Technical Appendix to:

Performance Contracting in After-Sales Service Supply Chains

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1 Introduction

In this technical appendix we show that all results and the insights derived in the paper remain valid when an alternative backorder measure is used, namely, the time average of stationary backorders. Assume that the repair processes have run for a sufficient amount of time such that the steady state is reached at time 0. Let τ be the horizon at which the total backorders are counted (averaged) and let $B(t)$ be the stationary backorder random variable measure at time t. For notational convenience we drop the subscript i denoting the suppliers. Define

$$\widetilde{B}(\tau) \equiv \frac{1}{\tau}\int_0^\tau B(t)dt.$$

Because of stationarity $E[\widetilde{B}(\tau)\,|\,s] = E[B(t)\,|\,s]$. The variance is

$$\mathrm{Var}[\widetilde{B}(\tau)\,|\,s] = \frac{2}{\tau}\int_0^\tau \omega(u\,|\,s)\left(1-\frac{u}{\tau}\right)du$$

where

$$\omega(u\,|\,s) \equiv \mathrm{Cov}[B(t), B(t+u)\,|\,s]$$

is the autocovariance of $B(t)$ (see Papoulis and Pillai [6], p. 525) and where u is the time lag. Note that $\omega(u\,|\,s)$ is a function of u only because of the stationarity assumption. The ergodicity of $B(t)$ means that $\mathrm{Var}[\widetilde{B}(\tau)\,|\,s]$ goes to zero as τ approaches infinity. In our problem context, however, τ may

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be comparable to $1/\lambda$ and in such instances the variance is not negligible.

We digress at this point and motivate why the distinction between the steady state variable B and the time average $\widetilde{B}(\tau)$ is important. At the heart of the discussion are the following two related questions:

(a) Which measure should be used in order to structure contracts?

(b) Which measure is used in practice in order to compute contractual payments?

In an ideal situation, the answers to these two questions should be the same. However, our inquiries into this matter revealed that a discrepancy exists in practice. The answer to (a) is quite unambiguous – various guidelines on the weapon system support and maintenance make it clear that it is the availability observed at a random point in time that practitioners are concerned with (e.g., make sure that a fleet of systems is available with 95% chance if a war breaks out tomorrow). The DoD Guide for Achieving Reliability, Availability, and Maintainability [2] defines availability as "a measure of a degree to which an item is in operable state and can be committed at the start of a mission when the mission is called for *at an unknown (random) point in time*". In addition, the U.S. Air Force currently uses a confidence interval of availability, which is defined in terms of steady state random variable $A = 1 - B/N$, in assessing mission readiness and optimizing the stocking levels accordingly (Slay et al. [7]). Under this definition B is the natural candidate for the performance metric that should be used to define contract parameters.

However, the implied method of measurement under this definition involves counting the number of backorders once, i.e., at a random point in time. Understandably, the supplier is unlikely to agree to this method of performance evaluation because of the perceived arbitrariness. Instead, the widely accepted method of computing subsystem availability is counting the number of hours the subsystems are ready for a mission (i.e., not grounded due to a backordered condition) and comparing that number to the planned usage hours, forming a ratio between the two. This procedure is equivalent to counting backorders over time and converting it to availability, i.e., $\widetilde{A} = 1 - \left(\int_0^\tau B(t)dt\right) / (\tau N) = 1 - \widetilde{B}(\tau)/N$. Thus, we are faced with a situation where there exists a gap between what needs to be measured for managerial purpose (B) and what is actually measured ($\widetilde{B}(\tau)$) to compute contractual payments. Indeed, our conversations with practitioners indicate that this discrepancy is a source of controversy that is generating ongoing debates in the aerospace and defense industries. Although this is a very interesting observation that merits further in-depth study, we chose to employ B in the paper since (1) it is consistent with the performance measure that practitioners intend to evaluate, and (2) it allows us to develop a model that highlights the tradeoff between risk and incentives, which is the focus of our paper. For completeness, however, we show the implications of using $\widetilde{B}(\tau)$ as the performance measure in the sequel.

