\lambda^{-1/3})/4$ of

$$[5(\theta^{AI})^2 - 14B\theta^{AI} + 8B^2]\sqrt{\lambda(1 - \theta^{AI}/B)} + 3Z(9\theta^{AI} - 6B - 4\sqrt{3(B - \theta^{AI})(2\theta^{AI} - B)} = 0. \quad (10)$$

Proposition 2a *Suppose that the firm offers at most two products and $A = 0$. Under asymmetric information the optimal segmentation, product line, batch sizes, and prices are as follows:*
(i) If $Z < 2\sqrt{\lambda}B/27$, then the firm offers two products, $\theta^{AI} \in (2B/3, 4B/5)$ and

$$\theta_H^\# = \theta^{AI},\ \theta_L^\# = \frac{B + \theta^{AI}}{3}, \quad (11)$$

$$q_H^\# = \frac{\theta_H^\# - Z\sqrt{B/\lambda(B - \theta_H^\#)}}{2a},\ q_L^\# = \frac{4\theta_H^\# - 2B - 3Z\sqrt{3B/\lambda(2\theta_H^\# - B)}}{6a}, \quad (12)$$

$$Q_H^\# = \sqrt{\frac{2K\lambda(B - \theta_H^\#)}{Bia}}\frac{1}{q_H^\#},\ Q_L^\# = \sqrt{\frac{2K\lambda(2\theta_H^\# - B)}{3Bia}}\frac{1}{q_L^*}, \quad (13)$$

$$p_H^\# = \theta_L^\# q_L^\# + \theta_H^\#(q_H^\# - q_L^\#),\ p_L^\# = \theta_L^\# q_L^\#. \quad (14)$$

(ii) If $Z \in [2\sqrt{\lambda}B/27, 2\sqrt{\lambda}B/3\sqrt{3})$, *then* $\theta_L^\# = \theta_H^\#$, $p_L^\# = q_L^\# = Q_L^\# = 0$, *the firm offers one product and*

$$\theta_H^\# = \frac{2B}{3} \tag{15}$$

$$q_H^\# = \frac{2B - 3Z\sqrt{3/\lambda}}{6a}, \tag{16}$$

$$Q_H^\# = \sqrt{\frac{2K\lambda}{3ia}}\frac{1}{q_H^\#}, \tag{17}$$

$$p_H^\# = \theta_H^\# q_H^\#. \tag{18}$$

Otherwise, no product is offered.

Proof: It is optimal to segment the market so that consumers of type $[\theta_H, B]$ purchase a product of quality q_H and consumers of type $[\theta_L, \theta_H)$ purchase a product of quality q_L, where $q_L \leq q_H$. The optimal prices that satisfy the self-selection constraints are $p_H = \theta_L q_L + \theta_H(q_H - q_L)$ and $p_L = \theta_L q_L$. Thus, the firm's problem is to choose θ_H, θ_L, q_H, q_L, Q_H and Q_L to maximize

$$\frac{\lambda(\theta_H - \theta_L)(\theta_L q_L - aq_L^2)}{B} + \frac{\lambda(B - \theta_H)[\theta_L q_L + \theta_H(q_H - q_L) - aq_H^2]}{B}$$
$$- \frac{\lambda(\theta_H - \theta_L)K}{BQ_L} - \frac{aiq_L^2 Q_L}{2} - \frac{\lambda(B - \theta_H)K}{BQ_H} - \frac{aiq_H^2 Q_H}{2},$$

where $B \geq \theta_H \geq \theta_L \geq 0$ and $q_H \geq q_L$. The optimal batch sizes are $Q_L = \sqrt{2\lambda(\theta_H - \theta_L)K/(Baiq_L^2)}$ and $Q_H = \sqrt{2\lambda(B - \theta_H)K/(Baiq_H^2)}$, and the resulting profit is

$$\frac{\lambda(\theta_H - \theta_L)(\theta_L q_L - aq_L^2)}{B} + \frac{\lambda(B - \theta_H)[\theta_L q_L + \theta_H(q_H - q_L) - aq_H^2]}{B}$$
$$- Z\sqrt{\lambda(\theta_H - \theta_L)/B}\,q_L - Z\sqrt{\lambda(1 - \theta_H/B)}\,q_H.$$

With $Y = Z\sqrt{B/\lambda}$, the problem is equivalent to choosing θ_H, θ_L, q_H and q_L to maximize

$$\begin{aligned}\Pi^{AI}(\theta_H, \theta_L, q_H, q_L) &= (\theta_H - \theta_L)(\theta q_L - aq_L^2) + (B - \theta_H)[\theta_L q_L + \theta_H(q_H - q_L) - aq_H^2] \\ &\quad - Y\sqrt{\theta_H - \theta_L}\,q_L - Y\sqrt{B - \theta_H}\,q_H.\end{aligned}$$

We begin by considering the relaxed problem in which the constraints $\theta_H \geq \theta_L$ and $q_H \geq q_L$ are ignored, and we will subsequently show that the resulting solution satisfies these constraints. $\Pi^{AI}(\theta_H, \theta_L, q_H, q_L)$ is jointly concave in (q_H, q_L) and is maximized at

$$q_H^\#(\theta_H) = \frac{[\theta_H - Y/\sqrt{B - \theta_H}]^+}{2a} \text{ and } q_L^\#(\theta_H, \theta_L) = \frac{[\theta_H + \theta_L - B - Y/\sqrt{\theta_H - \theta_L}]^+}{2a}.$$

Because

$$(\partial/\partial\theta_L)\Pi^{AI}(\theta_H, \theta_L, q_H^\#(\theta_H), q_L^\#(\theta_H, \theta_L)) = (B + \theta_H - 3\theta_L)q_L^\#(\theta_H, \theta_L),$$

