TECHNICAL APPENDIX TO:

STRATEGIC TECHNOLOGY CHOICE AND CAPACITY INVESTMENT UNDER DEMAND UNCERTAINTY

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In this technical appendix, we detail the solutions to the production and the capacity games without imposing Assumptions A-1 and A-2. Further, we formally show the existence of equilibrium in the capacity game for all three subgames and prove that the symmetric equilibrium in the choice of capacity is unique for the pure flexible and the pure dedicated subgames. We formally show the nonnegativity of ex-post prices in the pure flexible subgame.[1] Finally, we obtain analytical conditions under which Assumptions A-1 and A-2 hold.

We recall that the price for product y is $P_y(Q_y, Q_{3-y}) = A_y - Q_y - \beta Q_{3-y}$. Profit expressions for the production game can be calculated as follows:

$$\widehat{\pi}_i^x = \sum_{y=1}^{2} P_y\left(\widehat{Q}_y, \widehat{Q}_{3-y}\right)\widehat{q}_{yi}^x = \sum_{y=1}^{2}\left(A_y - \left(\widehat{q}_{yi}^x + \widehat{q}_{yj}^x\right) - \beta\left(\widehat{q}_{(3-y)i}^x + \widehat{q}_{(3-y)j}^x\right)\right)\widehat{q}_{yi}^x \tag{1}$$

where $x = f, d, m$ depending on the subgame in which the firm operates (flexible, dedicated or mixed). The designation ˆ denotes the optimal values of profits/decision variables. Bold letters denote vectors. All vectors are column vectors, and the superscript T denotes the transpose. For example, $\mathbf{A}^T$ represents the vector (A_1, A_2). All vectors are compared component-wise.

1 The flexible subgame

Suppose both firms invest in flexible technology that can produce both products and consider the last stage of the game (the production game). Assume without loss of generality that firm j has higher

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[1] The nonnegativity of prices is shown in the pure flexible subgame for illustrative purposes only. Proofs for the other subgames are along similar lines.

capacity than firm i, i.e., that the outcome of the capacity game is such that $K_{fi}^f \leq K_{fj}^f$. Given these capacities and a vector of demand intercept realizations $\mathbf{A}^T$, firms decide upon production quantities. The decision for one firm in isolation has been obtained by Chod and Rudi (2005). For two firms, the last-stage optimization problem can be formulated using Lagrange multipliers as follows:

$$\max\ L_i^f(u_i, q_{1i}^f, q_{2i}^f) = \sum_{y=1}^{2} \left(A_y - (q_{yi}^f + q_{yj}^f) - \beta\left(q_{(3-y)i}^f + q_{(3-y)j}^f\right)\right) q_{yi}^f - u_i\left(q_{1i}^f + q_{2i}^f - K_{fi}^f\right),\ i = 1, 2. \tag{2}$$

Combinations of the Lagrange multipliers and the slack variables give rise to 9 different optimization problems. It is convenient to represent the possible outcomes of the production game using the state-space diagram in Figure 1.[2]

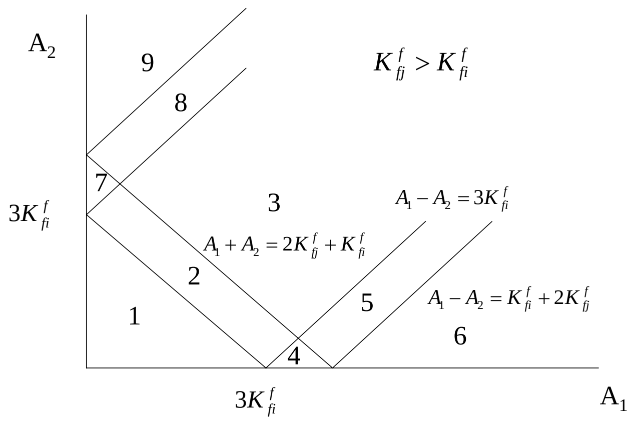

Figure 1. Pure flexible subgame (asymmetric), $\beta = 0$

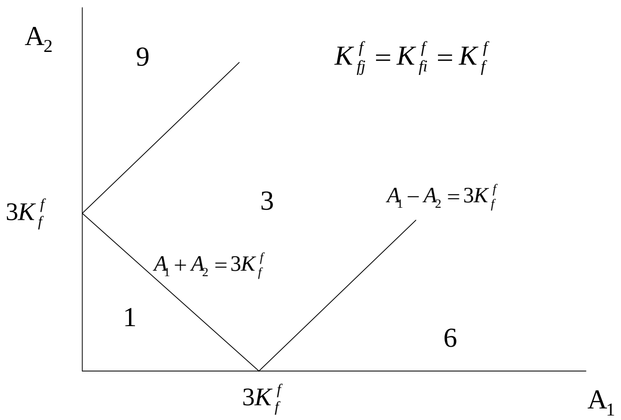

Figure 2. Pure flexible subgame (symmetric), $\beta = 0$

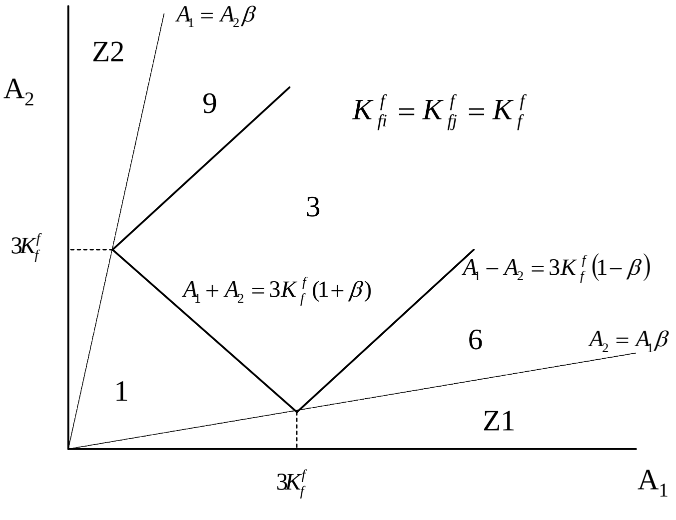

Figure 3. Pure flexible subgame (symmetric), $\beta > 0$

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[2] Note that although Figure 1 is for $\beta = 0$, the solutions are expressed for $\beta \neq 0$. Figure 2 represents a symmetric subgame with $\beta = 0$, and Figure 3 represents a symmetric subgame with $\beta > 0$. In the interest of simplicity, we do not provide pictures for $\beta < 0$.

The various areas of these figures can be explained intuitively. For instance, area Ω_1 represents the set of demand realizations such that no firm is capacity-constrained. Similarly, area Ω_3 represents the case in which both firms are capacity-constrained. For area Ω_2, only firm i is capacity-constrained. Areas Ω_6 and Ω_9 arise when the demand for one product is so high that, when the firms are capacity-constrained, they prefer to manufacture only one product. In areas Ω_4 and Ω_7, firm i is capacity-constrained while firm j is not, whereas in areas Ω_5 and Ω_8 both firms are capacity-constrained. Moreover, in these last four areas, firm i finds it economical to produce only one product, but firm j produces both products. A mathematical description of the areas follows (we assume $A_y \geq \beta A_{3-y}, y = 1, 2$ to ensure nonnegativity of quantities):

$$
\begin{aligned}
\Omega_1 &= \left\{\mathbf{A} : \mathbf{1}^T\mathbf{A} \leq 3K_{fi}^f (1+\beta)\right\}, \\
\Omega_2 &= \left\{\mathbf{A} : \mathbf{1}^T\mathbf{A} \geq 3K_{fi}^f (1+\beta), \mathbf{1}^T\mathbf{A} \leq \left(2K_{fj}^f + K_{fi}^f\right)(1+\beta), |A_1 - A_2| \leq 3K_{fi}^f (1-\beta)\right\}, \\
\Omega_3 &= \left\{\mathbf{A} : \mathbf{1}^T\mathbf{A} \ \geq \left(2K_{fj}^f + K_{fi}^f\right)(1+\beta), |A_1 - A_2| \leq 3K_{fi}^f (1-\beta)\right\}, \\
\Omega_{4,7} &= \left\{\mathbf{A} : |A_1 - A_2| \geq 3K_{fi}^f (1-\beta); \mathbf{1}^T\mathbf{A} \geq 3K_{fi}^f (1+\beta), \mathbf{1}^T\mathbf{A} \leq \left(2K_{fj}^f + K_{fi}^f\right)(1+\beta)\right\}, \\
\Omega_{5,8} &= \left\{\mathbf{A} : 3K_{fi}^f (1-\beta) \leq |A_1 - A_2| \leq \left(2K_{fj}^f + K_{fi}^f\right)(1-\beta); \mathbf{1}^T\mathbf{A} \geq \left(2K_{fj}^f + K_{fi}^f\right)(1+\beta)\right\}, \text{ and} \\
\Omega_{6,9} &= \left\{\mathbf{A} : |A_1 - A_2| \geq \left(2K_{fj}^f + K_{fi}^f\right)(1-\beta)\right\}.
\end{aligned}
$$

In each area, the production game can be solved in closed form (and the SPNE is trivially unique). The first-order KKT conditions are:

$$
\begin{aligned}
A_1 - 2q_{1i}^f - q_{1j}^f - \beta\left(q_{2i}^f + q_{2j}^f\right) - \beta q_{2i}^f - u_i + v_{1i} &= 0, \\
v_{1i}q_{1i}^f &= 0, \\
A_2 - 2q_{2i}^f - q_{2j}^f - \beta\left(q_{1i}^f + q_{1j}^f\right) - \beta q_{1i}^f - u_i + v_{2i} &= 0, \\
v_{2i}q_{2i}^f &= 0, \\
q_{1i}^f + q_{2i}^f + v_{3i} &= K_{fi}^f, \\
u_i v_{3i} &= 0,
\end{aligned}
$$

where v_{li} are the slack variables (it is easily verified that the objective function is concave so these conditions are also sufficient). For firm j we have similar expressions with the Lagrange multipliers and the slack variables labeled as u_j, v_{lj}, $l = 1, 2, 3$. The expressions for optimal quantities and profits for the various areas of the state-space diagram are obtained by taking appropriate values of the various Lagrange multipliers and the slack variables. Unless specified otherwise, we assume that all quantities are positive and hence the slack variables $v_{li} = v_{lj} = 0$ for $l = 1, 2$.

