Chapter 4 : Oligopoly

Oligopoly is the term typically used to describe the situation where a few firms dominate a particular market. The defining characteristic of this type of market structure is that the competing firms are interdependent. This occurs when the behaviour of one firm affects the profits earned by other firms in the industry. It is this characteristic of interdependence that makes oligopoly suitable for game theory analysis. Oligopoly can be contrasted with the two extremes of market structure. Under perfect competition all firms are price takers, and so they are assumed to be independent of each other. In the case of monopoly, as there is only one firm in the market, there is again no interdependence.

Initially in examining the behaviour of oligopolistic firms we consider one-off games where firms are assumed to interact with each other only once. In this context we present the now classic models of Cournot, Stackelberg, and Bertrand competition. Having discussed the outcome of these models, we then analyse how repeated interaction between firms can change their predicted behaviour. In particular we discuss the extent to which oligopolies can maintain a non-cooperative collusive outcome, that maximizes joint profits. In discussing these issues many of the ideas presented in the previous two chapters are directly applied. One important contrast between the games previously discussed and those presented in this chapter is that now we represent the players making choices over a continuous, rather than discrete, strategic variable. For example, we examine how ideas presented in the previous two chapters can be applied when firms must choose one particular output / price combination from a continuum of such choices as represented by the demand curve.

4.1. Three Models of Oligopoly.

A number of different models have been developed that try to explain and predict the behaviour of oligopolistic firms. In this section we will examine three such models, each named after their originator. The distinguishing feature of these models of oligopoly is their underlying market structure. In Cournot competition firms simultaneously compete in terms of quantity supplied to the market. In Stackelberg competition one or more firms are able to initially precommit themselves to a particular output level. The remaining firms observe this level of output and then simultaneously determine their own optimal output levels. Finally, in Bertrand competition firms simultaneously compete in terms of the price they charge consumers. These differences fundamentally alter the underlying structure of the game between firms, and dramatically change the predicted behaviour of the firms involved. To illustrate these differences we initially focus our attention on the case of one-off competition between just two firms. More general models of oligopoly, where there are more than two interdependent firms, are examined in various exercises, while repeated interaction between firms is analysed in the next section.

a. Cournot Competition.

As stated above in Cournot competition firms simultaneously decide how much of the good they will supply to the market. Once aggregate supply is determined the price is set so that the market clears. To examine this type of competition we initially assume that two firms produce an identical product. As the output decisions are taken simultaneously each firm supplies the market without observing the other firm’s level of supply. The market price, \( P \), is determined so that aggregate supply, \( Q \), is just demanded. We assume that total demand for the product is determined according to the inverse demand curve \( P = a - Q \), where \( a \) is a positive constant. There are assumed to be constant marginal costs equal to \( c \) and no fixed costs. Each firm is assumed to maximize profits.
From this informal description we can identify the three basic requirements for a normal form game.

(i) **The players.**

These are the two firms. We will call them firm A and firm B.

(ii) **The strategies available to each player.**

As this is a static game the available strategies are the same as the possible actions of the two players. The available strategies are therefore the possible quantities of the good each firm can supply to the market. We will assume that firms A and B can supply any positive level of output, and denote these as \(q_A\) and \(q_B\) respectively.

(iii) **The payoffs.**

These are the profits each firm receives. These are denoted as \(\Pi_A\) and \(\Pi_B\) for firms A and B respectively. Furthermore we also know how much information each player has when deciding upon their optimal strategy, and the timing of these events. We can therefore represent this competitive structure as an extensive form game. This is shown in Figure 4.1.

![Figure 4.1. Cournot Duopoly: Extensive Form Game.](image)

This extensive form game illustrates the essential elements of Cournot competition. In particular it shows that both firms make their supply decisions before knowing how much the other firm has supplied to the market. This is represented by having all of firm B’s decision nodes in the same information set. This is also true for firm A as it is depicted as only having one decision node, the initial node. It should be noted, however, that Figure 4.1 does simplify the game described above by assuming that each firm has only three possible levels of output. In contrast we assumed above that the firms can supply any positive level of output. This means there are an infinite number of actions available to each firm. It is clearly impossible to fully represent all these possibilities diagrammatically, and so the extensive form game depicts only a representative sample of the available strategies. Once each firm has chosen its optimal level of supply, the market price is determined, and the firms receive...
the corresponding level of profit. As we are restricting ourselves to one-off competition between firms in this section, the game is assumed to end when firms have received these profit levels.

Having depicted Cournot competition as a game between interdependent firms we can now apply the solution techniques discussed in Chapter 2 to predict the outcome of this market structure. An obvious solution technique to use in this static game is the concept of Nash equilibrium. This involves determining each firm’s optimal strategy dependent on what it expects the other firm to do. Diagrammatically this involves drawing each firm’s so called reaction function. This shows a firm’s optimal supply of output for every possible quantity chosen by the other firm. (The term “reaction function” is a misnomer because strictly it is not possible for either firm to react to each others output decision in this static game.)