$\Pi^{AI}(\theta_H, \theta_L, q_H^{\#}(\theta_H), q_L^{\#}(\theta_H, \theta_L))$ is maximized at $\theta_L^{\#}(\theta_H) = (B + \theta_H)/3$. Thus, the profit function can be written as a function of a single variable $\Pi^{AI}(\theta_H, \theta_L^{\#}(\theta_H), q_H^{\#}(\theta_H), q_L^{\#}(\theta_H, \theta_L^{\#}(\theta_H)))$, which we write more compactly as $\Pi^{AI}(\theta_H)$. We begin by assuming that $Y \leq 2B^{3/2}/3\sqrt{3}$. This implies that there exist $\underline{\theta}$ and $\overline{\theta}$ such that

$$\theta_H - Y/\sqrt{B - \theta_H} \text{ if and only if } \theta_H \in [\underline{\theta}, \overline{\theta}].$$

Further $\underline{\theta} \leq 2B/3 \leq \overline{\theta}$. We begin by restricting attention to $\theta_H \in [\underline{\theta}, \overline{\theta}]$. Let $\tilde{\theta} = (2B + 3\ 2^{1/3}Y^{2/3})/4$. Note $q_L^{\#}(\theta_H, \theta_L^{\#}(\theta_H)) > 0$ if and only if $\theta > \tilde{\theta}$. It is straightforward to verify that $\tilde{\theta} > \underline{\theta}$ and

$$\Pi^{AI}(\theta_H) = \begin{cases} [Y^2 - 2Y\sqrt{B - \theta_H} + (B - \theta_H)\theta_H^2]/4a & \text{if } \theta_H \in [\underline{\theta}, \min(\tilde{\theta}, \overline{\theta})] \\ [54Y^2 - 6Y[2\sqrt{3}(2\theta_H - B)^{3/2} + 9\theta_H\sqrt{B - \theta_H}] & \text{if } \theta_H \in [\min(\tilde{\theta}, \overline{\theta}), \overline{\theta}]. \\ \quad +(2B - \theta_H)^2(5\theta_H - B)]/108a & \end{cases}$$

In the region $\theta_H \in [\underline{\theta}, \min(\tilde{\theta}, \overline{\theta})]$, Π^{AI} is maximized at $\theta_H = \min(2B/3, \tilde{\theta})$. Further $2B/3 \leq \tilde{\theta}$ if and only if $Y \geq 2B^{3/2}/27$. In the region $\theta_H \in [\min(\tilde{\theta}, \overline{\theta}), \overline{\theta}]$, $(\partial^3/\partial\theta_H^3)\Pi^{AI}(\theta_H) > 0$, so profit is either concave, convex, or concave-convex in θ_H. Further,

$$\Pi^{AI}(\overline{\theta}) \leq \Pi^{AI}(2B/3) = \frac{([2B^{3/2} - 3\sqrt{3}Y]^+)^2}{108a} + \frac{([2B^{3/2} - 27Y]^+)^2}{2916a}.$$

If $Y < 2B^{3/2}/27$, then $\tilde{\theta} < \overline{\theta}$ and the optimal solution is on $[\tilde{\theta}, \overline{\theta}]$ and in this region the first order condition is given by (10). Because $(\partial/\partial\theta_H)\Pi^{AI}(2B/3) > 0$ and $(\partial/\partial\theta_H)\Pi^{AI}(4B/5) < 0$, $\theta^{AI} \in (2B/3, 4B/5)$. Therefore, if $Y < 2\sqrt{2}B^{3/2}/27$, then the optimal solution is given by (11)-(14); because $\theta^{AI} > B/2$ implies $\theta^{AI} > \theta_L^{\#}(\theta^{AI})$ and because $\theta^{AI} < 4B/5$ implies $q_H^{\#}(\theta^{AI}) > q_L^{\#}(\theta^{AI}, \theta_L^{\#}(\theta^{AI}))$, the solution to the relaxed problem satisfies the constraints of the original problem. If $Y \in [2\sqrt{2}B^{3/2}/27, 2B^{3/2}/3\sqrt{3}]$, then an optimal solution is given by (15)-(18), $\theta_L^{\#} = \theta_H^{\#}$ and $p_L^{\#} = q_L^{\#} = Q_L^{\#} = 0$; this solution trivially satisfies the constraints of the original problem. We conclude by relaxing the restrictions on θ_H and Y imposed above. First, if $\theta_H \notin [\underline{\theta}, \overline{\theta}]$, then $q_H^{\#}(\theta_H) = 0$; the single-product product line in (15)-(18) yields weakly greater profit than any product line with $\theta_H \notin [\underline{\theta}, \overline{\theta}]$. Second, if $Y = 2B^{3/2}/3\sqrt{3}$, then the profit under the optimal product line is zero; this continues to be true when $Y > 2B^{3/2}/3\sqrt{3}$. ■

Propositions 1a and 2a are similar to Propositions 1 and 2, but with some differences. The main distinction is that when $A = 0$, it is never optimal to offer a composite product (i.e., serve the entire market with a single product). More generally, it is never optimal to serve the entire market because a portion of the market has very low valuations. Accordingly, the results in §4 that rely on the firm offering a composite product for at least some values of B and Z do not extend to the case with $A = 0$. However, an extensive numerical study indicates that when A is sufficiently large, offering a composite product becomes attractive and the results in §4 extend to this setting where consumer types are distributed uniformly. In particular, we conducted a numerical study of the 36, 800 parameter combinations of $A = \{1, 2, 4, 8\}$, $B = \{A + 0.2, A + 0.4, .., A + 2.0\}$, and $Z = \{0.00, 0.02, .., 9.00\}$, where λ is normalized to unity. This parameter set allows for considering the impact of the cost-related parameters a, i and K individually; to examine the impact of a, when i and K are fixed, we fixed the product iK at unity. Because the segments are endogenous, the second part of Result 1, Result 5, and the first part of Result 6 are not meaningful. All of the remaining assertions in Results 1-8 are consistent with the numerical results. However, in contrast to the two-type case, the value of information about consumer preferences will always be positive. Figures 1a and 2a depict the optimal product line under full information and asymmetric information when $A = 1$ as a function of the production cost parameters and the upper limit of the consumer type distribution B, which can be interpreted as how heterogenous the consumers are. The "sell high quality product" region corresponds to the case where the optimal product line consists of a single product which is priced sufficiently high that some consumers are excluded.