Capacity is not binding (area Ω_1)

$u_i = u_j = 0$ and $v_{3i}, v_{3j} > 0$ by complementary slackness. Solving for quantities we obtain

$$\begin{aligned} \widehat{q}^{f}_{yi} &= \widehat{q}^{f}_{yj} = \frac{A_y - A_{3-y}\beta}{3\left(1-\beta^2\right)}, \\ P_y &= \frac{A_y}{3}. \end{aligned}$$

The quantities are nonnegative as long as $A_y \geq A_{3-y}\beta$. As shown in Figure 3, this is true as long as we are outside areas Z_1 and Z_2. The prices are, of course, positive.

Capacity is binding for firm i but not for firm j (area Ω_2)

$u_i > 0$ while $u_j = 0$. From complementary slackness, we have

$$\begin{aligned} \widehat{q}^{f}_{yi} &= \frac{A_y - A_{3-y}}{6\left(1-\beta\right)} + \frac{K^f_{fi}}{2},\ \widehat{q}^{f}_{yj} = \frac{\left(\left(5-\beta\right)A_y + \left(1-5\beta\right)A_{3-y}\right)}{12\left(1-\beta^2\right)} - \frac{K^f_{fi}}{4}, \\ P_y &= \frac{5A_y + A_{3-y} - 3\left(1+\beta\right)K^f_{fi}}{12} \text{ for } y = 1, 2. \end{aligned}$$

It can also be shown that $u_i = (1/4)\left(A_1 + A_2 - 3K^f_{fi}\left(1+\beta\right)\right) > 0$ and $v_{3j} > 0 \Rightarrow (A_1 + A_2) \leq \left(2K^f_{fj} + K^f_{fi}\right)\left(1+\beta\right)$, which gives the defining equation for Ω_2. The price is nonnegative if $5A_y + A_{3-y} \geq 3\left(1+\beta\right)K^f_{fi}$. However, we know that $\left(A_1 + A_2 - 3K^f_{fi}\left(1+\beta\right)\right) > 0$ since $u_i > 0$. Hence prices are nonnegative.

Capacity is binding for both firms (area Ω_3)

$u_i, u_j > 0$. Solving for quantities we obtain

$$\begin{aligned} \widehat{q}^{f}_{yi} &= \frac{A_y - A_{3-y}}{6\left(1-\beta\right)} + \frac{K^f_{fi}}{2},\ \widehat{q}^{f}_{yj} = \frac{A_y - A_{3-y}}{6\left(1-\beta\right)} + \frac{K^f_{fj}}{2}, \\ P_y &= \frac{4A_y + 2A_{3-y} - 3\left(1+\beta\right)\left(K^f_{fi} + K^f_{fj}\right)}{6}, y = 1, 2. \end{aligned}$$

$u_j = (1/2)\left(A_1 + A_2 - \left(K^f_{fi} + 2K^f_{fj}\right)\left(1+\beta\right)\right) > 0$ gives the defining equation for Ω_3 in Figure 3. Quantities are nonnegative if $A_{3-y} - A_y \leq 3K^f_{fi}\left(1-\beta\right)$. To show that prices are nonnegative is a bit more involved. We show the nonnegativity of prices for $\beta \geq 0$ but if prices are nonnegative for $\beta \geq 0$, then they are nonnegative elsewhere. The prices are nonnegative if $4A_y + 2A_{3-y} \geq 3\left(1+\beta\right)\left(K^f_{fi} + K^f_{fj}\right)$. We know that $A_y + A_{3-y} \geq 3\left(1+\beta\right)K^f_{fi}$ and $A_y + A_{3-y} \geq \left(K^f_{fi} + 2K^f_{fj}\right)\left(1+\beta\right)$ from $u_j > 0$. After adding these two inequalities we obtain

$$2\left(A_y + A_{3-y}\right) \geq \left(1+\beta\right)\left(4K^f_{fi} + 2K^f_{fj}\right). \tag{3}$$

Also, we know from the geometry of the state space (Figure 3) that the minimum value of A_y (call it $A^{\min}_y$) is obtained from the intersection of lines $A_{3-y} - A^{\min}_y = 3K^f_{fi}\left(1-\beta\right)$ and $A_{3-y} + A^{\min}_y =$

$\left(K^f_{fi}+2K^f_{fj}\right)(1+\beta)$. From these two equations we obtain $A_y^{\min}=(1+\beta)\,K^f_{fj}-(1-2\beta)\,K^f_{fi}>0$. Hence,

$$2A_y \geq (1+\beta)\,K^f_{fj}-(1-2\beta)\,K^f_{fi}. \tag{4}$$

Adding inequalities (3) and (4), we obtain

$$4A_y+2A_{3-y} \geq 3(1+\beta)\left(K^f_{fi}+K^f_{fj}\right)+3\beta K^i_{fi} \geq 3(1+\beta)\left(K^f_{fi}+K^f_{fj}\right), \tag{5}$$

since $\beta \geq 0$. This proves the nonnegativity of prices in the two markets in Ω_3.

Difference in demand intercept realizations is very large with capacity constraint for firm i (areas $\Omega_{4,7}$)

We solve in Ω_4 first. Firm i has a capacity constraint and the difference in the demand for the two products is so large that firm i manufactures only one product. Firm j has no capacity constraint and can manufacture both products. The values of various variables for firm i are as follows: $q^f_{2i}=v_{3i}=v_{1i}=0$ with $q^f_{1i}, u_i, v_{2i}>0$. For firm j, $u_j=v_{1j}=v_{2j}=0$ with the corresponding duals being positive. Solving with the above parameters gives us $v_{2i}=\left(A_1-A_2-3K^f_{fi}(1-\beta)\right)/2>0$. From this condition we obtain the defining equation for the region as $A_1-A_2>3K^f_{fi}(1-\beta)$. Solving similarly for area Ω_7, we obtain

$$\begin{aligned}
\widehat{q}^f_{yi} &= K^f_{fi},\ \widehat{q}^f_{(3-y)i}=0,\ \widehat{q}^f_{yj}=\frac{A_y-\beta A_{3-y}}{2(1-\beta^2)}-\frac{K^f_{fi}}{2},\ \widehat{q}^f_{(3-y)j}=\frac{A_{3-y}-\beta A_y}{2(1-\beta^2)},\\
P_y &= \frac{A_y-K^f_{fi}}{2},\ P_{3-y}=\frac{A_{3-y}-\beta K^f_{fi}}{2}
\end{aligned}$$

and $\mathbf{A}\in\Omega_{4,7}$ where $y=1$ for $\mathbf{A}\in\Omega_4$ and $y=2$ for $\mathbf{A}\in\Omega_7$. It is relatively straightforward to see that the quantities and prices are nonnegative.

