ON-LINE SUPPLEMENT FOR: POSITIVE VS. NEGATIVE EXTERNALITIES IN INVENTORY MANAGEMENT: IMPLICATIONS FOR SUPPLY CHAIN DESIGN

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1 Model with externality through sales

In this Supplement, we look closely at demand model (*i*) that is a counterpart of the two-stage demand model widely used in stock-out based demand substitution problems (see Parlar [5], Wang and Parlar [6], Anupindi and Bassok [1], Netessine and Rudi [4] and other papers cited therein). The main feature of this demand model is that it reflects supply-side externality through sales and hence captures the nature of several examples described in the introduction.

Specifically, in this model the effective demand D_i^e is a sum of two components. The *direct* demand for retailer i is a nonnegative random variable denoted by D_i. The vector of direct demand $D = (D_1, D_2, \cdots, D_N)$ has a known, continuous multivariate distribution. To model externality, let $-1 \leq \alpha_{ij} \leq 1$ be the deterministic proportion of customers who buy from retailer i and will (or will not if $\alpha_{ij} < 0$) also attempt to buy from retailer j (*indirect* demand). Hence, α_{ij} is the measure of the degree of externality between retailers i and j. Assume $\alpha_{ii} = 0$ for all i. The effective demand for retailer i is the sum of direct demand and indirect demand. According to the above demand model,

$$D_i^e = D_i + \sum\nolimits_{j \neq i} \alpha_{ji} \min(Q_j, D_j) \qquad i = 1, 2, \cdots, N. \tag{1}$$

Note that as long as $\alpha_{ij} \geq 0$ in demand model (1), for each realization of the demand vector D, D_i^e is concave in Q. In addition, D_i^e is stochastically increasing in Q_{-i}. Therefore, the two-stage demand model is a special case of the general demand model with complements discussed previously so that all properties of the general model apply. We are particularly interested in comparing supply chain efficiency for complements and substitutes. For that purpose, we will, for a part of our analysis, assume that

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$\alpha_{ij} = b, \forall i, j$ so that by changing b from -1 to 1 we can analyze the impact of the relative level of substitutability/complementarity on the system inventory and supply chain efficiency.

We will first assume that the wholesaler does not have price-setting power but later relax this assumption. We will analyze competition and centralization at the retail level separately, and then use these results to do the comparison. We begin by considering the decentralized model where retailers compete and the wholesaler sells the product at fixed prices w_i. The next proposition establishes that the model is well-behaved and that the total inventory stocked by competing retailers is increasing in b, i.e., a higher level of complementarity leads to higher inventory.

Proposition 8 *In the decentralized model with demand model (1),*
1) There is a unique, globally stable Nash equilibrium Q^d under decentralized inventory management if either $\sum_{i \neq j} \alpha_{ji} < 1$, all j, or $\sum_{j \neq i} \alpha_{ji} < 1$, all i.
Furthermore, if we let $\alpha_{ij} = b$, $|b| < 1/(N-1)$, $\forall i, j$, the following results hold:
2) The game is supermodular for $b > 0$ and submodular for $b < 0$.
3) The total inventory stocked by competing retailers in equilibrium (and hence the wholesaler's profit) is monotonically increasing in b if either $b \geq 0$ or $b < 0$ and $n = 2$.

Proof. 1) The existence of equilibrium follows from the concavity of player i's objective function in Q_i. To show uniqueness and global stability of the Nash equilibrium it suffices to show that the matrix of slopes of the best-response functions is diagonally dominant (see Netessine and Rudi [4] for details), which implies that the best-response mapping is a contraction and hence there is a unique fixed point of such a mapping. To show this result, we can bound the slopes of the best responses as follows:

$$\left| \frac{\partial Q_i^d}{\partial Q_j} \right| = \left| -\frac{\frac{\partial^2 \pi_i^d}{\partial Q_i \partial Q_j}}{\frac{\partial^2 \pi_i^d}{\partial Q_i^2}} \right| = \left| -\frac{r_i \alpha_{ji} f_{D_i^e | D_j > Q_j}(Q_i) \Pr(D_j < Q_j)}{-r_i f_{D_i^e}(Q_i)} \right| \leq |\alpha_{ji}|.$$

Hence, to ensure diagonal dominance, we need to have either $\sum_{i \neq j} \alpha_{ji} < 1$, all j, or $\sum_{j \neq i} \alpha_{ji} < 1$, all i.