To see that employing $\widetilde{B}(\tau)$ does not change any qualitative results, we note that $\mathrm{Var}[B \,|\, s]$ appears in places where we discuss the impact of risk aversion with unobservable s. These sections are on general problem formulation (Sections 4.2 and 4.3, including Proposition 2 and Corollary 1) and on the analysis of the single supplier case (Section 5.3, including Lemma 1 and Proposition 5). All results in these sections depend on two facts: (1) $\mathrm{Var}[B \,|\, s]$ decreases with s, and (2) the supplier's utility function is unimodal (as shown in the proof of Proposition 2). Specifically, we rely on the following results (equations (15) and (16) appearing in the proof of Proposition 2):

$$\begin{aligned} d\mathrm{Var}[B \,|\, s]/ds &= -2F(s)E[B \,|\, s] \leq 0, \text{ and} \qquad (1) \\ d^2\mathrm{Var}[B \,|\, s]/ds^2 &= -2f(s)E[B \,|\, s] + 2F(s)[1-F(s)]. \qquad (2) \end{aligned}$$

When computing the optimal s we use (1) that appears in the first-order condition of the supplier's problem, and derive the expression for $v(\alpha, s)$ (equation (13) in the paper). To show that the supplier's utility U is unimodal in s under the condition specified in Proposition 2, we use (2) and show that the second derivative of the supplier's utility is always negative at a critical point, ensuring that this point represents a global maximum.

The exact expressions in (1) and (2) are not essential. Rather, what matters are the signs of $d\mathrm{Var}[B \,|\, s]/ds$ and of $\partial^2 U/\partial s^2\big|_{s=s^*}$. The reason is that all of our results use comparative statics after identifying the optimal supplier choice s^*. Therefore, once we show that both signs continue to be negative using the alternative measure $\widetilde{B}(\tau)$, we can simply substitute the terms on the right-hand sides of (1) and (2) that appear anywhere in the paper by $d\mathrm{Var}[\widetilde{B}(\tau) \,|\, s]/ds$ and $d^2\mathrm{Var}[\widetilde{B}(\tau) \,|\, s]/ds^2$ without changing any qualitative insights.

To prove that $d\mathrm{Var}[B \,|\, s]/ds \leq 0$ for all s, one only needs to show that $\partial\omega(u \,|\, s)/\partial s \leq 0$. This result relies upon the multivariate stochastic dependence concept of *association* because $\omega(u \,|\, s)$ is a function of two identical but dependent random variables. We were able to show $\partial\omega(u \,|\, s)/\partial s \leq 0$, and the proof is given in the next section. Showing that U is unimodal turns out to be much more difficult. However, we find that U has to have an interior solution whenever the mild condition in Proposition 2 is met, and we also obtain the lower and upper bounds on s^*. Since the supplier will choose an interior maximum, we are assured that $\partial^2 U/\partial s^2\big|_{s=s^*} \leq 0$. In fact, our numerical analysis indicates that U is unimodal, although an analytical proof of this result is hard to obtain. We discuss this difficulty in the last section.

2 Backorder variance decreases in s

Because of stationarity, $\widetilde{B}(t \mid s)$ and $\widetilde{B}(t+u \mid s)$ are identically distributed, and the autocovariance is a function of the lag u only. That is,

$$\omega(u \mid s) = \operatorname{Cov}[B(t), B(t+u) \mid s] = \operatorname{Cov}[B(0), B(u) \mid s]$$