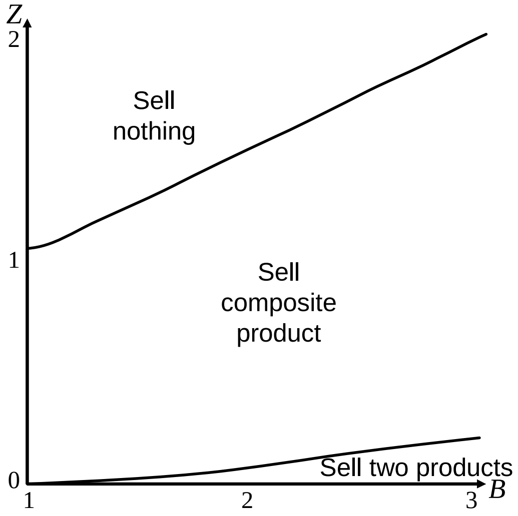

Figure 1a. Full information.

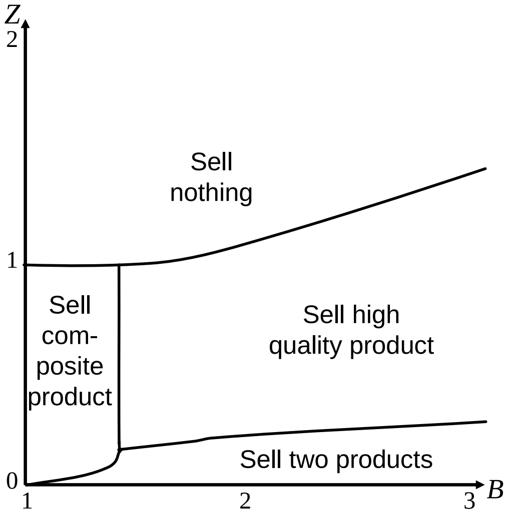

Figure 2a. Asymmetric information.

Appendix B: Generalized Valuation and Quality Cost Functions

The goal of this appendix is to demonstrate that a majority of our results are preserved under more general valuation and quality cost functions. In order to demonstrate this we will assume that consumers have a continuous, twice-differentiable valuation function $V(\theta, q)$ which satisfies the following (rather standard) technical assumptions: $V_q > 0$, $V_\theta > 0$, $V_{q\theta} > 0$ and $V(0, q) = V(\theta, 0) = 0$, where subscripts denote partial derivatives. Furthermore, we assume that the quality cost $g(q)$ is a continuous, twice-differentiable and increasing. Finally, we assume that the profit functions are unimodal in q_L, q_H and q_C (a simple sufficient condition for this would be $2g''(q)\,g(q) > (g'(q))^2$) and that $V_q(\theta_L, 0) > g'(0)$, so that the low-type segment is viable. We proceed by demonstrating how Figures 2-3, which reflect the structure of the optimal product line, change when valuation function and quality cost are generalized. With mild abuse of notation, define $Z = \sqrt{2iK}$.

Full Information

It is straightforward to verify that, after solving for the optimal batch sizes and prices, the objective functions for the two products and the composite product become:

$$\begin{aligned}
\pi^{FI2} &= [V(\theta_L, q_L) - g(q_L)]\,\lambda_L - \sqrt{g(q_L)}\sqrt{\lambda_L}Z + [V(\theta_H, q_H) - g(q_H)]\,\lambda_H - \sqrt{g(q_H)}\sqrt{\lambda_H}Z, \\
\pi^{FI1} &= [V(\theta_L, q_C) - g(q_C)]\,\lambda_L + [V(\theta_H, q_C) - g(q_C)]\,\lambda_H - \sqrt{g(q_C)}\sqrt{\lambda_L + \lambda_H}Z.
\end{aligned}$$

Optimal quality for each product can be found by solving the following optimality conditions:

$$\begin{aligned}
\frac{\partial \pi^{FI2}}{\partial q_L} &= [V_q(\theta_L, q_L) - g'(q_L)]\,\lambda_L - \frac{g'(q_L)\sqrt{\lambda_L}Z}{2\sqrt{g(q_L)}} = 0, \\
\frac{\partial \pi^{FI2}}{\partial q_H} &= [V_q(\theta_H, q_H) - g'(q_H)]\,\lambda_H - \frac{g'(q_H)\sqrt{\lambda_H}Z}{2\sqrt{g(q_H)}} = 0, \\
\frac{\partial \pi^{FI1}}{\partial q_C} &= [V_q(\theta_L, q_C) - g'(q_C)]\,\lambda_L + [V_q(\theta_H, q_C) - g'(q_C)]\,\lambda_H - \frac{g'(q_C)\sqrt{\lambda_L + \lambda_H}Z}{2\sqrt{g(q_C)}} = 0.
\end{aligned}$$

The rest of the proof proceeds in four steps.

(1) Note that for $Z = 0$ and any $\theta_H > \theta_L$ we replicate the classical adverse selection problem in which it is beneficial to manufacture two products. Likewise, for $Z > 0$ and $\theta_H = \theta_L$ it is advantageous to offer one composite product (or no products at all for large Z). Hence, behavior on the boundaries of Figure 2 is preserved.