2) Supermodularity/submodularity is demonstrated by showing that the second-order cross-partial derivative is positive/negative

$$\frac{\partial^2 \pi_i}{\partial Q_i \partial Q_j} = r_i b f_{D_i^e | D_j > Q_j}(Q_i) \Pr(D_j < Q_j) \begin{cases} \geq 0 \text{ if } b \geq 0 \\ \leq 0 \text{ if } b < 0 \end{cases}.$$

3) It is straightforward to see that

$$\frac{\partial^2 \pi_i}{\partial Q_i \partial b} = r_i \frac{\partial}{\partial b} \Pr(D_i^e > Q_i) > 0, \tag{2}$$

and hence for $b > 0$, the game is supermodular in decision variables as well as in parameter b, which implies that equilibrium decisions are monotonically increasing in b (see Theorem 4.2.2 in Topkis). To demonstrate the sensitivity of equilibrium inventory when $n = 2$ and $b < 0$ we apply the Implicit Function Theorem to the system of first-order conditions to obtain the matrix equality (see proof of Proposition 4). Denote by

$|H|$ the determinant of the Hessian matrix on the left-hand side. This system of linear equations can be solved in closed form as follows:

$$\frac{\partial Q_i^*}{\partial b} = -\frac{\frac{\partial^2 \pi_i}{\partial Q_i \partial b}\frac{\partial^2 \pi_j}{\partial Q_j^2} - \frac{\partial^2 \pi_i}{\partial Q_i \partial Q_j}\frac{\partial^2 \pi_j}{\partial Q_j \partial b}}{|H|},\ i,j = 1,2. \tag{3}$$

Taken together, these solutions yield

$$\frac{\partial \left(Q_1^* + Q_2^*\right)}{\partial b} = \frac{\frac{\partial^2 \pi_1}{\partial Q_1 \partial b}\left(\frac{\partial^2 \pi_2}{\partial Q_2 \partial Q_1} - \frac{\partial^2 \pi_2}{\partial Q_2^2}\right) + \frac{\partial^2 \pi_2}{\partial Q_2 \partial b}\left(\frac{\partial^2 \pi_1}{\partial Q_1 \partial Q_2} - \frac{\partial^2 \pi_1}{\partial Q_1^2}\right)}{|H|}.$$

Note that each bracket in the numerator is positive since we have seen above that $\left|\frac{\partial^2 \pi_i}{\partial Q_j \partial Q_i}\right| \leq \left|\frac{\partial^2 \pi_i}{\partial Q_i^2}\right|$ and $\frac{\partial^2 \pi_i}{\partial Q_i^2} > 0$. Further, $|H| \geq 0$ using the same results. Finally, using (2) we obtain $\frac{\partial\left(Q_1^* + Q_2^*\right)}{\partial b} > 0$. ■

The condition sufficient for the uniqueness of the equilibrium in the decentralized case is essentially identical to the one derived by Netessine and Rudi [4] for substitutes. We can now summarize both results: the equilibrium outcome of the game is unique when externalities among competitors are not too strong. In particular, one additional sale at retailer i should not increase/decrease the demand at all other retailers taken together by more than one unit.

The last result establishes that total inventory in the channel is increasing in the level of complementarity captured by b. One can notice that each retailer's individual inventory is increasing in b when $b > 0$ but for $b < 0$ we only show that total inventory is increasing in b. A similar result has been obtained by Anupindi and Bassok [1] for substitutes in a different model. Although we only show that this is the case in a model with two players, this result can also be demonstrated in a symmetric model with any number of players. We now analyze the same problem but under centralized inventory control at the retail level.