(see Heyman and Sobel [4], p. 365). Thus one can restrict attention to $B(0)$ and $B(u)$ (and hence, also $O(0)$ and $O(u)$), which are, in general, not independent. Whenever there is no ambiguity we omit the argument and write B and O. We need the following multivariate dependence concept. Two random variables X and Y are called (positively) *associated* if

$$\operatorname{Cov}[\zeta(X,Y), \xi(X,Y)] = E[\zeta(X,Y)\xi(X,Y)] - E[\zeta(X,Y)]E[\xi(X,Y)] \geq 0$$

for all nondecreasing functions (in each argument) $\zeta, \xi : \mathbb{R}^2 \to \mathbb{R}$ (see, for example, Müller and Stoyan [5]). Eick, Massey, and Whitt [3] deomstrate that $O(0)$ and $O(u)$ are associated for each u. Let $\mathbf{1}_S(x)$ be the indicator function, which is one if $x \in S$ and zero otherwise. Since nondecreasing functions of associated random variables are associated (Barlow and Proschan [1], p. 30), and the functions $(x-s)^+$ and $\mathbf{1}_{(s,\infty)}(x)$ are nondecreasing in x, $B(0) = (O(0)-s)^+$, $B(u) = (O(u)-s)^+$, $\mathbf{1}_{(s,\infty)}(O(0))$, $\mathbf{1}_{(s,\infty)}(O(u))$ are associated among themselves. In particular,

$$\begin{aligned} E[B(0)B(u) \mid s] &\geq E[B \mid s]^2, && (3)\\ E[\mathbf{1}_{(s,\infty)}(O(0)) \cdot B(u) \mid s] &\geq E[\mathbf{1}_{(s,\infty)}(O(0))] \cdot E[B(u) \mid s] = [1-F(s)]E[B \mid s], \text{ and} && (4)\\ E[\mathbf{1}_{(s,\infty)}(O(u)) \cdot B(0) \mid s] &\geq [1-F(s)]E[B \mid s], && (5) \end{aligned}$$

where we have used the fact that $O(0)$ and $O(u)$ are identically distributed.

As Eick, Massey, and Whitt [3] point out, $O(0)$ and $O(u)$ can be decomposed into three independent random variables. Let $A_1(u)$ be the number of arrivals in the repair facility up to time 0 that depart by time u; let $A_2(u)$ be the number of arrivals between times 0 and u that remain in the repair facility at time u; and let $A_3(u)$ be the number of arrivals up to time 0 that remain in the facility at time u. Clearly, these three variables are independent, and $O(0) = A_1(u) + A_3(u)$, $O(u) = A_2(u) + A_3(u)$. Thus we can write

$$\omega(u \mid s) = E[(A_1(u) + A_3(u) - s)^+ (A_2(u) + A_3(u) - s)^+] - E[(O-s)^+]^2.$$

Since the second term is fully characterized by the cdf F, we are mainly concerned with the first term. Let $\psi_i(\cdot\,|\,u)$ be the pdf of $A_i(u)$. For notational convenience we use the shorthand notation $\psi_i(\cdot)$ and A_i when u is fixed. The first term is

$$\begin{aligned}
& E[(A_1 + A_3 - s)^+(A_2 + A_3 - s)^+] \\
= & \int_0^\infty \int_{(s-z)^+}^\infty \int_{(s-z)^+}^\infty (x+z-s)(y+z-s)\psi_1(x)\psi_2(y)\psi_3(z)dxdydz \\
= & \int_0^s \int_{s-z}^\infty \int_{s-z}^\infty (x+z-s)(y+z-s)\psi_1(x)\psi_2(y)\psi_3(z)dxdydz \\
& + \int_s^\infty \int_0^\infty \int_0^\infty (x+z-s)(y+z-s)\psi_1(x)\psi_2(y)\psi_3(z)dxdydz.
\end{aligned}$$