Firm i manufactures one product and firm j has a capacity constraint (areas $\Omega_{5,8}$)

We solve in Ω_5 first. Firm i has a capacity constraint and manufactures only one product whereas firm j, though capacity-constrained, manufactures both products. The values of various variables for firm i are as follows: $q^f_{2i}=v_{3i}=v_{1i}=0$ with $q^f_{1i}, u_i, v_{2i}>0$. For firm j, $v_{3j}=v_{1j}=v_{2j}=0$ with the corresponding duals being positive. Again by forcing $v_{2i}>0$ we obtain the defining equation similar to the one derived above. The complete solution for $\Omega_{5,8}$ is

$$\begin{aligned}
\widehat{q}^f_{yi} &= K^f_{fi},\ \widehat{q}^i_{3-y}=0,\ \widehat{q}^j_y=\frac{A_y-A_{3-y}}{4(1-\beta)}-\frac{K^f_{fi}}{4}+\frac{K^f_{fj}}{2},\ \widehat{q}^j_{3-y}=\frac{A_{3-y}-A_y}{4(1-\beta)}+\frac{K^f_{fi}}{4}+\frac{K^f_{fj}}{2},\\
P_y &= \frac{3A_y+A_{3-y}-(3+\beta)\,K^f_{fi}-2(1+\beta)\,K^f_{fj}}{4}, P_{3-y}=\frac{A_y+3A_{3-y}-(1+3\beta)\,K^f_{fi}-2(1+\beta)\,K^f_{fj}}{4}
\end{aligned}$$

and $\mathbf{A} \in \Omega_{5,8}$ where $y = 1$ for $\mathbf{A} \in \Omega_5$ and $y = 2$ for $\mathbf{A} \in \Omega_8$. It is easily shown that the quantities are nonnegative (which follows from the boundary equations for these areas). Now $P_y \geq 0$, if $3A_y + A_{3-y} \geq (3+\beta) K_{fi}^f + 2(1+\beta) K_{fj}^f$. We know that for Ω_5, $A_y - A_{3-y} \geq 3K_{fi}^f (1-\beta)$ and $A_y + A_{3-y} \geq \left(K_{fi}^f + 2K_{fj}^f\right)(1+\beta)$. Adding these two inequalities, we obtain

$$2A_y \geq K_{fi}^f (4-2\beta) + K_{fj}^f (2+2\beta). \tag{6}$$

Next, after adding (6) and $A_y + A_{3-y} \geq \left(K_{fi}^f + 2K_{fj}^f\right)(1+\beta)$ we obtain

$$3A_y + A_{3-y} \geq K_{fi}^f (5-\beta) + 2K_{fj}^f (2+2\beta) \geq (3+\beta) K_{fi}^f + 2(1+\beta) K_{fj}^f. \tag{7}$$

To show $P_{3-y} \geq 0$, we need $A_y + 3A_{3-y} \geq (1+3\beta) K_{fi}^f + 2(1+\beta) K_{fj}^f$. This follows from $A_y + A_{3-y} \geq \left(K_{fi}^f + 2K_{fj}^f\right)(1+\beta)$ and $2A_{3-y} \geq 2\beta K_{fi}^f$.

Both firms are capacity-constrained and manufacture one product (areas $\Omega_{6,9}$)

We solve in Ω_6 first. Both firms are capacity-constrained and the difference in demands is so high that each manufactures only one product. The values of the parameters for firm i are $q_{2i}^f = v_{3i} = v_{1i} = 0$ and for firm j are $q_{2j}^f = v_{3j} = v_{1j} = 0$, with the corresponding duals being positive. The optimal quantities and prices are

$$\begin{aligned} \widehat{q}_y^i &= K_{fi}^f,\ \widehat{q}_y^j = K_{fj}^f,\ \widehat{q}_{3-y}^i = \widehat{q}_{3-y}^j = 0, \\ P_y &= A_y - \left(K_{fi}^f + K_{fj}^f\right),\ P_{3-y} = A_{3-y} - \beta\left(K_{fi}^f + K_{fj}^f\right), \end{aligned}$$

where $\mathbf{A} \in \Omega_{6,9}$ with $y = 1$ for $\mathbf{A} \in \Omega_6$ and $y = 2$ for $\mathbf{A} \in \Omega_9$. Moreover, $v_{2i} = \left(A_1 - A_2 - (1-\beta)\left(2K_{fi}^f + K_{fj}^f\right)\right) > 0$ gives us $A_1 - A_2 > \left(2K_{fi}^f + K_{fj}^f\right)(1-\beta)$ and forcing $v_{2j} > 0$ gives us $A_1 - A_2 > \left(K_{fi}^f + 2K_{fj}^f\right)(1-\beta)$. Since $K_{fj}^f > K_{fi}^f$, the defining equation for the areas is $A_1 - A_2 > \left(K_{fi}^f + 2K_{fj}^f\right)(1-\beta)$. It is straightforward to see that the quantities and the prices are nonnegative.

The optimality conditions in the capacity game

The first-order condition for firm i in the capacity game can be expressed as[3]

$$E\frac{\partial \pi_i^f}{\partial K_{fi}^f} = c_{fi} \Rightarrow \sum_l \iint_{\Omega_l} \frac{\partial \pi_i^f}{\partial K_{fi}^f} dF(x_1, x_2) = c_{fi}.$$

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[3] Note that we interchanged differentiation and integration. This is justified if the function under differentiation/integration is Lipschitz-continuous of order one (see Glasserman 1994). This is easily verified since the first derivative is clearly bounded.

Differentiating the profit function w.r.t. K^f_{fi} in each area and using Leibnitz's rule gives us the first-order condition for firm i:

$$\begin{aligned} c_{fi} &= \iint\limits_{\Omega_2} \frac{1}{4}\left(x_1+x_2-2K^f_{fi}(1+\beta)\right)dF(x_1,x_2) \\ &+(1/2)\iint\limits_{\Omega_3}\left(x_1+x_2-\left(2K^f_{fi}+K^f_{fj}\right)(1+\beta)\right)dF(x_1,x_2) \\ &+(1/2)\iint\limits_{\Omega_4}\left(x_1-2K^f_{fi}\right)dF(x_1,x_2) \\ &+(1/4)\iint\limits_{\Omega_5}\left(3x_1+x_2-2(3+\beta)K^f_{fi}-2K^f_{fj}(1+\beta)\right)dF(x_1,x_2) \\ &+\iint\limits_{\Omega_6}\left(x_1-2K^f_{fi}-K^f_{fj}\right)dF(x_1,x_2)+(1/2)\iint\limits_{\Omega_7}\left(x_2-2K^f_{fi}\right)dF(x_1,x_2) \\ &+(1/4)\iint\limits_{\Omega_8}\left(x_1+3x_2-2(3+\beta)K^f_{fi}-2K^f_{fj}(1+\beta)\right)dF(x_1,x_2) \\ &+\iint\limits_{\Omega_9}\left(x_2-2K^f_{fi}-K^f_{fj}\right)dF(x_1,x_2), \end{aligned} \tag{8}$$

and similarly for firm j:

$$\begin{aligned} c_{fj} &= (1/2)\iint\limits_{\Omega_3}\left(x_1+x_2-\left(K^f_{fi}+2K^f_{fj}\right)(1+\beta)\right)dF(x_1,x_2) \\ &+(1/2)\iint\limits_{\Omega_{5,8}}\left(x_1+x_2-(1+\beta)\left(2K^f_{fj}+K^f_{fi}\right)\right)dF(x_1,x_2) \\ &+\iint\limits_{\Omega_6}\left(x_1-2K^f_{fj}-K^f_{fi}\right)dF(x_1,x_2) \\ &+\iint\limits_{\Omega_9}\left(x_2-2K^f_{fj}-K^f_{fi}\right)dF(x_1,x_2). \end{aligned}$$

In the following proposition, we show the uniqueness of a symmetric equilibrium. Proving uniqueness for an asymmetric capacity investment is difficult because one needs to differentiate the first-order conditions, and these are not continuous at the boundaries of the various regions when capacities are asymmetric. Since the boundaries are themselves functions of the capacities of the two firms, differentiating the first-order conditions does not result in tractable expressions. However, we did obtain the optimality conditions above for the capacity game without assuming symmetry. Therefore, all asymmetric equilibria can be found numerically (or in closed form for some probability distributions). The same comment applies for the pure dedicated subgame analyzed in the next section.

Proposition TA 1 *Equilibrium in the capacity game for the pure flexible subgame exists, and the symmetric equilibrium is unique* $\forall\beta\in(-1,1)$.

Proof. The concavity of the objective functions was demonstrated by Chod and Rudi (2005), which immediately implies existence of equilibrium. Uniqueness is proven by showing that the slope of the best-response function for each firm is less than one (Cachon and Netessine 2004). Using implicit differentiation, the absolute value of the slope of the best-response function for, say firm j, is found as $\left|\partial^2\Pi_j^f/\partial K_{fi}^f \partial K_{fj}^f\right| / \left|\partial^2\Pi_j^f/\partial\left(K_{fj}^f\right)^2\right|$.