Proposition 9 *In the centralized problem with demand model (1)*

1) The optimal solution Q^c is characterized by the following optimality conditions:

$$\Pr\left(D_i^e < Q_i^c\right) - \sum_{j \neq i} \alpha_{ij} \frac{r_j}{r_i} \Pr\left(D_i > Q_i^c, D_j^e < Q_j^c\right) = \frac{r_i - w_i}{r_i},\ i = 1, ..., N. \tag{4}$$

For $\alpha_{ij} \geq 0$ this solution is unique.

Furthermore, if we let $\alpha_{ij} = b$, $|b| < 1/\left(N-1\right)$, $\forall i,j$ the following results hold:

2) The objective function is supermodular when $b > 0$ and submodular when $b < 0$.

3) In a symmetric problem, all Q_i^c are monotonically increasing in b and hence the wholesaler's profit is increasing in b as well.

Proof. 1) Uniqueness of the solution for $b > 0$ follows from Proposition 2 and the fact that effective demand in model (1) is concave in stocking quantities. Derivation of the optimality condition follows along the lines of Netessine and Rudi [4]. Since $\min(Q_i, D_i^e) = Q_i - (Q_i - D_i^e)^+$, we rewrite the profit function as

$$\pi(Q) = E\left[\sum\nolimits_j \left(r_j - w_j\right) Q_j\right] - E\left[\sum\nolimits_j r_j (Q_j - D_j^e)^+\right],$$

and focus on differentiating the second term. We have

$$\frac{\partial E\left[\sum_j r_j (Q_j - D_j^e)^+\right]}{\partial Q_i} = r_i \frac{\partial E(Q_i - D_i^e)^+}{\partial Q_i} + \frac{\partial E\left[\sum_{j\neq i} r_j (Q_j - D_j^e)^+\right]}{\partial Q_i}.$$

It is easy to show for the first term $\partial E(Q_i - D_i^e)^+ / \partial Q_i = \Pr(Q_i > D_i^e)$. The difficulty lies in the second term, which breaks down to the derivative of $E(Q_j - D_j^e)^+$ $(j \neq i)$ w.r.t. Q_i. Since $(Q_j - D_j^e)^+$ is integrable and has a bounded derivative, it satisfies the Lipschits condition of order one and hence the expectation and derivative operations can be interchanged:

$$\frac{\partial E(Q_j - D_j^e)^+}{\partial Q_i} = E\left(\frac{\partial (Q_j - D_j^e)^+}{\partial D_j^e}\frac{\partial D_j^e}{\partial Q_i}\right).$$

Let $1_{\{A\}}$ be the indicator function of event A, i.e., $1_{\{A\}} = 1$ if A is true and $1_{\{A\}} = 0$ if A is false. From the above we get

$$\frac{\partial E(Q_j - D_j^e)^+}{\partial Q_i} = E[(-1_{\{D_j^e < Q_j\}})(\alpha_{ij} 1_{\{D_i > Q_i\}})] = -\alpha_{ij}\Pr(D_i > Q_i,\, D_j^e < Q_j).$$

So together we have

$$\frac{\partial \pi(Q)}{\partial Q_i} = (r_i - w_i) - r_i \Pr(Q_i > D_i^e) + \textstyle\sum_{j\neq i} \alpha_{ij} r_j \Pr(D_i > Q_i, D_j^e < Q_j).$$

Equating the above to zero gives the FOC.

2) The cross-partial derivative is

$$\frac{\partial^2 \pi_i}{\partial Q_i \partial Q_j} = rbf_{D_i^e|D_j>Q_j}(Q_i)\Pr(D_j > Q_j) + rbf_{D_j^e|D_i>Q_i}(Q_j)\Pr(D_i > Q_i) \begin{cases} \geq 0 \; if \; b \geq 0 \\ < 0 \; if \; b < 0 \end{cases}$$

and the result follows.