(2) Note that quality of all products decreases in Z and becomes zero for large enough Z. Hence, for large enough Z, $\pi^{FI1} = 0$ and $\pi^{FI2} = 0$. Thus, we obtain the "Sell nothing" area in Figure 2. To define its boundaries, we set quality of each product to zero in the optimality conditions in order to obtain the threshold values of Z corresponding to zero quality.

$$\begin{aligned}
\widehat{Z}_L^{FI} &= [V_q(\theta_L, 0) - g'(0)]\,\frac{2\sqrt{g(0)}\sqrt{\lambda_L}}{g'(0)}, \text{(so that for } Z > \widehat{Z}_L^{FI},\ q_L^* = 0), \\
\widehat{Z}_H^{FI} &= [V_q(\theta_H, 0) - g'(0)]\,\frac{2\sqrt{g(0)}\sqrt{\lambda_H}}{g'(0)}, \text{ (so that for } Z > \widehat{Z}_H^{FI},\ q_H^* = 0), \\
\widehat{Z}_C^{FI} &= ([V_q(\theta_L, 0) - g'(0)]\,\lambda_L + [V_q(\theta_H, 0) - g'(0)]\,\lambda_H)\,\frac{2\sqrt{g(0)}}{g'(0)\sqrt{\lambda_L + \lambda_H}}, \\
&\qquad \text{(so that for } Z > \widehat{Z}_C^{FI},\ q_C^* = 0).
\end{aligned}$$

Note that $\widehat{Z}_L^{FI}$ is independent of θ_H and hence it is a straight horizontal line separating areas (*i*) and (*iii*) as in Figure 2: the low-quality product is not offered for $Z < \widehat{Z}_L^{FI}$. Next, observe that both $\widehat{Z}_H^{FI}$ and $\widehat{Z}_C^{FI}$ are increasing in θ_H but at different rates:

$$\frac{\partial \widehat{Z}_H^{FI}}{\partial \theta_H} = V_{q\theta}(\theta_H, 0)\frac{2\sqrt{g\left(0\right)}\sqrt{\lambda_H}}{g'\left(0\right)} > \frac{\partial \widehat{Z}_C^{FI}}{\partial \theta_H} = V_{q\theta}(\theta_H, 0)\frac{2\sqrt{g\left(0\right)}\lambda_H}{g'\left(0\right)\sqrt{\lambda_L + \lambda_H}},$$

and, furthermore, $\widehat{Z}_C^{FI}(\theta_H = \theta_L) > \widehat{Z}_L^{FI}$ so that we obtain boundaries of the “Sell nothing” area that are similar to Figure 2 except they are not necessarily linear in θ_H. In fact, these boundaries are convex in θ_H if $V_{q\theta\theta} > 0$ and concave otherwise.

(3) We can calculate $\widehat{\theta}_H^{FI}$, the unique solution to $\widehat{Z}_C^{FI}\left(\widehat{\theta}_H^{FI}\right) = \widehat{Z}_H^{FI}\left(\widehat{\theta}_H^{FI}\right)$ as follows:

$$\widehat{\theta}_H^{FI} = \left\{\theta_H : V_q(\theta_H, 0) - g'(0) = [V_q(\theta_L, 0) - g'(0)]\left(1 + \sqrt{1 + \lambda_L/\lambda_H}\right)\right\},$$

which is a straight vertical line independent of Z. However, as we demonstrate below, $\widehat{\theta}_H^{FI}$ may not be the line that separates areas (*ii*) and (*iii*).

(4) Consider optimal profits as functions of Z.

$$\left.\frac{d\pi^{FI2}}{dZ}\right|_{q_L^*, q_H^*} = \frac{\partial \pi^{FI2}}{\partial Z} = -\sqrt{g\left(q_L^*\right)}\sqrt{\lambda_L} - \sqrt{g\left(q_H^*\right)}\sqrt{\lambda_H} < 0.$$

Note further that both $g\left(q_L^*\right)$ and $g\left(q_H^*\right)$ decrease in Z because q_t^* decreases in Z. Thus, π^{FI2} is a convex (because its first derivative increases in Z) and decreasing function of Z. Likewise,

$$\left.\frac{d\pi^{FI1}}{dZ}\right|_{q_C^*} = \frac{\partial \pi^{FI1}}{\partial Z} = -\sqrt{g\left(q_C^*\right)}\sqrt{\lambda_L + \lambda_H} < 0,$$

and $g(q_C^*)$ decreases in Z so that π^{FI1} is a convex and decreasing function of Z. Consider two cases. First, assume $\theta_H < \widehat{\theta}_H^{FI}$. Then at $Z = 0$, $\pi^{FI2} > \pi^{FI1}$, and for $\widehat{Z}_H^{FI} < Z < \widehat{Z}_C^{FI}$, $\pi^{FI2} < \pi^{FI1}$. Clearly, since both objectives are decreasing convex in Z, they will only cross once. We denote their intersection z^{FI}, which as in Figure 2, is the boundary between areas (*ii*) and (*iii*); the shape of this boundary may differ from that in Figure 2.

Next, assume $\theta_H > \widehat{\theta}_H^{FI}$. In this case at $Z = 0$, $\pi^{FI2} > \pi^{FI1}$, and for $\widehat{Z}_C^{FI} < Z < \widehat{Z}_H^{FI}$, $\pi^{FI2} > \pi^{FI1}$. Clearly, since both objectives are decreasing convex in Z, they will either cross never or twice. If they never cross then the boundary between areas (*ii*) and (*iii*) is a straight line $\widehat{\theta}_H^{FI}$. If they cross twice, then the boundary between areas (*ii*) and (*iii*) is a quasi-concave (because there are exactly two intersections) curve in Z. We have therefore demonstrated that the structure of the optimal product line as captured in Figure 2 holds for the generalized functions V and g, although the shapes of the boundaries between regions may differ.