Note that for a symmetric case ($c_{fi} = c_{fj} = c_f$), Figure 1 simplifies to Figure 2 (which is a special case of Figure 3 for $\beta = 0$), and hence areas 4, 5, 7 and 8 disappear. It is easy to check that the integrands in equation (8) are continuous at the boundaries of the areas once we assume symmetry. For instance, the boundary of areas 3 and 6 is $A_1 - A_2 = \left(2K_{fj}^f + K_{fi}^f\right)(1-\beta)$. Evaluating the integrands of Ω_3 and Ω_6 (say, for firm j) at this boundary gives $(1/2)\left(2x_2 + \left(2K_{fj}^f + K_{fi}^f\right)(1-\beta-1-\beta)\right) = \left(x_2 - \beta\left(2K_{fj}^f + K_{fi}^f\right)\right)$ for Ω_3 and $\left(x_2 - \left(2K_{fj}^f + K_{fi}^f\right)(1-1+\beta)\right) = \left(x_2 - \beta\left(2K_{fj}^f + K_{fi}^f\right)\right)$ for Ω_6. Hence, we can safely ignore differentiating the limits of the integrals when we apply Leibnitz's rule as the corresponding terms cancel out.

$$\begin{aligned}
\left|\frac{\partial^2\Pi_j^f}{\partial\left(K_{fj}^f\right)^2}\right| &= \iint\limits_{\Omega_3}(1+\beta)\,dF(x_1,x_2) + 2\iint\limits_{\Omega_{6,9}} dF(x_1,x_2),\\
\left|\frac{\partial^2\Pi_j^f}{\partial K_{fi}^f \partial K_{fj}^f}\right| &= \frac{1}{2}\iint\limits_{\Omega_3}(1+\beta)\,dF(x_1,x_2) + \iint\limits_{\Omega_{6,9}} dF(x_1,x_2).
\end{aligned}$$

Clearly, $\left|\partial^2\Pi_j^f/\partial K_{fi}^f \partial K_{fj}^f\right| / \left|\partial^2\Pi_j/\partial\left(K_{fj}^f\right)^2\right| = 1/2 < 1$. The same result holds for firm i. ■

2 The dedicated subgame

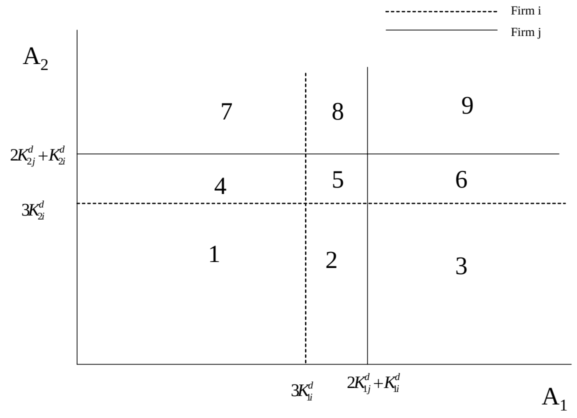

Figure 4. Pure dedicated subgame - asymmetric solution

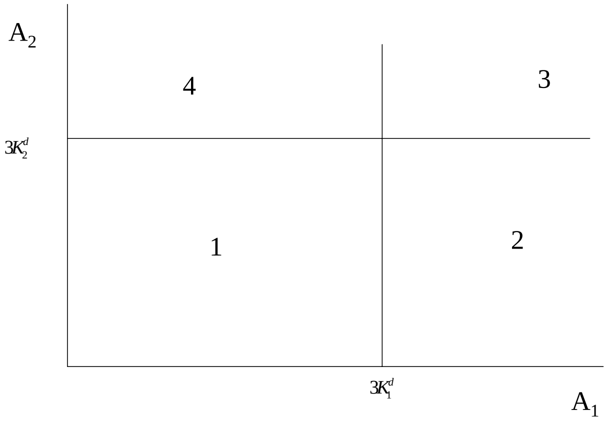

Figure 5. Pure dedicated subgame - symmetric solution

Suppose that both firms invest in dedicated technology, i.e., there is a dedicated production line for each product (see Figures 4 and 5)[4]. Note the assumption we make here is that the capacities of firm j are higher than corresponding capacities of firm i. We emphasize that this is not a unique state representation. For instance, the capacity of firm j could be higher than that of firm i in one market but lower in another. We could solve the production game in closed form for any of these scenarios, so the above assumption is for expositional convenience only.

Compared to the pure flexible subgame, the interpretations of the areas in Figure 4 are much simpler. For instance, area Ω_1 represents no capacity constraint for either firm. In areas Ω_2 and Ω_4, firm i has capacity constraints for products 1 and 2 respectively, whereas firm j has no capacity constraint for either product. Since in each area the production game can be solved uniquely, the SPNE in the production game is trivially unique. The Lagrangian can be written as:

$$\max L_i^d(q_{1i}^d, q_{2i}^d, \mathbf{u}) = \sum_{y=1}^{2} \left(A_y - \left(q_{yi}^d + q_{yj}^d + \beta\left(q_{(3-y)i}^d + q_{(3-y)j}^d\right)\right)\right) q_{yi}^d - u_{yi}(q_{yi}^d - K_{yi}^d).$$

A similar expression can be obtained for firm j with the Lagrange multipliers u_{1j} and u_{2j}. The KKT conditions for firm i are:

$$\begin{aligned}
A_1 - 2q_{1i}^d - q_{1j}^d - \beta\left(q_{2i}^d + q_{2j}^d\right) - \beta q_{2i}^d - u_{1i} + v_{1i} &= 0, \\
q_{1i}^d v_{1i} &= 0, \\
A_2 - 2q_{2i}^d - q_{2j}^d - \beta\left(q_{1i}^d + q_{1j}^d\right) - \beta q_{1i}^d - u_{2i} + v_{2i} &= 0, \\
q_{2i}^d v_{2i} &= 0, \\
q_{1i}^d + v_{3i} &= K_{1i}^d, \\
v_{3i} u_{1i} &= 0, \\
q_{2i}^d + v_{4i} &= K_{2i}^d, \text{ and} \\
v_{4i} u_{2i} &= 0,
\end{aligned}$$

where v_{li}, $l = 1, 2, 3, 4$ are the slack variables. We now provide closed-form solutions for the optimal quantities. All quantities are positive and hence, unless specified otherwise, $v_{li} = v_{lj} = 0$ for all l.

Capacity is not binding (area Ω_1)

$u_{li} = u_{lj} = 0$ for $l = 1, 2$. The optimal production quantities are

$$\widehat{q}_{yi}^d = \widehat{q}_{yj}^d = \frac{A_y - A_{3-y}\beta}{3\left(1 - \beta^2\right)}.$$

---

[4] The figures are drawn for $\beta = 0$. In the interest of simplicity, the modified state-space representation for $\beta \neq 0$ is omitted here.

Capacity is binding for one product for firm i (areas $\Omega_{2,4}$)

For Ω_2, $u_{2i}=0$ and $u_{kj}=0$ for $k=1,2$. However, since capacity binds for product 1 for firm i, $u_{1i}>0$. The corresponding duals are positive by complementary slackness. Solving for the optimal quantities we obtain

$$\begin{aligned}\widehat{q}^{d}_{yi} &= K^{d}_{yi},\ \widehat{q}^{d}_{(3-y)i}=\frac{A_{3-y}-3\beta K^{d}_{yi}}{3},\ \widehat{q}^{d}_{yj}=\frac{A_y-\beta A_{3-y}-K^{d}_{yi}\left(1-\beta^2\right)}{2\left(1-\beta^2\right)},\\ \widehat{q}^{d}_{(3-y)j} &= \frac{A_{3-y}\left(2+\beta^2\right)-3\beta A_y}{6\left(1-\beta^2\right)}+\frac{\beta K^{d}_{yi}}{2}\end{aligned}$$

and $\mathbf{A}\in\Omega_{2,4}$ where $y=1$ for $\mathbf{A}\in\Omega_2$ and $y=2$ for $\mathbf{A}\in\Omega_4$.

Capacity is binding for both products for firm i (area Ω_5)

We have $u_{1i},u_{2i}>0$, $u_{1j}=u_{2j}=0$. Other variables are positive by complementary slackness and production quantities are

$$\widehat{q}^{d}_{1i}=K^{d}_{1i},\ \widehat{q}^{d}_{2i}=K^{d}_{2i},\ \widehat{q}^{d}_{1j}=\frac{A_1-\beta A_2-\left(1-\beta^2\right)K^{d}_{1i}}{2\left(1-\beta^2\right)},\ \widehat{q}^{d}_{2j}=\frac{A_2-\beta A_1-\left(1-\beta^2\right)K^{d}_{2i}}{2\left(1-\beta^2\right)}.$$

Capacity is binding for both firms for the same product (areas $\Omega_{3,7}$)

For Ω_3 we have $u_{1i},u_{1j}>0$, $u_{2i}=u_{2j}=0$ and more generally:

$$\widehat{q}^{d}_{yi}=K^{d}_{yi},\ \widehat{q}^{d}_{(3-y)i}=\frac{A_{3-y}-3\beta K^{d}_{yi}}{3},\ \widehat{q}^{d}_{yj}=K^{d}_{yj},\ \widehat{q}^{d}_{(3-y)j}=\frac{A_{3-y}-3\beta K^{d}_{yj}}{3},$$

and $\mathbf{A}\in\Omega_{3,7}$ where $y=1$ for $\mathbf{A}\in\Omega_3$ and $y=2$ for $\mathbf{A}\in\Omega_7$.

Capacity is binding for one product for firm j and both products for firm i (areas $\Omega_{6,8}$)

For Ω_6 we have $u_{1i},u_{2i}>0$, $u_{2j}=0,u_{1j}>0$. Other variables are nonzero by complementary slackness, and we obtain

$$\widehat{q}^{d}_{yi}=K^{d}_{yi},\ \widehat{q}^{d}_{(3-y)i}=K^{d}_{(3-y)i},\ \widehat{q}^{d}_{yj}=K^{d}_{yj},\ \widehat{q}^{d}_{(3-y)j}=\frac{A_{3-y}-K^{d}_{(3-y)i}-\beta(K^{d}_{yi}+2K^{d}_{yj})}{2},$$

and $\mathbf{A}\in\Omega_{6,8}$ where $y=1$ for $\mathbf{A}\in\Omega_6$ and $y=2$ for $\mathbf{A}\in\Omega_8$.