3) In a symmetric problem, there is a single optimality condition

$$\frac{\partial \pi}{\partial Q} = r\Pr(D_i^e > Q) + br\sum_{j\neq i}\Pr\left(D_j^e < Q, D_i > Q\right) - w = 0.$$

To show that $\partial Q/\partial b > 0$ it is sufficient to verify that $\partial^2\pi/\partial Q\partial b > 0$:

$$\begin{aligned} \frac{\partial^2 \pi_i}{\partial Q \partial b} &= r\frac{\partial}{\partial b}\Pr(D_i^e > Q) + r\sum_{j\neq i}\Pr\left(D_i > Q, D_j^e < Q\right) + rb\sum_{j\neq i}\frac{\partial}{\partial b}\Pr\left(D_j^e < Q, D_i > Q\right) \\ &\geq r\frac{\partial}{\partial b}\Pr(D_i^e > Q) + rb\sum_{j\neq i}\frac{\partial}{\partial b}\Pr\left(D_j^e < Q, D_i > Q\right) \\ &= r\frac{\partial}{\partial b}\left(\Pr(D_i^e > Q, D_j > Q) + \Pr(D_i^e > Q, D_j < Q)\right) + rb\sum_{j\neq i}\frac{\partial}{\partial b}\Pr\left(D_j^e < Q, D_i > Q\right) \end{aligned}$$

$$\begin{aligned} &\geq r\frac{\partial}{\partial b}\Pr\left(D_i^e > Q, D_j > Q\right) + rb\sum_{j\neq i}\frac{\partial}{\partial b}\Pr\left(D_j^e < Q, D_i > Q\right) \\ &= r\frac{\partial}{\partial b}\Pr\left(D_i^e > Q, D_j > Q\right) + rb\left(n-1\right)\frac{\partial}{\partial b}\Pr\left(D_i^e < Q, D_j > Q\right), \end{aligned}$$

where the last equality is due to the assumed symmetry. Evidently, if $b < 0$ the result follows. If $b > 0$, recall that $b\left(n-1\right) < 1$ and further simplify

$$\begin{aligned} \frac{\partial^2 \pi_i}{\partial Q \partial b} &\geq r\frac{\partial}{\partial b}\Pr\left(D_i^e > Q, D_j > Q\right) + r\frac{\partial}{\partial b}\Pr\left(D_i^e < Q, D_j > Q\right) \\ &= r\frac{\partial}{\partial b}\left(\Pr\left(D_i^e > Q, D_j > Q\right) + \Pr\left(D_i^e < Q, D_j > Q\right)\right) \\ &= r\frac{\partial}{\partial b}\Pr\left(D_j > Q\right) = 0. \end{aligned}$$

The proof is complete. ■

The solution in the case of centralized inventory management mirrors the solution in the case of substitutes (see Netessine and Rudi [4]) except for the sign in front of the second term on the left of (4). Analysis of this expression provides some additional intuition behind the result that retailers understock under complementarity, the effect captured by the second term on the left. Notice an apparent difficulty in guaranteeing uniqueness of the solution when products are substitutable. This is reminiscent of the non-uniqueness demonstrated by Netessine and Rudi [4] in another substitution setting. Both of these problems arise due to non-concavity of the effective demand in stocking quantities when products are substitutes. However, this difficulty does not preclude us from deriving further results since we operate with the first-order necessary conditions that have to hold at any optimum. As in the competitive setting, we see that total channel inventory is increasing in the level of complementarity.

It is quite clear that the centralized supply chain model can be analyzed in exactly the same way as the model with centralized retailers: the only difference is that $w_i = c$. One observation we can make immediately is that $Q_i^o \geq Q_i^c$: due to double-marginalization, retailers understock in the centralized model as compared to the system-optimal quantity. The next step is to compare stocking policies under centralization Q^c and competition Q^d at the retail level, and to further compare Q_i^o and Q_i^d. From Proposition 3 we know that both retailers will understock in the centralized model under complementarity (i.e., when $b > 0$) as compared to the decentralized model. We will also verify that in this model retailers will overstock under substitutability (i.e., when $b < 0$). While this result has been demonstrated previously in other models, it has not been established in general (and hence it is not clear if it holds for our model) and moreover, examples exist in which overstocking does not occur (see Netessine and Rudi [4]). In the next proposition we confirm that in the decentralized model retailers indeed overstock under substitution and moreover, we show that this overstocking can compensate for the double-marginalization effect and can result in a system-optimal performance.