Differentiating with respect to s using Leibnitz's rule, we obtain

$$\begin{aligned}
& \frac{d}{ds}E[(A_1 + A_3 - s)^+(A_2 + A_3 - s)^+] \\
= & \psi_3(s)\int_0^\infty \int_0^\infty xy\psi_1(x)\psi_2(y)dxdy - \int_0^s \int_{s-z}^\infty \int_{s-z}^\infty (x+y+2z-2s)\psi_1(x)\psi_2(y)\psi_3(z)dxdydz \\
& -\psi_3(s)\int_0^\infty \int_0^\infty xy\psi_1(x)\psi_2(y)dxdy - \int_s^\infty \int_0^\infty \int_0^\infty (x+y+2z-2s)\psi_1(x)\psi_2(y)\psi_3(z)dxdydz \\
= & -\int_0^\infty \int_{(s-z)^+}^\infty \int_{(s-z)^+}^\infty (x+y+2z-2s)\psi_1(x)\psi_2(y)\psi_3(z)dxdydz \\
= & -E[\mathbf{1}_{\{(s,\infty),(s,\infty)\}}(O(0),O(u))\cdot(O(0)+O(u)-2s)].
\end{aligned}$$

After combining, we obtain

$$\frac{\partial\omega(u\,|\,s)}{\partial s} = -E[\mathbf{1}_{\{(s,\infty),(s,\infty)\}}(O(0),O(u))\cdot(O(0)+O(u)-2s)] + 2[1-F(s)]E[(O-s)^+]. \qquad (6)$$

The association property is needed to determine the sign of this quantity. Observe, for the first term, that

$$\begin{aligned}
& E[\mathbf{1}_{\{(s,\infty),(s,\infty)\}}(O(0),O(u))\cdot(O(0)+O(u)-2s)] \\
= & \int_0^\infty \int_{(s-z)^+}^\infty \int_{(s-z)^+}^\infty (y+z-s)\psi_1(x)\psi_2(y)\psi_3(z)dxdydz \\
& + \int_0^\infty \int_{(s-z)^+}^\infty \int_{(s-z)^+}^\infty (x+z-s)\psi_1(x)\psi_2(y)\psi_3(z)dxdydz \\
= & \int_0^\infty E[\mathbf{1}_{((s-z)^+,\infty)}(A_1)\cdot(A_2+z-s)^+]\psi_3(z)dz + \int_0^\infty E[\mathbf{1}_{((s-z)^+,\infty)}(A_2)\cdot(A_1+z-s)^+]\psi_3(z)dz \\
= & E\left[E[\mathbf{1}_{(s,\infty)}(A_1+A_3)\cdot(A_2+A_3-s)^+\,|\,A_3]\right] + E\left[E[\mathbf{1}_{(s,\infty)}(A_2+A_3)\cdot(A_1+A_3-s)^+\,|\,A_3]\right] \\
= & E[\mathbf{1}_{(s,\infty)}(O(0))\cdot(O(u)-s)^+] + E[\mathbf{1}_{(s,\infty)}(O(u))\cdot(O(0)-s)^+] \\
\geq & 2[1-F(s)]E[(O-s)^+] \qquad (7)
\end{aligned}$$

where the last inequality comes from (4) and (5). Therefore we have shown that

$$\frac{\partial \omega(u \mid s)}{\partial s} \leq 0$$

as desired.

3 Unimodality of the supplier's utility function

Using a technique similar to the above, one can show that

$$\begin{aligned}
&\frac{d^2}{ds^2} E[(A_1 + A_3 - s)^+ (A_2 + A_3 - s)^+] \\
= \; & 2\int_0^\infty \int_{(s-z)^+}^\infty \int_{(s-z)^+}^\infty \psi_1(x)\psi_2(x)\psi_3(z) dx dy dz \\
& + \int_0^s \left[\psi_1(s-z)\left(\int_{s-z}^\infty (y+z-s)\psi_2(y)dy\right) + \psi_2(s-z)\left(\int_{s-z}^\infty (x+z-s)\psi_1(x)dx\right)\right]\psi_3(z)dz \\
= \; & 2\Pr(O(0) > s, O(u) > s) + 2\int_0^s \psi_1(s-z)E[(A_1 + z - s)^+]\psi_3(z)dz
\end{aligned}$$

which yields

$$\begin{aligned}
\frac{\partial^2 \omega(u \mid s)}{\partial s^2} = \; & 2\Pr(O(0) > s, O(u) > s) + 2\int_0^s \psi_1(s - z \mid u)E[(A_1(u) + z - s)^+]\psi_3(z \mid u)dz \\
& -2f(s)E[(O-s)^+] - 2[1 - F(s)]^2. \qquad (8)
\end{aligned}$$