Capacity is binding for both products for both firms (area Ω_9)

We have $u_{li},u_{lj}>0$ for $l=1,2$. The optimal solution is simply

$$\widehat{q}^{d}_{yi}=K^{d}_{yi};\ \widehat{q}^{d}_{yj}=K^{d}_{yj}.$$

The optimality conditions in the capacity game

For firm i, the first-order condition can be expressed as $\partial\left(E\pi_i^d\right)/\partial K_{1i}^d = c_i$ which translates into

$$\begin{aligned}
c_i &= (1/2)\iint\limits_{\Omega_2}\left(x_1 - x_2\beta - 2\left(1-\beta^2\right)K_{1i}^d\right)dF(x_1,x_2) \\
&+ \iint\limits_{\Omega_3}\left(x_1 - \beta x_2 - \left(1-\beta^2\right)\left(2K_{1i}^d + K_{1j}^d\right)\right)dF(x_1,x_2) \\
&+ (1/2)\iint\limits_{\Omega_5}\left(x_1 - 2K_{1i}^d - 2\beta K_{2i}^d\right)dF(x_1,x_2) \\
&+ \iint\limits_{\Omega_6}\left(x_1 - \left(2K_{1i}^d + K_{1j}^d\right) + \beta\left(-x_2 + 2\beta\left(K_{1i}^d + K_{1j}^d\right) - 2K_{2i}^d\right)/2\right)dF(x_1,x_2) \\
&+ (1/2)\iint\limits_{\Omega_8}\left(x_1 - 2\beta K_{2i}^d - 2K_{1i}^d\right)dF(x_1,x_2) \\
&+ \iint\limits_{\Omega_9}\left(x_1 - \left(2K_{1i}^d + K_{1j}^d\right) - \beta\left(2K_{2i}^d + K_{2j}^d\right)\right)dF(x_1,x_2).
\end{aligned}$$

Analogously, $\partial E\left(\pi_i^d\right)/\partial K_{2i}^f = c_i$ yields:

$$\begin{aligned}
c_i &= (1/2)\iint\limits_{\Omega_4}\left(x_2 - x_1\beta - 2\left(1-\beta^2\right)K_{2i}^d\right)dF(x_1,x_2) \\
&+ (1/2)\iint\limits_{\Omega_5}\left(x_2 - 2K_{2i}^d - 2\beta K_{1i}^d\right)dF(x_1,x_2) \\
&+ (1/2)\iint\limits_{\Omega_6}\left(x_2 - 2\beta K_{1i}^d - 2K_{2i}^d\right)dF(x_1,x_2) \\
&+ \iint\limits_{\Omega_7}\left(x_2 - \beta x_1 - \left(1-\beta^2\right)\left(2K_{2i}^d + K_{2j}^d\right)\right)dF(x_1,x_2) \\
&+ \iint\limits_{\Omega_8}\left(x_2 - \left(2K_{2i}^d + K_{2j}^d\right) + \beta\left(-x_1 + 2\beta\left(K_{2i}^d + K_{2j}^d\right) - 2K_{1i}^d\right)/2\right)dF(x_1,x_2) \\
&+ \iint\limits_{\Omega_9}\left(x_2 - \left(2K_{2i}^d + K_{2j}^d\right) - \beta\left(2K_{1i}^d + K_{1j}^d\right)\right)dF(x_1,x_2).
\end{aligned}$$

For firm j, $E\left(\partial\pi_j^d/\partial K_{1j}^d\right) = c_j$ translates into:

$$\begin{aligned}
c_j &= \iint\limits_{\Omega_{3,6}}\left[x_1 - \beta x_2 - \left(1-\beta^2\right)\left(2K_{1j}^d + K_{1i}^d\right)\right]dF(x_1,x_2) \\
&+ \iint\limits_{\Omega_9}\left[x_1 - \left(2K_{1j}^d + K_{1i}^d\right) - \beta\left(2K_{2j}^d + K_{2i}^d\right)\right]dF(x_1,x_2),
\end{aligned} \tag{9}$$

and finally from $E\left(\partial\pi_j^d/\partial K_{2j}^d\right) = c_j$ we obtain

$$c_j = \iint\limits_{\Omega_{7,8}}\left[x_2 - \beta x_1 - \left(1-\beta^2\right)\left(2K_{2j}^d + K_{2i}^d\right)\right]dF(x_1,x_2)$$

$$+\iint\limits_{\Omega_9}\left[x_2-\left(2K^d_{2j}+K^d_{2i}\right)-\beta\left(2K^d_{1j}+K^d_{1i}\right)\right]dF(x_1,x_2).$$

In the following proposition we demonstrate the existence and the uniqueness of equilibrium in the capacity game.

Proposition TA 2 *Equilibrium in the capacity game for the pure dedicated subgame exists* $\forall\beta\in(-1,1)$, *and the symmetric equilibrium is unique for* $\beta\in(-1,1/3)$.

Proof. It can be easily verified that each objective function is concave so a pure strategy Nash equilibrium exists. The Hessian for this game can be written as:

$$H^d=\begin{pmatrix}\frac{\partial^2\Pi^d_i}{\partial\left(K^d_{1i}\right)^2} & \frac{\partial^2\Pi^d_i}{\partial K^d_{1i}\partial K^d_{2i}} & \frac{\partial^2\Pi^d_i}{\partial K^d_{1i}\partial K^d_{1j}} & \frac{\partial^2\Pi^d_i}{\partial K^d_{1i}\partial K^d_{2j}}\\ \frac{\partial^2\Pi^d_i}{\partial K^d_{1i}\partial K^d_{2i}} & \frac{\partial^2\Pi^d_i}{\partial\left(K^d_{2i}\right)^2} & \frac{\partial^2\Pi^d_i}{\partial K^d_{2i}\partial K^d_{1j}} & \frac{\partial^2\Pi^d_i}{\partial K^d_{2i}\partial K^d_{2j}}\\ \frac{\partial^2\Pi^d_j}{\partial K^d_{1j}\partial K^d_{1i}} & \frac{\partial^2\Pi^d_j}{\partial K^d_{1j}\partial K^d_{2i}} & \frac{\partial^2\Pi^d_j}{\partial\left(K^d_{1j}\right)^2} & \frac{\partial^2\Pi^d_j}{\partial K^d_{1j}\partial K^d_{2j}}\\ \frac{\partial^2\Pi^d_j}{\partial K^d_{2j}\partial K^d_{1i}} & \frac{\partial^2\Pi^d_j}{\partial K^d_{2j}\partial K^d_{2i}} & \frac{\partial^2\Pi^d_j}{\partial K^d_{2j}\partial K^d_{1j}} & \frac{\partial^2\Pi^d_j}{\partial\left(K^d_{2j}\right)^2}\end{pmatrix}.$$

Following Cachon and Netessine (2004), a condition sufficient for the uniqueness of the Nash equilibrium is the diagonal dominance that translates into

$$\left|\frac{\partial^2\Pi^d_i}{\partial\left(K^d_{yi}\right)^2}\right|>\left|\frac{\partial^2\Pi^d_i}{\partial K^d_{yi}\partial K^d_{(3-y)i}}\right|+\left|\frac{\partial^2\Pi^d_i}{\partial K^d_{yi}\partial K^d_{yj}}\right|+\left|\frac{\partial^2\Pi^d_i}{\partial K^d_{yi}\partial K^d_{(3-y)j}}\right|,\ y=1,2, \tag{10}$$

$$\left|\frac{\partial^2\Pi^d_j}{\partial\left(K^d_{yj}\right)^2}\right|>\left|\frac{\partial^2\Pi^d_j}{\partial K^d_{yj}\partial K^d_{(3-y)j}}\right|+\left|\frac{\partial^2\Pi^d_j}{\partial K^d_{yj}\partial K^d_{yi}}\right|+\left|\frac{\partial^2\Pi^d_j}{\partial K^d_{yj}\partial K^d_{(3-y)i}}\right|,\ y=1,2. \tag{11}$$

Because of symmetry assumption, we show the analysis for one firm only (say, firm j). We rewrite the first-order conditions (9) for firm j for the symmetric case (note that Figure 4 transforms into Figure 5 and $c_i=c_j=c$). These are

$$\begin{aligned}c&=\iint\limits_{\Omega_2}\left[x_1-\beta x_2-\left(1-\beta^2\right)\left(2K^d_{1j}+K^d_{1i}\right)\right]dF(x_1,x_2)\\&\quad+\iint\limits_{\Omega_3}\left[x_1-\left(2K^d_{1j}+K^d_{1i}\right)-\beta\left(2K^d_{2j}+K^d_{2i}\right)\right]dF(x_1,x_2),\end{aligned} \tag{12}$$

and

$$\begin{aligned}c&=\iint\limits_{\Omega_4}\left[x_2-\beta x_1-\left(1-\beta^2\right)\left(2K^d_{2j}+K^d_{2i}\right)\right]dF(x_1,x_2)\\&\quad+\iint\limits_{\Omega_3}\left[x_2-\left(2K^d_{2j}+K^d_{2i}\right)-\beta\left(2K^d_{1j}+K^d_{1i}\right)\right]dF(x_1,x_2).\end{aligned}$$

We now number the areas with respect to Figure 5. We again verify that the integrands are continuous along boundaries. To illustrate a specific case, consider areas 2 and 3 in Figure 5. The boundary condition is $A_2 = \beta\left(K_{1i}^d + 2K_{1j}^d\right) + \left(K_{2i}^d + 2K_{2j}^d\right)$. The integrand of Ω_2 in equation (12) reduces to $x_1 - \left(2K_{1j}^d + K_{1i}^d\right) - \beta\left(2K_{2j}^d + K_{2i}^d\right)$, which is the same as the integrand of Ω_3. Hence, we ignore the limits while differentiating using Leibnitz's rule.