Proposition 10 *Let $\alpha_{ij} = b$, $\forall i, j$. Then with demand model (1) the following results hold:*
1) $Q_i^c \geq Q_i^d$ $\forall i$ *if* $b > 0$, $Q_i^c = Q_i^d$ $\forall i$ *if* $b = 0$, *and for* $n = 2$ $Q_1^c + Q_2^c \leq Q_1^d + Q_2^d$ *if* $b < 0$.
2) $\Pi^c \geq \Pi^d$ *if* $b > 0$, $\Pi^c = \Pi^d$ *if* $b = 0$, *and for* $n = 2$, *there is* $\Pi^c \leq \Pi^d$ *if* $b < 0$.

3) In the decentralized model for $b \leq 0$ there always exist wholesale prices $c \leq w_i \leq r_i$ that achieve channel coordination under a simple wholesale price contract. No wholesale price contract exists that coordinates the supply chain for $b > 0$.

Proof. 1) The result for $b > 0$ follows from Proposition 3. Next consider $n = 2$ and $b < 0$. From the optimality condition (4), it is straightforward to see that $\Pr(D_i^e > Q_i^c) > \Pr(D_i^e > Q_i^d)$ or

$$\Pr\left(D_i + b\min\left(Q_j^c, D_j\right) > Q_i^c\right) > \Pr\left(D_i + b\min\left(Q_j^d, D_j\right) > Q_i^d\right), \ i, j = 1, 2.$$

Consider all possible scenarios:

1. $Q_i^c > Q_i^d$ and $Q_j^c > Q_j^d$. Since $b < 0$, the above inequalities cannot hold. Therefore, $Q_i^c > Q_i^d$ and $Q_j^c > Q_j^d$ cannot be true.
2. $Q_i^c > Q_i^d$ and $Q_j^c \leq Q_j^d$. Note $-1 \leq b < 0$ and $\min(Q_j^d, D_j) - \min(Q_j^c, D_j) \leq Q_j^d - Q_j^c$ for all sample paths of D_j. Thus there must be $Q_j^d - Q_j^c \geq Q_i^c - Q_i^d$, which yields $Q_i^c + Q_j^c \leq Q_i^d + Q_j^d$.
3. $Q_j^c > Q_j^d$ and $Q_i^c \leq Q_i^d$. Similarly we can show $Q_i^c + Q_j^c \leq Q_i^d + Q_j^d$.
4. $Q_i^c \leq Q_i^d$ and $Q_j^c \leq Q_j^d$. The desired result follows.

2) Follows immediately from the previous result.

3) It is straightforward to verify that the system-optimal solution satisfies the following conditions:

$$\frac{\partial \pi}{\partial Q_i} = r_i \Pr(D_i^e(Q_{-i}) \geq Q_i) + b \textstyle\sum_{j \neq i} r_j \Pr(D_j^e(Q_{-j}) < Q_j, D_i > Q_i) - c = 0, \ \forall i. \tag{5}$$

Recall that the solution in the case with competing retailers is

$$r_i \Pr(D_i^e(Q_{-i}^d) \geq Q_i^d) - w_i = 0, \text{ all } i.$$

We need to show that we can choose $c \leq w_i \leq r_i$ such that $w_i = r_i \Pr\left(D_i^e\left(Q_{-i}^o\right) \geq Q_i^o\right)$. This is possible as long as $b < 0$ (since otherwise $r_i \Pr\left(D_i^e\left(Q_{-i}^o\right) \geq Q_i^o\right) < c$ so that $w_i < c$). Hence, if $b \leq 0$ the supply chain optimal solution can always be achieved by some $c \leq w_i \leq r_i$. ■