Asymmetric Information

It is straightforward to verify that, after solving for the optimal batch sizes and prices, the objective functions for the two products and the composite product become:

$$\begin{aligned}
\pi^{AI2} &= \left[V(\theta_L, q_L) - g(q_L)\right]\lambda_L - \sqrt{g\left(q_L\right)}\sqrt{\lambda_L}Z \\
&\quad + \left[V(\theta_H, q_H) + V\left(\theta_L, q_L\right) - V\left(\theta_H, q_L\right) - g(q_H)\right]\lambda_H - \sqrt{g\left(q_H\right)}\sqrt{\lambda_H}Z, \\
\pi^{AI1} &= \left[V(\theta_L, q_C) - g(q_C)\right]\left(\lambda_L + \lambda_H\right) - \sqrt{g\left(q_C\right)}\sqrt{\lambda_L + \lambda_H}Z.
\end{aligned}$$

Optimal quality for each product can be found by solving the following optimality conditions:

$$\begin{aligned}
\frac{\partial \pi^{AI2}}{\partial q_L} &= \left[V_q(\theta_L, q_L) - g'(q_L)\right]\lambda_L - \left[V_q\left(\theta_H, q_L\right) - V_q\left(\theta_L, q_L\right)\right]\lambda_H - \frac{g'\left(q_L\right)\sqrt{\lambda_L}Z}{2\sqrt{g\left(q_L\right)}} = 0, \\
\frac{\partial \pi^{AI2}}{\partial q_H} &= \left[V_q(\theta_H, q_H) - g'(q_H)\right]\lambda_H - \frac{g'\left(q_H\right)\sqrt{\lambda_H}Z}{2\sqrt{g\left(q_H\right)}} = 0, \\
\frac{\partial \pi^{AI1}}{\partial q_C} &= \left[V_q(\theta_L, q_C) - g'(q_C)\right]\left(\lambda_L + \lambda_H\right) - \frac{g'\left(q_C\right)\sqrt{\lambda_L + \lambda_H}Z}{2\sqrt{g\left(q_C\right)}} = 0.
\end{aligned}$$

The rest of the proof proceeds in four steps.

(1) Note that for $Z = 0$ and any $\theta_H > \theta_L$ we replicate the classical adverse selection problem in which it is beneficial to manufacture two products for small θ_H and one product for large θ_H. Likewise, for $Z > 0$ and $\theta_H = \theta_L$ it is advantageous to offer one composite product (or no products at all for large Z). Hence, behavior on the boundaries of Figure 3 is preserved.

(2) Note that quality of all products decreases in Z and becomes zero for large enough Z. Hence, for large enough Z, $\pi^{FI1} = 0$ and $\pi^{FI2} = 0$. Thus, we obtain the "Sell nothing" area in Figure 3. To define its boundaries, we set quality of each product to zero in the optimality conditions in order to obtain the threshold values of Z corresponding to zero quality.

$$\begin{aligned}
\widehat{Z}_L^{AI} &= \left(\left[V_q(\theta_L, 0) - g'(0)\right]\lambda_L - \left[V_q\left(\theta_H, 0\right) - V_q\left(\theta_L, 0\right)\right]\lambda_H\right)\frac{2\sqrt{g\left(0\right)}}{g'\left(0\right)\sqrt{\lambda_L}}, \\
&\qquad\qquad\qquad\qquad \text{(so that for } Z > \widehat{Z}_L^{AI},\ q_L^{\#} = 0), \\
\widehat{Z}_H^{AI} &= \left[V_q(\theta_H, 0) - g'(0)\right]\frac{2\sqrt{g\left(0\right)}\sqrt{\lambda_H}}{g'\left(0\right)}, \text{ (so that for } Z > \widehat{Z}_H^{AI},\ q_H^{\#} = 0), \\
\widehat{Z}_C^{AI} &= \left[V_q(\theta_L, 0) - g'(0)\right]\frac{2\sqrt{g\left(0\right)}\sqrt{\lambda_L + \lambda_H}}{g'\left(0\right)}, \text{ (so that for } Z > \widehat{Z}_C^{AI},\ q_C^{\#} = 0).
\end{aligned}$$

Furthermore, note that $\widehat{Z}_L^{AI}$ is a decreasing function of θ_H (convex if $V_{q\theta\theta}\left(\theta_H, 0\right) < 0$ and concave otherwise) and $\widehat{Z}_C^{AI}$ is a straight line independent of θ_H. Finally, $\widehat{Z}_H^{AI}$ is an increasing function of θ_H, convex in θ_H if $V_{q\theta\theta}(\theta_H, 0) > 0$ and concave otherwise. All of these findings are consistent with

Figure 3. We therefore established curves separating the "Sell nothing" area as well as the curve separating areas (*i*) and (*iii*).

(3) We can calculate $\widehat{\theta}_H^{AI}$, the unique solution to $\widehat{Z}_C^{AI}\left(\widehat{\theta}_H^{AI}\right) = \widehat{Z}_H^{AI}\left(\widehat{\theta}_H^{AI}\right)$ as follows:

$$\widehat{\theta}_H^{AI} = \left\{\theta_H : V_q(\theta_H, 0) - g'(0) = [V_q(\theta_L, 0) - g'(0)]\sqrt{1 + \lambda_L/\lambda_H}\right\},$$

which is a straight vertical line independent of Z. However, as we demonstrate below, $\widehat{\theta}_H^{AI}$ may not be the line that separates areas (*ii*) and (*iii*).

(4) Consider optimal profits as functions of Z.

$$\left.\frac{d\pi^{FI2}}{dZ}\right|_{q_L^{\#}, q_H^{\#}} = \frac{\partial \pi^{FI2}}{\partial Z} = -\sqrt{g\left(q_L^{\#}\right)}\sqrt{\lambda_L} - \sqrt{g\left(q_H^{\#}\right)}\sqrt{\lambda_H} < 0.$$

Note further that both $g(q_L^{\#})$ and $g(q_H^{\#})$ decrease in Z. Thus, π^{FI2} is a convex (because its first derivative increases in Z) and decreasing function of Z. Likewise,

$$\left.\frac{d\pi^{FI1}}{dZ}\right|_{q_C^{\#}} = \frac{\partial \pi^{FI1}}{\partial Z} = -\sqrt{g\left(q_C^{\#}\right)}\sqrt{\lambda_L + \lambda_H} < 0,$$

and $q_C^{\#}$ decreases in Z so that π^{FI1} is a convex and decreasing function of Z. Consider two cases. First, assume $\theta_H < \widehat{\theta}_H^{AI}$. Then at $Z = 0$, $\pi^{FI2} > \pi^{FI1}$, and for $\widehat{Z}_H^{AI} < Z < \widehat{Z}_C^{AI}$, $\pi^{FI2} < \pi^{FI1}$. Clearly, since both objectives are decreasing and convex in Z, they will only cross once. We denote their intersection z^{AI}, which as in Figure 3, is the boundary between areas (*ii*) and (*iii*); the shape of this boundary may differ from that in Figure 3.