Next we derive conditions for (11) to hold. We first show the result for $y = 1$:

$$\begin{aligned}
\left|\frac{\partial^2 \Pi_j^d}{\partial\left(K_{1j}^d\right)^2}\right| &= \iint_{\Omega_2} 2\left(1-\beta^2\right) dF(x_1,x_2) + \iint_{\Omega_3} 2dF(x_1,x_2), \\
\left|\frac{\partial^2 \Pi_j^d}{\partial K_{1j}^d \partial K_{2j}^d}\right| &= \iint_{\Omega_3} 2\beta dF(x_1,x_2), \\
\left|\frac{\partial^2 \Pi_j^d}{\partial K_{1j}^d \partial K_{1i}^d}\right| &= \iint_{\Omega_2} \left(1-\beta^2\right) dF(x_1,x_2) + \iint_{\Omega_3} 1dF(x_1,x_2), \\
\left|\frac{\partial^2 \Pi_j^d}{\partial K_{1j}^d \partial K_{2i}^d}\right| &= \iint_{\Omega_3} \beta dF(x_1,x_2).
\end{aligned}$$

It is easily verified that the inequality holds for each of the areas except for Ω_3. For the inequality to hold in Ω_3, we need $\beta < 1/3$. The same result for $y = 2$ can be shown analogously. ■

3 The mixed subgame

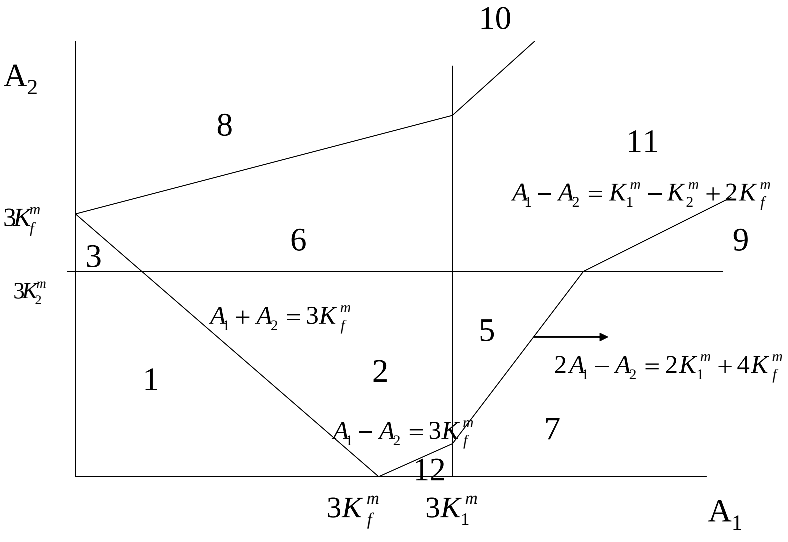

Figure 6. The mixed subgame with $K_1^m > K_f^m > K_2^m$

Suppose that firm i decides to invest in dedicated capacity and firm j decides to invest in flexible capacity. We alter our notation for the purpose of this section only: for clarity we remove the subscripts

i and j (subscripts d and f will be used if necessary for the flexible and the dedicated firm). For instance, K_1^m represents dedicated capacity for product 1 and q_{1f}^m represents the quantity of product 1 produced by the flexible firm. As in the previous two cases, a number of areas arise due to the capacity constraints of both firms (see Figure 6),[5] and the production game can be solved uniquely for each area. However, this is not a unique representation (i.e., there are other possibilities that could give rise to a different set of areas). Figure 6 is based on the assumption that $K_1^m > K_f^m > K_2^m$. For instance, there could be an area 4 similar to area 3 instead of area 12 in the figure. The presence of these areas depends on the assumptions about the capacity levels for the firms. Hence, Figure 6 is only a schematic representation of how the areas are placed with respect to each other. However, because there are multiple representations of the state-space, we suspect that there might be multiple equilibria in the capacity game of the mixed subgame associated with each such representation.

The areas in Figure 6 have intuitive explanations. For instance, in area Ω_1 no firm has a capacity constraint, whereas in area Ω_2 the flexible firm has a capacity constraint. In area Ω_3, the dedicated firm has a capacity constraint for product 1, etc.

Using the methodology below, we can solve for all possible ways of representing the mixed subgame. For ease of understanding, some of the areas can be referenced back to Figure 6. Others (like area Ω_4), which do not find representation in Figure 6, can be understood from the text detailing what they stand for. The Lagrangian formulation for the flexible firm is

$$\max L_f^m(u_f, q_{1f}^m, q_{2f}^m) = \sum_{y=1}^{2} \left(A_y - (q_{yf}^m + q_{yd}^m + \beta \left(q_{(3-y)f}^m + q_{(3-y)d}^m\right))\right) q_{yf}^m - u_f(q_{1f}^m + q_{2f}^m - K_f^m).$$

The KKT conditions (which, due to the concavity of the objective function, are sufficient) are:

$$\begin{aligned}
A_1 - 2q_{1f}^m - q_{1d}^m - \beta \left(q_{2f}^m + q_{2d}^m\right) - \beta q_{2f}^m - u_f + v_{1f} &= 0, \\
v_{1f} q_{1f}^m &= 0, \\
A_2 - 2q_{2f}^m - q_{2d}^m - \beta \left(q_{1f}^m + q_{1d}^m\right) - \beta q_{1f}^m - u_f + v_{2f} &= 0, \\
v_{2f} q_{2f}^m &= 0, \\
q_{1f}^m + q_{2f}^m + v_{3f} &= K_f^m, \\
u_f v_{3f} &= 0,
\end{aligned}$$

where v_{lf} are the slack variables for $l = 1, 2, 3$. For the dedicated firm, the Lagrangian is

$$\max L_d^m(q_{1d}^m, q_{2d}^m, \mathbf{u}) = \sum_{y=1}^{2} \left(\left(A_y - (q_{yd}^m + q_{yf}^m + \beta \left(q_{(3-y)d}^m + q_{(3-y)f}^m\right))\right) q_{yi}^d - u_{yd}(q_{yd}^m - K_y^m)\right).$$

---

[5] Figure 6 represents the state-space for a mixed subgame with $\beta = 0$. The modified state-space representation for $\beta \neq 0$ is omitted for simplicity.

The KKT conditions for the firm employing dedicated technology are:

$$\begin{aligned}
A_1 - 2q_{1d}^m - q_{1f}^m - \beta\left(q_{2d}^m + q_{2f}^m\right) - \beta q_{2d}^m - u_{1d} + v_{1d} &= 0, \\
q_{1d}^m v_{1d} &= 0, \\
A_2 - 2q_{2d}^m - q_{2f}^m - \beta\left(q_{1d}^m + q_{1f}^m\right) - \beta q_{1d}^m - u_{2d} + v_{2d} &= 0, \\
q_{2d}^m v_{2d} &= 0, \\
q_{1d}^m + v_{3d} &= K_1^m, \\
v_{3d} u_{1d} &= 0, \\
q_{2d}^m + v_{4d} &= K_2^m, \\
v_{4d} u_{2d} &= 0,
\end{aligned}$$

where v_{ld}, $l = 1, 2, 3, 4$ are the slack variables. We proceed by finding the optimal production quantities. Unless otherwise specified, the quantities are all positive and hence $v_{lf} = v_{ld} = 0$, $l = 1, 2$. Similar in spirit to the case of a pure flexible subgame, the Lagrange multipliers often define the boundary conditions for the various areas of integration. For the sake of simplicity, we do not show the values of the Lagrange multipliers whenever they are positive. We do, however, show some interesting cases below in which the slack variables are positive.