Several observations can be made from the last proposition. First, centralization at the retail level may or may not be beneficial for supply chain performance. Specifically, competition at the retail level is beneficial for the supply chain in the case of substitutes, but is detrimental in the case of complements. In fact, retail competition on substitutes can coordinate the supply chain since it compensates the effect of double-marginalization. In contrast, competition on complements cannot coordinate the supply chain since it exacerbates the effect of double-marginalization. In a different model, Mahajan and van Ryzin [3] have shown that, when the number of retailers is very large, retail competition on substitutes can result in system optimal performance. We, however, have shown that the requirement for a large number of retailers may not be mandatory: in our model coordination is possible even with two retailers, which is similar to an observation made by Anupindi and Bassok [1] with a different demand model. As a result, the wholesaler may or may not prefer the centralization of retailers. Namely, under complementarity the wholesaler is in favor of retail centralization, while under substitutability the wholesaler prefers competition at the retail level. This last statement seems to contradict a result of Anupindi and Bassok [1], who find that the wholesaler may prefer centralization of retailers under substitution. This apparent difference in findings is caused by the difference in what is implied by centralization in our model and in Anupindi and Bassok [1]. Namely, we refer to the centralization of *decision rights* only, while Anupindi and Bassok [1] centralize physical retail stocks as well, which leads to perfect substitution under centralization and a different implication.

To illustrate the results of the previous propositions, we conduct numerical experiments in a setting with two identical retailers and symmetric demand distribution. The following parameters are used: $r = 150$, $w = 110$, $c = 80$. Demand is assumed to have Normal distribution (truncated at 0) with parameters $\mu = 200$, $\sigma = 100$, $\rho = 0$. Monte-Carlo integration is used to evaluate gradients and calculate a solution. Figure 3 illustrates supply chain inventory as a function of b. As we have shown analytically, inventory in all three models (supply chain optimal, with centralized, and with competing retailers) increases in the level of complementarity. Since the wholesaler's profit is equal to inventory multiplied by the wholesaler's margin, the wholesaler is better off with decentralized retailers under substitutability and with centralized retailers under complementarity. At $b = 0$, the inventory levels are the same for the centralized and decentralized models since there are no externalities. Interestingly, the same is true when $b = 1$. Mathematically, this result can be seen by letting $Q_1 = Q_2$ (due to symmetry) so it becomes easy to verify that the centralized and decentralized models result in the same optimality conditions. Centralized inventory management exhibits the double-marginalization gap indicated in Figure 3, resulting in understocking of inventory (note that the double-marginalization gap is defined as the difference between the supply chain optimal and the centralized models). However, when retailers compete, the magnitude of the double-marginalization effect is alleviated when there are negative externalities (substitution) and is aggravated when there are positive externalities (complementarity). As Figure 3 indicates, at $b = -0.8$ the decentralized model results in the same performance as the supply chain optimal model (consistent with our last proposition) and for $b < -0.8$ decentralized retailers stock even more than is system-optimal. We can also see from Figure 4 that supply chain efficiency is dependent on the nature of externalities. It is clear that b has a much more significant impact on supply chain efficiency when retailers are decentralized than when they are centrally controlled.

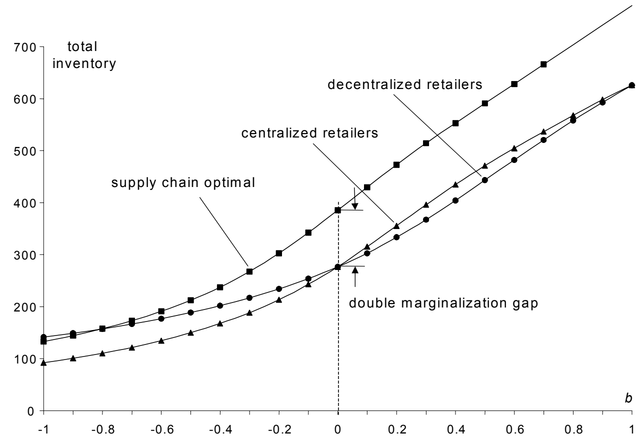

Figure 3. Inventory in the channel.