Next we assume $\theta_H > \widehat{\theta}_H^{AI}$. Clearly, at $Z = 0$, $\pi^{FI2} > \pi^{FI1}$, and for $\widehat{Z}_C^{AI} < Z < \widehat{Z}_H^{AI}$, $\pi^{FI2} > \pi^{FI1}$. Therefore, since both objectives are decreasing and convex in Z, they will either never cross or will cross exactly twice. If they never cross then the boundary between areas (*ii*) and (*iii*) is a straight line $\widehat{\theta}_H^{AI}$. If they cross twice, then the boundary between areas (*ii*) and (*iii*) is a quasi-concave (because there are exactly two intersections) curve in Z. We have therefore demonstrated that the structure of the optimal product line as captured in Figure 3 holds for the generalized functions V and g, although the shapes of the boundaries between regions may differ.

Comparisons and Implications

We first turn to the impact on the number of products.

Result 1b *As production cost parameters increase, the number of products offered (weakly) decreases. When consumer segments are close (far), this happens without (with) excluding low-valuation consumers.*

Proof: Follows trivially from derivations above. ■

The next Lemma is useful in the proofs of Results 2b and 3b.

Lemma 1 *The quality of the composite product is higher than the quality of the low-quality product:* $q_C^* \geq q_L^*$*, where the inequality is strict if* $q_C^* > 0$*, and* $q_C^{\#} \geq q_L^{\#}$*, where the inequality is strict if* $q_C^{\#} > 0$.

Proof: First, we show that $q_C^* \geq q_L^*$, where the inequality is strict if $q_C^* > 0$, as follows:

$$\begin{aligned}
\left.\frac{\partial \pi^{FI1}}{\partial q_C}\right|_{q_L^*} &= \left[V_q(\theta_L, q_L^*) - g'\left(q_L^*\right)\right]\lambda_L + \left[V_q(\theta_H, q_L^*) - g'\left(q_L^*\right)\right]\lambda_H - \frac{g'\left(q_L^*\right)\sqrt{\lambda_L+\lambda_H}Z}{2\sqrt{g\left(q_L^*\right)}} \\
&= \frac{g'\left(q_L^*\right)\sqrt{\lambda_L}Z}{2\sqrt{g\left(q_L^*\right)}} + \left[V_q(\theta_H, q_L^*) - g'\left(q_L^*\right)\right]\lambda_H - \frac{g'\left(q_L^*\right)\sqrt{\lambda_L+\lambda_H}Z}{2\sqrt{g\left(q_L^*\right)}} \\
&> \frac{g'\left(q_L^*\right)\sqrt{\lambda_L}Z}{2\sqrt{g\left(q_L^*\right)}} + \left[V_q(\theta_L, q_L^*) - g'\left(q_L^*\right)\right]\lambda_H - \frac{g'\left(q_L^*\right)\sqrt{\lambda_L+\lambda_H}Z}{2\sqrt{g\left(q_L^*\right)}} \\
&= \frac{g'\left(q_L^*\right)\sqrt{\lambda_L}Z}{2\sqrt{g\left(q_L^*\right)}} + \frac{\lambda_H}{\lambda_L}\frac{g'\left(q_L^*\right)\sqrt{\lambda_L}Z}{2\sqrt{g\left(q_L^*\right)}} - \frac{g'\left(q_L^*\right)\sqrt{\lambda_L+\lambda_H}Z}{2\sqrt{g\left(q_L^*\right)}} \\
&= \frac{g'\left(q_L^*\right)Z}{2\sqrt{g\left(q_L^*\right)}}\left(\frac{\lambda_L+\lambda_H-\sqrt{\lambda_L^2+\lambda_L\lambda_H}}{\sqrt{\lambda_L}}\right) > 0.
\end{aligned}$$

Likewise, for the asymmetric information case we can demonstrate that $q_C^{\#} \geq q_L^{\#}$, where the inequality is strict if $q_C^{\#} > 0$, as follows:

$$\begin{aligned}
\left.\frac{\partial \pi^{AI1}}{\partial q_C}\right|_{q_L^{\#}} &= \left[V_q(\theta_L, q_L^{\#}) - g'(q_L^{\#})\right](\lambda_L+\lambda_H) - \frac{g'\left(q_L^{\#}\right)\sqrt{\lambda_L+\lambda_H}Z}{2\sqrt{g\left(q_L^{\#}\right)}} \\
&= \left(\left[V_q\left(\theta_H, q_L^{\#}\right) - V_q\left(\theta_L, q_L^{\#}\right)\right]\frac{\lambda_H}{\lambda_L} + \frac{g'\left(q_L^{\#}\right)Z}{2\sqrt{\lambda_L}\sqrt{g\left(q_L^{\#}\right)}}\right)(\lambda_L+\lambda_H) - \frac{g'\left(q_L^{\#}\right)\sqrt{\lambda_L+\lambda_H}Z}{2\sqrt{g\left(q_L^{\#}\right)}} \\
&> \frac{g'\left(q_L^{\#}\right)Z}{2\sqrt{g\left(q_L^{\#}\right)}}\left(\frac{(\lambda_L+\lambda_H)-\sqrt{\lambda_L^2+\lambda_L\lambda_H}}{\sqrt{\lambda_L}}\right) > 0. \blacksquare
\end{aligned}$$

Result 2b *The optimal number of products when the cannibalization problem is present can be larger, the same, or smaller than is efficient.*