Capacity is not binding (area Ω_1)

$u_f = u_{1d} = u_{2d} = 0$. The unconstrained solution is

$$\widehat{q}_{yf}^m = \widehat{q}_{yd}^m = \frac{A_y - A_{3-y}\beta}{3\left(1-\beta^2\right)},\ y = 1, 2.$$

Only the flexible firm is capacity-constrained (area Ω_2)

For the flexible firm, $u_f > 0$. For the dedicated firm, $u_{kd} = 0$ for $k = 1, 2$. The optimal production quantities are:

$$\begin{aligned}
\widehat{q}_{yf}^m &= \frac{A_y - A_{3-y}}{6\left(1-\beta\right)} + \frac{K_f^m}{2} \text{ for } y = 1, 2, \\
\widehat{q}_{yd}^m &= \frac{\left(5-\beta\right)A_y + A_{3-y}\left(1-5\beta\right)}{12\left(1-\beta^2\right)} - \frac{K_f^m}{4} \text{ for } y = 1, 2.
\end{aligned}$$

The flexible firm is not capacity-constrained whereas the dedicated firm is capacity-constrained for both products (area $\Omega_{\tilde{2}}$)

Note that this area is not represented in Figure 6. Here, $u_f = 0$ and $u_{kd} > 0$ for $k = 1, 2$. The optimal production quantities are

$$\begin{aligned}
\widehat{q}_{yf}^m &= \frac{A_y - \beta A_{3-y} - K_y^m\left(1-\beta^2\right)}{2},\ \text{for } y = 1, 2, \\
\widehat{q}_{yd}^m &= K_y^m.
\end{aligned}$$

The flexible firm is not capacity-constrained whereas the dedicated firm is constrained for product 2 for area Ω_3 and product 1 for area Ω_4

Area Ω_4 is not in Figure 6. For the flexible firm, $u_f = 0$ with the corresponding duals being nonzero. For the dedicated firm we have $u_{yd} = 0$ and $u_{(3-y)d} > 0$. In what follows, $y = 1$ for area Ω_3 and $y = 2$ for area Ω_4. Solving for quantities we obtain:

$$\begin{aligned} \widehat{q}^m_{yf} &= \frac{A_y\left(2+\beta^2\right) - 3\beta A_{3-y} + 3\beta K^m_{(3-y)}\left(1-\beta^2\right)}{6\left(1-\beta^2\right)}, \\ \widehat{q}^m_{(3-y)f} &= \frac{A_{3-y} - \beta A_y - K^m_{(3-y)}\left(1-\beta^2\right)}{2\left(1-\beta^2\right)},\ \widehat{q}^m_{yd} = \frac{A_y - 3\beta K^m_{(3-y)}}{3}, \widehat{q}^m_{(3-y)d} = K^m_{(3-y)}. \end{aligned}$$

Capacity is binding for the flexible firm and for one product for the dedicated firm (area $\Omega_{5,6}$)

For the flexible firm, $u_f > 0$. For the dedicated firm, $u_{yd} > 0$ and $u_{(3-y)d} = 0$ where $y = 1$ for $\mathbf{A} \in \Omega_5$ and $y = 2$ for $\mathbf{A} \in \Omega_6$. The solution is:

$$\begin{aligned} \widehat{q}^m_{yf} &= \frac{2A_y - A_{(3-y)}\left(1+\beta\right) - 2K^m_y(1-\beta^2) + 3K^m_f\left(1-\beta\right)}{7-\beta\left(6+\beta\right)}, \\ \widehat{q}^m_{(3-y)f} &= \frac{-2A_y + A_{3-y}\left(1+\beta\right) + 2K^m_y\left(1-\beta^2\right) + K^m_f\left(4-3\beta-\beta^2\right)}{7-\beta\left(6+\beta\right)}, \\ \widehat{q}^m_{yd} &= K^m_y,\ \widehat{q}^m_{(3-y)d} = \frac{A_y + 3A_{3-y} - K^m_y(1+7\beta) - 2K^m_f\left(1+\beta\right)}{7+\beta}, \end{aligned}$$

and $\mathbf{A} \in \Omega_{5,6}$ where $y = 1$ for $\mathbf{A} \in \Omega_5$ and $y = 2$ for $\mathbf{A} \in \Omega_6$.

Capacity is binding for the flexible firm and for one product for the dedicated firm. The flexible firm manufactures one product (areas $\Omega_{7,8}$)

In area Ω_7 the difference in demand realizations is so high that the flexible firm manufactures only product 1. Hence, for the flexible firm, $u_f > 0$ and $v_{2f} > 0$ so that $\widehat{q}^m_{2f} = 0$. For the dedicated firm, $u_{1d} > 0$. Upon solving we obtain $v_{2f} = (1/2)\left(2A_1 - A_2\left(1+\beta\right) - 2K^m_1\left(1-\beta^2\right) - K^m_f\left(4-3\beta-\beta^2\right)\right) > 0$. For $\beta = 0$ this reduces to the boundary condition $2A_1 - A_2 > 2K^m_1 + 4K^m_f$ as is evident in Figure 6. The optimal quantities are:

$$\widehat{q}^m_{yf} = K^m_f,\ \widehat{q}^m_{(3-y)f} = 0,\ \widehat{q}^m_{yd} = K^m_y,\ \widehat{q}^m_{(3-y)d} = \left(A_{3-y} - 2\beta K^m_y - \beta K^m_f\right)/2,$$

and $\mathbf{A} \in \Omega_{7,8}$ where $y = 1$ for $\mathbf{A} \in \Omega_7$ and $y = 2$ for $\mathbf{A} \in \Omega_8$.

Capacity is binding for the flexible firm and for both products for the dedicated firm. The flexible firm manufactures one product (areas $\Omega_{9,10}$)

The only change from the preceding case is that for the dedicated firm we have $u_{1d}, u_{2d} > 0$. Solving in area Ω_9 we obtain $v_{2f} = A_1 - A_2 - (K_1^m - K_2^m + 2K_f^m)(1-\beta) > 0$. From here we obtain the boundary condition for these areas as shown in Figure 6. After solving for the optimal quantities, we obtain:

$$\widehat{q}_{yf}^m = K_f^m,\ \widehat{q}_{(3-y)f}^m = 0,\ \widehat{q}_{yd}^m = K_y^m,\ \widehat{q}_{(3-y)d}^m = K_{(3-y)}^m,$$

and $\mathbf{A} \in \Omega_{9,10}$ where $y = 1$ for $\mathbf{A} \in \Omega_9$ and $y = 2$ for $\mathbf{A} \in \Omega_{10}$.

Both firms are capacity-constrained (area Ω_{11})

All slack variables are zero for both firms. Solving for quantities we obtain

$$\widehat{q}_{yf}^m = \frac{A_y - A_{3-y}}{4(1-\beta)} + \frac{\left(K_{(3-y)}^m - K_y^m\right)}{4} + \frac{K_f^m}{2},\ \widehat{q}_{yd}^m = K_y^m,\ y = 1, 2.$$

The flexible firm is capacity-constrained and manufactures one product: product 1 for area Ω_{12} and product 2 for area Ω_{13}

Area Ω_{13} is not in Figure 6. Let $y = 1$ for area Ω_{12} and $y = 2$ for area Ω_{13}. For the flexible firm, $u_f > 0$ and $v_{(3-y)f} > 0$ so that $\widehat{q}_{(3-y)f}^m = 0$. For the dedicated firm, $u_{ld} = 0$ for $l = 1, 2$. Solving we obtain $v_{(3-y)f} = (1/2)\left(A_y - A_{3-y} - 3K_f^m(1-\beta)\right) > 0$, which gives us the boundary condition for this area. The optimal production quantities are:

$$\widehat{q}_{yf}^m = K_f^m,\ \widehat{q}_{(3-y)f}^m = 0,\ \widehat{q}_{yd}^m = \frac{A_y - A_{(3-y)}\beta - K_f^m\left(1-\beta^2\right)}{2\left(1-\beta^2\right)},\ \widehat{q}_{(3-y)d}^m = \frac{A_{3-y} - A_y\beta}{2\left(1-\beta^2\right)}.$$

The optimality conditions in the capacity game

For the flexible firm, the FOC is given by $E\left(\partial \pi_f^m / \partial K_f^m\right) = c_{fj}$, which translates into:

$$\begin{aligned}
c_{fj} \quad = \quad & \iint\limits_{\Omega_2} \frac{1}{4}\left(x_1 + x_2 - 2K_f^m(1+\beta)\right) dF(x_1, x_2) \\
& + \iint\limits_{\Omega_5} \frac{1}{(7+\beta)^2} \left(\begin{array}{c} (17-\beta)x_1 + (16 - \beta(1-\beta))x_2 \\ +(-17+\beta)(1+\beta)\left((1-\beta)K_1^m + 2K_f^m\right) \end{array} \right) dF(x_1, x_2) \\
& + \iint\limits_{\Omega_6} \frac{1}{(7+\beta)^2} \left(\begin{array}{c} (17-\beta)x_2 + (16 - \beta(1-\beta))x_1 \\ +(-17+\beta)(1+\beta)\left((1-\beta)K_2^m + 2K_f^m\right) \end{array} \right) dF(x_1, x_2) \\
& + \iint\limits_{\Omega_7} \left(x_1 - (\beta x_2/2) - \left(1-\beta^2\right)K_1^m - (2-\beta^2)K_f^m\right) dF(x_1, x_2) \\
& + \iint\limits_{\Omega_8} \left(x_2 - (\beta x_1/2) - \left(1-\beta^2\right)K_2^m - (2-\beta^2)K_f^m\right) dF(x_1, x_2) \\
& + \iint\limits_{\Omega_9} \left(x_1 - K_1^m - \beta K_2^m - 2K_f^m\right) dF(x_1, x_2)
\end{aligned}$$

$$
\begin{aligned}
&+\iint\limits_{\Omega_{10}} \left(x_2 - K_2^m - \beta K_1^m - 2K_f^m\right) dF(x_1,x_2) \\
&+(1/2)\iint\limits_{\Omega_{11}} \left(x_1 + x_2 - (1+\beta)\left(K_1^m + K_2^m + 2K_f^m\right)\right) dF(x_1,x_2) \\
&+\iint\limits_{\Omega_{12}} \frac{1}{2}\left(x_1 - 2K_f^m\right) dF(x_1,x_2) + \iint\limits_{\Omega_{13}} \frac{1}{2}\left(x_2 - 2K_f^m\right) dF(x_1,x_2).
\end{aligned}
$$