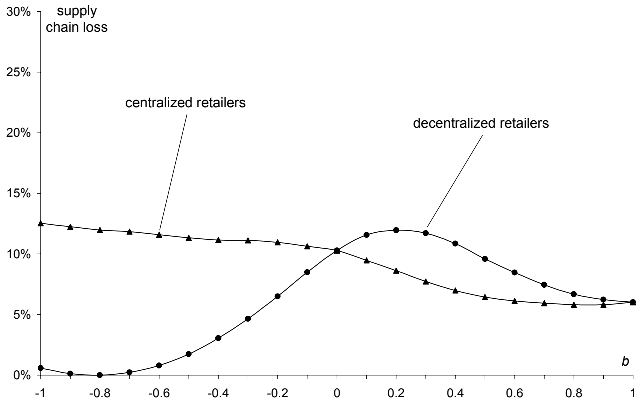

Figure 4. Supply chain efficiency loss.

In what follows, we numerically analyze the impact of the level of complementarity on supply chain inventory/efficiency under the assumption that the wholesaler acts as a Stackelberg leader and sets wholesale price optimally (later we will conduct the same analysis analytically for a different model). This is similar to the setting used by Lariviere and Porteus [2].

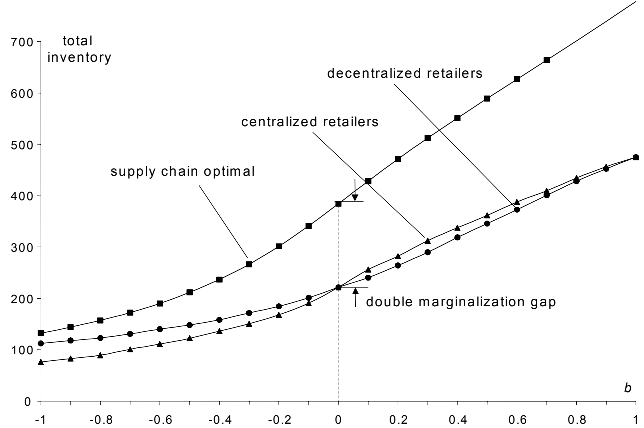

Figure 5. Inventory in the channel.

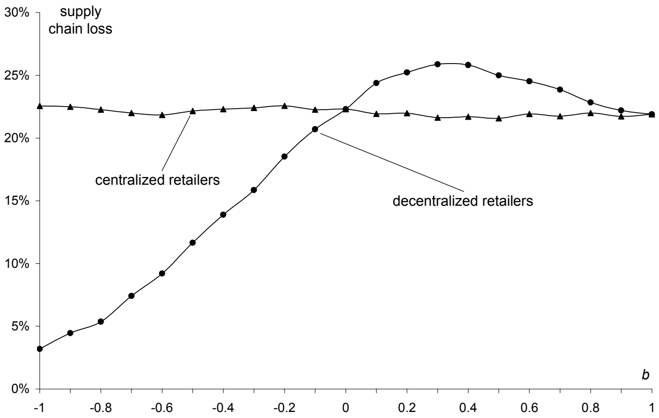

Figure 6. Supply chain efficiency loss.

Figures 5 and 6 depict supply chain inventory and efficiency loss, respectively. Although overall these figures appear to be quite similar to Figures 3 and 4, in which the wholesale prices are exogenous, a couple of differences should be noted. First, endowing the wholesaler with price-setting power results in significant inefficiencies. When retailers are centralized, the inefficiency is nearly the same for all b, but for competing retailers inefficiency varies significantly. We did not find situations in which competition on substitutes resulted in supply chain optimal performance in this setting. Hence, even for a very low b under competition there are significant inefficiencies. Overall, however, all the analytical results obtained earlier for the exogenous wholesale price case appear to hold: inefficiencies are highest under competition on complements and lowest under competition on substitutes, while centralized retailers exhibit intermediate levels of inefficiencies.

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