Proof: We need to verify that areas A and B in Figure 4 continue to exist for the generalized valuation and quality cost functions. To show that area A exists it is sufficient to show that $z^{FI}(\theta_H) < z^{AI}(\theta_H)$. We take an alternative approach and show that $\theta_H^{FI}(Z) > \theta_H^{AI}(Z)$ where $\theta_H^{FI}, \theta_H^{AI}$ are boundaries between regions (i) and (ii): θ_H^{FI} is the unique value of θ_H such that $\pi^{FI2} = \pi^{FI1}$, and θ_H^{AI} is the unique value of θ_H such that $\pi^{AI2} = \pi^{AI1}$. Furthermore, at $\theta_H = \theta_L$, $\pi^{FI2} = \pi^{AI2} <$

$\pi^{FI1} = \pi^{AI1}$. Next, we evaluate how profit differences change in θ_H :

$$\begin{aligned}\frac{d\left(\pi^{FI2}-\pi^{FI1}\right)}{d\theta_H} &= \left[V_\theta\left(\theta_H,q_H^*\right)-V_\theta\left(\theta_H,q_C^*\right)\right]\lambda_H,\\ \frac{d\left(\pi^{AI2}-\pi^{AI1}\right)}{d\theta_H} &= \left[V_\theta\left(\theta_H,q_H^{\#}\right)-V_\theta\left(\theta_H,q_L^{\#}\right)\right]\lambda_H,\end{aligned}$$

and because $q_H^* = q_H^{\#}$ and $q_C^* > q_C^{\#} > q_L^{\#}$ (where the last inequality is from Lemma 1) it is clear that $\frac{d\left(\pi^{FI2}-\pi^{FI1}\right)}{d\theta_H} < \frac{d\left(\pi^{AI2}-\pi^{AI1}\right)}{d\theta_H}$. It immediately follows that $\theta_H^{FI} > \theta_H^{AI}$: the function $\pi^{AI2}(\theta_H) - \pi^{AI1}(\theta_H)$ crosses zero faster than function $\pi^{FI2}(\theta_H) - \pi^{FAI1}(\theta_H)$ because it decreases faster. Thus, area A in Figure 4 is guaranteed to exist. To see that area B exists, note that at $\theta_H = \theta_L$, $\widehat{Z}_C^{FI} = \widehat{Z}_C^{AI}$, and recall that $\widehat{Z}_C^{FI}$ is increasing in θ_H while $\widehat{Z}_C^{AI}$ is invariant to θ_H. ■

We next consider the impact on prices and product qualities.

Result 3b *The quality of the product purchased by low-valuation consumers may increase in the production cost parameters.*

Proof: Consider the full information case first. When the regime changes from offering two products to offering one composite product $q_C^* > 0$, so from Lemma 1, $q_C^* > q_L^*$. Thus, quality of product purchased by low-valuation consumers increases when the firm switches from offering two products to offering one composite product. The proof for the asymmetric information follows similarly. ■

Result 3 states that quality of the product purchased by high-valuation consumers is decreasing in the costliness of quality. This might not be the case for arbitrary valuation and costliness of quality. To see why, consider the case where the boundary between areas (*ii*) and (*iii*) is non-linear. In this case we will show that, when the firm switches from offering one composite product to offering a high-quality product as θ_H increases (this will only occur for some range of $\theta_H > \widehat{\theta}_H^{FI}$ for the full information case and $\theta_H > \widehat{\theta}_H^{AI}$ for the asymmetric information case), $q_C^* < q_H^*$ and $q_C^{\#} < q_H^{\#}$, so that quality of the product purchased by high-valuation consumers jumps up in both cases. Thus, as Z increases, the quality of the product purchased by high-valuation consumers jumps down and then up. To see this, consider the full information case and observe that

$$\begin{aligned}\left.\frac{d\pi^{FI2}}{d\theta_H}\right|_{q_H^*} &= \frac{\partial\pi^{FI2}}{\partial\theta_H} = V_\theta\left(\theta_H,q_H^*\right)\lambda_H \geq 0,\\ \left.\frac{d\pi^{FI1}}{d\theta_H}\right|_{q_C^*} &= \frac{\partial\pi^{FI1}}{\partial\theta_H} = V_\theta\left(\theta_H,q_C^*\right)\lambda_H \geq 0.\end{aligned}$$

We know that profit functions π^{FI2} and π^{FI1} cross twice in this region and that at the second time they cross we have

$$\left.\frac{d\pi^{FI2}}{d\theta_H}\right|_{q_H^*} > \left.\frac{d\pi^{FI1}}{d\theta_H}\right|_{q_C^*}$$

which implies $V_\theta\left(\theta_H, q_H^*\right) > V_\theta\left(\theta_H, q_C^*\right);$ this implies that $q_H^* > q_C^*$ when the regime changes, which completes the proof in the full information case. For the asymmetric information case, it is straightforward to verify that $q_C^\# \leq q_C^*$ and $q_H^* = q_H^\#$ so that the analogous result holds: $q_C^\# < q_H^\#$. These derivations rely on the assumption that the boundary between areas (ii) and (iii) is non-linear which may not be the case. Thus, we do not state this result more formally.

To summarize, there are three changes in Result 3b compared to Result 3. First, there is no longer a costliness of quality parameter, so there is no associated result. Second, in Result 3 only the quality of the product purchased by low-valuation consumers was non-monotone in production cost parameters while with the generalized valuation and quality cost functions, the quality of both product may not be monotone. Finally, it appears difficult to establish the result regarding the average quality. However, given that the average quality is a function of λ_L and λ_H it seems intuitive that, for large enough λ_L, the weight of the lower-quality product is big enough so as to result in an average quality increase as the production cost parameters increase.