The sign of this expression is ambiguous. Substituting (8) in $d^2\mathrm{Var}[\widetilde{B}(\tau) \mid s]/ds^2$ and hence in $\partial^2 U/\partial s^2$ does not lead to an analytically tractable expression that would allow us to show that $\partial^2 U/\partial s^2\big|_{s=s^*} \leq 0$.

Despite the difficulty, we can infer the shape of the utility function U in two nonoverlapping domains that include $s = 0$ and $s = \infty$. Recall that

$$U(s) = w - (1-\alpha)(cs - a) - vE[B \mid s] - ka^2/2 - r(1-\alpha)^2\mathrm{Var}[\varepsilon]/2 - rv^2\mathrm{Var}[\widetilde{B}(\tau) \mid s]/2.$$

Define

$$\begin{aligned}
U^1(s) &\equiv w - (1-\alpha)(cs - a) - vE[B \mid s] - ka^2/2 - r(1-\alpha)^2\mathrm{Var}[\varepsilon]/2, \\
U^2(s) &\equiv w - (1-\alpha)(cs - a) - vE[B \mid s] - ka^2/2 - r(1-\alpha)^2\mathrm{Var}[\varepsilon]/2 - rv^2\mathrm{Var}[B \mid s]/2.
\end{aligned}$$

They differ from $U(s)$ only by the last term, the performance premium. Specifically, $U^1(s)$ does not have this term, whereas the corresponding term in $U^2(s)$ is the maximum value, since $\max_\tau\{\mathrm{Var}[\widetilde{B}(\tau) \mid s]\} =$

$\lim_{\tau\to\infty}\mathrm{Var}[\widetilde{B}(\tau)\,|\,s] = \mathrm{Var}[B\,|\,s]$. With these definitions, we can bound $U(s)$ by $U^1(s)$ and $U^2(s)$ and also the first derivative of $U(s)$ by the first derivatives of $U^1(s)$ and $U^2(s)$.

Lemma 1 *For* $s \in [0,\infty)$,

$$U^1(s) \ge U(s) \ge U^2(s)\ , \tag{9}$$

$$dU^1(s)/ds \le dU(s)/ds \le dU^2(s)/ds\ . \tag{10}$$

Proof. (9) is trivial. To see (10), note that $d\mathrm{Var}[B(s)]/ds = -2F(s)E[B(s)]$. Then

$$\begin{aligned}
dU(s)/ds &= -(1-\alpha)c + v[1-F(s)] - \frac{rv^2}{\tau}\int_0^\tau \frac{\partial\omega(u\,|\,s)}{\partial s}\left(1-\frac{u}{\tau}\right)du \\
&= -(1-\alpha)c + v[1-F(s)] - rv^2[1-F(s)]E[B\,|\,s] \\
&\quad + \frac{rv^2}{\tau}\int_0^\tau E[\mathbf{1}_{\{(s,\infty),(s,\infty)\}}(O(0),O(u))\cdot(O(0)+O(u)-2s)]\left(1-\frac{u}{\tau}\right)du, \qquad (11)\\
dU^1(s)ds &= -(1-\alpha)c + v[1-F(s)], \\
dU^2(s)/ds &= -(1-\alpha)c + v[1-F(s)] + rv^2F(s)E[B\,|\,s]
\end{aligned}$$

where we have used $\partial\mathrm{Var}[B(s)]/\partial s = -2F(s)E[B(s)]$ to obtain the last equation. Observe that $dU^1(s)/ds \le dU(s)/ds$, since $\partial\omega(u\,|\,s)/\partial s \le 0$. To compare $dU(s)/ds$ and $dU^2(s)/ds$, note that