For the dedicated firm, we have $\partial E\left(\pi_d^m\right)/\partial K_1^m = c_i$, which translates into

$$
\begin{aligned}
c_i \;=\;& (1/2)\iint\limits_{\Omega_{\hat{2}}} \left(x_1 - 2\left(K_1^m + \beta K_2^m\right)\right) dF(x_1,x_2) + (1/2)\iint\limits_{\Omega_4} \left(x_1 - x_2\beta - 2\left(1-\beta^2\right)K_1^m\right) dF(x_1,x_2) \\
&+\frac{1}{(7+\beta)^2}\iint\limits_{\Omega_5} \left(\begin{array}{c} (33-\beta(2-\beta))\,x_1 + (1-\beta(34-\beta))\,x_2 \\ -(17-\beta)\left(1-\beta^2\right)\left(4K_1^m + K_f^m\right) \end{array}\right) dF(x_1,x_2) \\
&+\iint\limits_{\Omega_7} \left(x_1 - \beta x_2 - \left(1-\beta^2\right)\left(2K_1^m + K_f^m\right)\right) dF(x_1,x_2) \\
&+\iint\limits_{\Omega_9} \left(x_1 - \left(2K_1^m + 2\beta K_2^m + K_f^m\right)\right) dF(x_1,x_2) \\
&+\iint\limits_{\Omega_{10}} \left(x_1 - \left(2K_1^m + 2\beta K_2^m + \beta K_f^m\right)\right) dF(x_1,x_2) \\
&+(1/4)\iint\limits_{\Omega_{11}} \left(3x_1 + x_2 - 2\left((3+\beta)K_1^m + (1+3\beta)K_2^m + (1+\beta)K_f^m\right)\right) dF(x_1,x_2),
\end{aligned}
$$

and $\partial E\left(\pi_d^m\right)/\partial K_2^m = c_i$ yields:

$$
\begin{aligned}
c_i \;=\;& (1/2)\iint\limits_{\Omega_{\hat{2}}} \left(x_2 - 2\left(\beta K_1^m + K_2^m\right)\right) dF(x_1,x_2) + (1/2)\iint\limits_{\Omega_3} \left(x_2 - x_1\beta - 2\left(1-\beta^2\right)K_2^m\right) dF(x_1,x_2) \\
&+\frac{1}{(7+\beta)^2}\iint\limits_{\Omega_6} \left(\begin{array}{c} (33-\beta(2-\beta))\,x_2 + (1-\beta(34-\beta))\,x_1 \\ -(17-\beta)\left(1-\beta^2\right)\left(4K_2^m + K_f^m\right) \end{array}\right) dF(x_1,x_2) \\
&+(1/4)\iint\limits_{\Omega_{11}} \left(3x_2 + x_1 - 2\left((3+\beta)K_2^m + (1+3\beta)K_1^m + (1+\beta)K_f^m\right)\right) dF(x_1,x_2) \\
&+\iint\limits_{\Omega_8} \left(x_2 - \beta x_1 - \left(1-\beta^2\right)\left(2K_2^m + K_f^m\right)\right) dF(x_1,x_2) \\
&+\iint\limits_{\Omega_9} \left(x_2 - 2K_2^m - 2\beta K_1^m - \beta K_f^m\right) dF(x_1,x_2) \\
&+\iint\limits_{\Omega_{10}} \left(x_2 - 2K_2^m - 2\beta K_1^m - K_f^m\right) dF(x_1,x_2).
\end{aligned}
$$

Proposition TA 3 *Equilibrium in the capacity game for the mixed subgame exists for all* $\beta \in (-1,1)$.

Existence follows from the concavity of the objective functions, which can be easily verified. Uniqueness is analytically difficult to show in this case as there is no symmetry argument that we can invoke. In fact, we conjecture that for a holdback strategy, the equilibrium in the mixed subgame may not be unique. This follows from the fact that there is more than one way to represent the capacity of the two firms as detailed by Figure 6.

4 Conditions sufficient for Assumptions A1 and A2 to hold

We develop appropriate analytical restrictions on the support of the distribution of the demand intercepts so that the Assumptions A-1 and A-2 hold. The lemma below details one such possibility for a special case of symmetric costs in which $\mu_1 = \mu_2 = \mu$, $\sigma_1 = \sigma_2 = \sigma$, $\rho = 0$ and $\beta = 0$.

Lemma 1 *If there exist* $A_{\max}$ *and* $A_{\min}$ *such that* $\Pr\{A \in (A_{\min}, A_{\max})\} = 1$ *and* $A_{\max} - A_{\min} \leq \min\left[c, (4/3)\left(\mu + c - 2c_f\right)\right]$*, then A-1 and A-2 hold with probability one. In other words, A-1 and A-2 hold if* c_f *does not exceed value* $c_{f(crit)} = (\mu + c)/2 - (3/8)\min\left((A_{\max} - A_{\min}), c\right).$

Proof. We prove the lemma in two stages. We first develop bounds on the realizations A_1 and A_2 such that A-1 holds with probability one for the pure flexible and the mixed subgames. We then develop conditions under which A-2 holds. Combining the two completes the proof.

For the pure flexible subgame, A-1 holds if $\Pr\left\{|A_1 - A_2| \leq 3K_f^{fh}\right\} = 1$ (we add a superscript h to indicate that this is the optimal capacity under holdback to distinguish it from the capacity derived under A-1 and A-2). It can be shown that $K_f^{fh} > K_f^f$. (Since Π_i^{fh} is concave, we can show that $\left(\partial\Pi_i^{fh}/\partial K_f^f\right)|_{K_f^f} \geq 0$; see Chod and Rudi 2005 for details.) Hence, $\Pr\left\{|A_1 - A_2| \leq 3K_f^f\right\} = 1 \Rightarrow \Pr\left\{|A_1 - A_2| \leq 3K_f^{fh}\right\} = 1$. For A-1 to hold in the pure flexible subgame, we must have

$\Pr\left\{|A_1 - A_2| \leq 2\left(\mu - c_f\right)\right\} = 1$ by Proposition 2. Similarly, for the mixed subgame it is sufficient to show that $\Pr\left\{A_1 - A_2 < K_1^{mh} - K_2^{mh} + 2K_f^{mh}\right\} = 1$ and $\Pr\left\{A_2 - A_1 < K_2^{mh} - K_1^{mh} + 2K_f^{mh}\right\} = 1$. Since for a symmetric distribution $K_1^{mh} = K_2^{mh}$, we have $\Pr\left\{|A_1 - A_2| < 2K_f^{mh}\right\} = 1$. Again we can show that $K_f^m \leq K_f^{mh}$. Hence, the necessary condition for A-1 to hold in the mixed subgame is $\Pr\left\{|A_1 - A_2| < (4/3)\left(\mu + c - 2c_f\right)\right\} = 1$ by Proposition 3. Given that $c < c_f < \mu$, it can easily be shown that $(4/3)\left(\mu + c - 2c_f\right) < 2\left(\mu - c_f\right)$. Hence, we have $\Pr\left\{|A_1 - A_2| < (4/3)\left(\mu + c - 2c_f\right)\right\} = 1$ for A-1 to hold. This condition holds with probability one if $|A_1 - A_2|_{\max} < (4/3)\left(\mu + c - 2c_f\right)$ or in other words if $A_{\max} - A_{\min} < (4/3)\left(\mu + c - 2c_f\right).$

If A-2 holds, then no realization of A_1 and A_2 falls in area Ω_1 in Figures 2 and 5, and clearance is optimal since it coincides with holdback. For this to happen we need $\Pr\left\{A_1 + A_2 > 3K_f^{mh}\right\} = \Pr\left\{A_1 + A_2 > 3K_f^{fh}\right\} = \Pr\left\{A_y > 3K_y^{dh}\right\} = 1$ for $y = 1, 2$. Take the case of a pure flexible subgame.

Let K_f^{fu} be the optimal capacity in the deterministic case when $A_1 = A_2 = A_{\max}$. Then, $K_f^{fu} = (2/3)(A_{\max} - c_f) \geq K_f^{fh}$ and $\Pr\{A_1 + A_2 > 2(A_{\max} - c_f)\} = 1 \Rightarrow \Pr\left\{A_1 + A_2 > 3K_f^{fh}\right\} = 1$. For this to hold for all realizations of A, we must have $A_{\min}+A_{\min} > 2(A_{\max} - c_f)$ or $A_{\max}-A_{\min} < c_f$. Similarly, for the mixed subgame, $A_{\max} - A_{\min} < 2c_f - c$, and for the dedicated subgame $A_{\max} - A_{\min} < c$. Hence, for clearance to be optimal we must have $A_{\max} - A_{\min} < c$. Taking the intersection of the conditions for A-1 and A-2, we finally obtain $A_{\max} - A_{\min} \leq \min[c, (4/3)(\mu + c - 2c_f)]$. ■

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