Result 4b *The individual product prices are decreasing in the production cost parameters.*

Proof: It is straightforward to demonstrate that quality of all products decreases in production cost parameters as long as the regime does not change. This immediately implies that $p_H^* = V\left(\theta_H, q_H^*\right), p_L^* = V\left(\theta_L, q_L^*\right), p_{CH}^* = V\left(\theta_H, q_C^*\right),\ p_{CL}^* = V\left(\theta_L, q_C^*\right), p_H^\# = V(\theta_H, q_H^\#) + V\left(\theta_L, q_L^\#\right) - V\left(\theta_H, q_L^\#\right)$ and $p_L^\# = V\left(\theta_L, q_L^\#\right)$ decrease as well. ■

Once again, in contrast with Result 4, it appears difficult to establish the result regarding the average prices.

Result 5b *The impact of information asymmetry is to distort the quality and price of the product serving low-valuation consumers downward from the efficient level. The quality of the product serving high-valuation consumers can be distorted upward, downward or not at all. The price of the product serving high-valuation consumers can be distorted downward or not at all.*

Proof: First, we establish that the quality of the product serving low-valuation consumers is distorted downward. By inspecting the optimality conditions it is straightforward to conclude that $q_H^* = q_H^\#$, $q_L^* \geq q_L^\#$, $q_C^* \geq q_C^\#$. This, together with $q_C^\# \geq q_L^\#$ (by Lemma 1), implies that $q_C^* \geq q_L^\#$. Note that, if it is optimal to offer a composite product under asymmetric information, then it is optimal to offer a composite product under full information as well (follows from the proof of Result 2b), so that $q_C^* \geq q_C^\#$ implies the result. If it is optimal to offer two products under asymmetric information, then the result follows from $q_L^* \geq q_L^\#$ and $q_C^* \geq q_L^\#$. If it is optimal to offer only a high quality product under asymmetric information, the result is immediate. That the price of the product serving low-valuation consumers is distorted downward follows.

Second, we establish that the quality and price of the product serving high-valuation consumers can be distorted downward or not at all. From the proof of Result 2b, we know that area B in Figure 4 exists; here $q_H^* > q_H^{\#} = 0$ and $p_H^* > p_H^{\#} = 0$, so the quality and price are distorted downward. To show that quality and price may not be distorted at all, consider the northeast region of Figure 7 where the firm only sells a high-quality product under both full and asymmetric information. This region is guaranteed to exist for high enough θ_H and Z.

Third, we establish that the quality of the product serving high-valuation consumers can be distorted upward. From the proof of Result 2b, we know that area A in Figure 4 exists; because the curve z^{FI} marks the bottom boundary of this region, it is sufficient to show that at this value of Z, $q_H^{\#} > q_C^*$, or equivalently that $q_H^* > q_C^*$ (because $q_H^{\#} = q_H^*$). Because at $Z = z^{FI}$, $\pi^{FI1} = \pi^{FI2}$, it follows from the proof of Result 2b that

$$\left.\frac{d\left(\pi^{FI2} - \pi^{FI1}\right)}{d\theta_H}\right|_{z^{FI}} = \left[U_\theta\left(\theta_H, q_H^*\right) - U_\theta\left(\theta_H, q_C^*\right)\right]\lambda_H > 0,$$

which implies $q_H^* > q_C^*$. ■

In contrast with Result 5, it appears difficult to establish the result regarding the average price and quality; also we are unable to establish that the price may be distorted upward. Finally, we believe that whether Result 6 (monotonicity of product line length in cost parameters) holds may depend on the functional form. To see this, assume $g(q) = aq^2$ for simplicity and observe that

$$\frac{\partial\left(q_H^* - q_L^*\right)}{\partial Z} = \frac{1}{\left(V_{qq}\left(\theta_H, q_H^*\right) - 2a\right)\sqrt{\lambda_H}} - \frac{1}{\left(V_{qq}\left(\theta_L, q_L^*\right) - 2a\right)\sqrt{\lambda_L}}.$$

The sign of this expression depends on λ_H, λ_L, q_H^*, q_L^*, and how $V_{qq}\left(\theta, q\right)$ changes in θ and q.

We next consider the impact of the production cost parameters on the total production cost. We are unable to extend Result 7: it might be functional form dependent. To see why, take one of the production costs (full information, two products):

$$g\left(q_L\right)\lambda_L + g\left(q_H\right)\lambda_H + \sqrt{g\left(q_L\right)}\sqrt{\lambda_L}Z + \sqrt{g\left(q_H\right)}\sqrt{\lambda_H}Z,$$

Clearly, first two terms decrease in Z, because $g(q_t)$ decreases in Z because q_t decreases in Z. However, it is unclear what happens with the last two terms because they may increase in Z (it probably depends on how steep $g(\cdot)$ is). To understand whether this happens we must evaluate $\partial g(q_t)/\partial Z$ which involves making assumptions about the relative sizes of V_{qq}, $g'(\cdot)$ and $g''(\cdot)$. For example, it can be demonstrated that for $g(q) = aq^2$ and for such valuation function that $V_{qq} = 0$, total production cost is, indeed, always decreasing in production cost parameters.

Finally, we turn to the impact of the production cost parameters on consumer welfare.

Result 8b *Consumers may prefer a firm with higher production cost parameters to a firm with lower production cost parameters.*

Proof: Consumer utility under information asymmetry is

$$\begin{aligned} W^{AI2} &= \left[V\left(\theta_H, q_L^{\#}\right) - V\left(\theta_L, q_L^{\#}\right)\right]\lambda_H, \\ W^{AI1} &= \left[V\left(\theta_H, q_C^{\#}\right) - V(\theta_L, q_C^{\#})\right]\lambda_H. \end{aligned}$$

Recall that the valuation function is supermodular so that the difference $V\left(\theta_H, q_L^{\#}\right) - V\left(\theta_L, q_L^{\#}\right)$ is increasing in $q_L^{\#}$. If $q_C^{\#} > 0$, then $q_C^{\#} > q_L^{\#}$ (by Lemma 1), which implies that $W^{AI1} > W^{AI2}$. Thus, consumers may prefer a composite product and therefore they may prefer a firm with higher production cost parameters. ■