$$\begin{aligned}
& E[\mathbf{1}_{\{(s,\infty),(s,\infty)\}}(O(0),O(u))\cdot(O(0)+O(u)-2s)] \\
= \;& E[\mathbf{1}_{\{(s,\infty),(s,\infty)\}}(O(0),O(u))\cdot((O(0)-s)^+ + (O(u)-s)^+)] \\
\le \;& E[(O(0)-s)^+ + (O(u)-s)^+] = 2E[B\,|\,s].
\end{aligned}$$

Using (11), we see that this implies that

$$\begin{aligned}
dU(s)/ds &\le -(1-\alpha)c + v[1-F(s)] - rv^2[1-F(s)]E[B\,|\,s] + rv^2E[B\,|\,s] \\
&= -(1-\alpha)c + v[1-F(s)] + rv^2F(s)E[B\,|\,s] = dU^2(s)/ds
\end{aligned}$$

since $\frac{1}{\tau}\int_0^\tau\left(1-\frac{u}{\tau}\right)du = 1/2$. ■

As in Proposition 2 of the paper, we assume that condition $v \ge v[1-F(0)] \ge (1-\alpha)c$, which

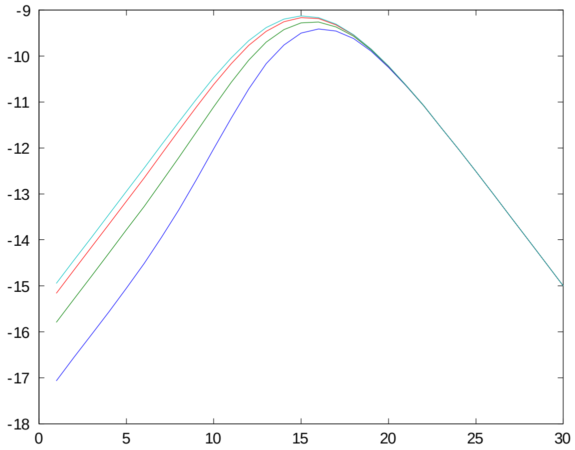

Figure 1: $U(s)$ for $\tau = 5, 10, 20, 30$ from bottom to top. $\lambda = 0.1$, $L = 150$, $c = 1$, $r = 1$, $\alpha = 0.5$, and $v = 1$ when exponential service time are chosen (note the scale in the y axis is negative because we have chosen $w = 0$ and zero fixed cost).

ensures that U^1 is increasing initially, is met. Notice that

$$\begin{aligned}
\lim_{s\to 0} dU^1(s)/ds &= -(1-\alpha)c + v[1 - F(0)] \geq 0, \\
\lim_{s\to 0} dU(s)/ds &= \lim_{s\to 0} dU^2(s)/ds = -(1-\alpha)c + v[1 - F(0)] + rv^2 F(0)\mu \geq 0, \\
\lim_{s\to\infty} dU(s)/ds &= \lim_{s\to\infty} dU^1(s)/ds = \lim_{s\to\infty} dU^2(s)/ds = -(1-\alpha)c \leq 0.
\end{aligned}$$

Hence, $U(s)$, $U^1(s)$, and $U^2(s)$ all start with positive slopes and approach the same negative slope as s goes to infinity. Thus, under the condition above, $U(s)$ is guaranteed to have an interior maximum.

Let s^*, s^1, and s^2 be the global maxima of $U(s)$, $U^1(s)$, and $U^2(s)$, respectively. Differentiation proves that $U^1(s)$ is concave. We show that $U^2(s)$ is quasiconcave in the proof of Proposition 2. Hence, they both have unique maxima. It is now also clear from (10) that

$$s^1 \leq s^* \leq s^2$$

since $U(s)$ is increasing on $[0, s^1]$ and decreasing on $[s^2, \infty)$.

Although we cannot show that $U(s)$ is quasiconcave analytically, numerical experiments indicate that this is indeed the case. Figure 1 shows a typical plot for $U(s)$, which is representative of many of the numerical experiments we attempted with a variety of problem parameters. In this particular example, $\lambda = 0.1$, $L = 150$, $c = 1$, $r = 1$, $\alpha = 0.5$, and $v = 1$ when exponential service times are chosen. We observe that $U(s)$ approaches $U^1(s)$ asymptotically as τ increases and unimodality is preserved.

4 Changes in contract terms

Although qualitative insights remain unchanged whether the steady state random variable B or the time average $\widetilde{B}(\tau)$ is used, the corresponding optimal contract parameters will assume different values. The following proposition summarizes how the the contract terms are modified in the single supplier case.

Proposition 1 *Suppose there is a single supplier such that the conditions specified in Proposition 5 are satisfied. Let* (α^{SB}, v^{SB}) *be the optimal contract parameters when the steady state random variable* B *is used as a backorder measure, and* $(\widetilde{\alpha}^{SB}(\tau), \widetilde{v}^{SB}(\tau))$ *be their counterparts when the time average* $\widetilde{B}(\tau)$ *is used. Then*

(i) $\widetilde{\alpha}^{SB}(\tau) \leq \alpha^{SB}$*, and*

(ii) $\widetilde{v}^{SB}(\tau) \geq v^{SB}$*.*

Proof. Under the single supplier assumption, the optimal stocking level s^{SB} is fixed, as it is determined from the binding backorder constraint $E[B \,|\, s] = E[\widetilde{B}(\tau) \,|\, s] = \widehat{B}_0$. Let $J(\tau) \equiv \mathrm{Var}\big[\frac{1}{\tau}\int_0^\tau B(t)dt \,|\, s^{SB}\big]$. With this definition, $J(\tau) = \mathrm{Var}[\widetilde{B}(\tau) \,|\, s^{SB}]$ and $\lim_{\tau \to 0} J(\tau) = \mathrm{Var}[B \,|\, s^{SB}]$. Since $\mathrm{Var}[\widetilde{B}(\tau) \,|\, s^{SB}] \leq \mathrm{Var}[B \,|\, s^{SB}]$, we have $J'(\tau) \leq 0$. From the Proof of Proposition 5 (found in the Appendix of the paper), the optimality condition for α is

$$\frac{\alpha}{k} + [(r_0 + r)\alpha - r]\,\mathrm{Var}[\varepsilon] - \frac{2(r_0 + r)c^2}{[1 - F(s)]^2}\frac{1}{r\gamma}\left(1 - \frac{1}{\sqrt{1 + r\gamma(1-\alpha)}}\right) J(\tau) = 0,$$

where $\gamma \equiv 4cF(s)E[B \,|\, s]/[1 - F(s)]^2$, which is independent of τ. Via implicit differentiation, we get

$$\frac{d\alpha}{d\tau} = \frac{\frac{2(r_0+r)c^2}{[1-F(s)]^2}\frac{1}{r\gamma}\left(1 - \frac{1}{\sqrt{1+r\gamma(1-\alpha)}}\right) J'(\tau)}{\frac{1}{k} + (r_0 + r)\mathrm{Var}[\varepsilon] + \frac{(r_0+r)c^2}{[1-F(s)]^2}\frac{1}{r\gamma}\frac{J(\tau)}{[1+r\gamma(1-\alpha)]^{3/2}}}.$$

Since the nuemrator is nonpositive and the denominator is positive, $d\alpha/d\tau \leq 0$, which in turn implies $dv/d\tau \geq 0$ since $sgn(\alpha) = -sgn(v)$ from Equation (13) of the paper. The results in (i) and (ii) follow since $(\widetilde{\alpha}^{SB}(\tau), \widetilde{v}^{SB}(\tau))$ are associated with larger τ. ■

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