To find each firm’s reaction function we differentiate the firm’s profit function with respect to its own output level and set this equal to zero. Rearranging this yields the first order condition for finding a maximum. The second order condition, that the second derivative is negative, is then checked to ensure that a maximum has indeed been found. These calculations are performed below for firm A and firm B, under the assumptions of the model being used.

\[
P_A = Pq_A - cq_A
\]
\[
\therefore \frac{d\Pi_A}{dq_A} = a - 2q_A - q_B - c = 0
\]
\[
\therefore q_A = \frac{a - q_B - c}{2}
\]
\[
\frac{d^2\Pi_A}{dq_A^2} = -2 < 0 \therefore \text{max}
\]

\[
P_B = Pq_B - cq_B
\]
\[
\therefore \frac{d\Pi_B}{dq_B} = a - q_A - 2q_B - c = 0
\]
\[
\therefore q_B = \frac{a - q_A - c}{2}
\]
\[
\frac{d^2\Pi_B}{dq_B^2} = -2 < 0 \therefore \text{max}
\]

The penultimate line of these calculations is the reaction function for each firm. These functions show that the optimal level of supply for each firm is negatively related to the expected level of supply of the other firm. As an expected increase in one firm’s supply causes the other firm to cut back on its supply the output of the two firms are said to be strategic substitutes for each other. (Conversely if strategic variables are positively related there are said to be strategic complements.) Diagrammatically strategic substitutes are illustrated by downward sloping reaction curves. The reaction functions for firm A and firm B are plotted in Figure 4.2.
Figure 4.2. Cournot - Nash Equilibrium.

This diagram also shows the relationship between each firm’s reaction curve and its isoprofit curves. In this diagram the isoprofit curve plots the different combinations of the two firms’ output that yield the same level of profit for one of the firms. Profits are greatest when each firm is the sole supplier to the market. If one of the firms acts as a monopolist it will supply \( \frac{a-c}{2} \) units of the product to the market. This is equal to the intercept value of the reaction functions. The further away the firm is from this monopoly outcome the smaller its level of profit. This is indicated by moving on to isoprofit curves further away from the relevant intercept. As firm A is assumed to maximizes profit given the expected level of supply by firm B, so its reaction function intersects its isoprofit curves where they are horizontal. Similarly firm B’s reaction function intersects its isoprofit curves where they are vertical. This represents maximum profit for firm B for each anticipated level of supply by firm A. In Nash equilibrium both firms must be maximising profits simultaneously, given their beliefs about the other firm’s level of supply. This means that both firms must be on their reaction curves simultaneously. In Figure 4.2 the reaction curves intersect only once, and so this corresponds to the unique Nash equilibrium for this model. From the diagram we can see that the Nash equilibrium is where each firm supplies an output equal to \( \frac{a-c}{3} \). This can be confirmed algebraically by setting the two equations for the reaction functions equal to each other.

Although we have identified the output levels of \( \frac{a-c}{3} \) as the unique solution to this model using the concept of Nash equilibrium it should be noted that Cournot (1838) also claimed this output combination to be the equilibrium. Cournot, however, identified this as the equilibrium to the model by analysing how the firms would react when they were out of equilibrium. Cournot assumed that each firm believes that if it changed its own output level the other firm would not react by changing its output level. To show this yields the same equilibrium as before consider Figure 4.3.
Firm A’s Reaction Function.

Firm B’s Reaction Function.

Cournot Equilibrium.

\[ q_B \]

\[ q_A \]

\[ \frac{1}{2}(a-c) \]

\[ (a-c)^0 \]

1/2(a-c)

(a-c)

Figure 4.3. Cournot Disequilibrium Dynamics.

This diagram again plots the two firms’ reaction functions. Assume initially that only firm A is producing in this market. As firm A is initially a monopoly it will produce the amount \( \frac{(a - c)}{2} \) and be at point A. If now firm B enters the market, and assumes that firm A will maintain its initial level of supply, then it will produce at point B, which is on firm B’s reaction curve vertically above point A. However at point B firm A is off its reaction curve and so, if it assumes firm B will not change its level of supply, it will change its output to point C. It can be seen that this process will continue until we reach the Nash equilibrium. A similar argument could be advanced starting from any point in Figure 4.3, with the two firms converging on the Nash equilibrium. As both methods yield the same equilibrium it is often called the Cournot-Nash equilibrium.

Although both the Cournot solution and the Nash equilibrium predict the same final output levels the concept of Nash equilibrium is theoretically much the stronger. Specifically Cournot’s methodology has two main weaknesses. First, it postulates that each firm is able to react to the other firm’s output level. This is inconsistent with the initial structure of the game where it was assumed both firms set their output levels simultaneously. Second, although each firm assumes the other will not respond to output changes, this would in fact be falsified by actual behaviour if the firms did repeatedly interact, and were initially away from Nash equilibrium. This means that each firm’s assumption about the behaviour of the other firm is not consistent with the model itself, and so is not a rational conjecture. Although Cournot’s equilibrium is reasonable his analysis of what happens out of equilibrium must be regarded as unsatisfactory.

In contrast the concept of Nash equilibrium does not introduce a dynamic process into an otherwise static model, nor does it introduce arbitrary behavioural assumptions. Instead each firm sets the equilibrium quantity based on rational beliefs about the behaviour of the other firm. It is via this rational decision making process, where each firm explicitly takes into account its interdependence with other firms, that equilibrium is attained.

Before going on to discuss other models of oligopoly it is worth noting that the Cournot-Nash equilibrium is Pareto inefficient. If the two firms could coordinate their supply decisions then they can potentially both earn greater profits. From Figure 4.2 the Nash equilibrium is seen to be inefficient because at this point the two firm’s isoprofit curves are not tangential. This implies there are other supply combinations where at least one firm is better off and the other firm no worse off. These combinations are shown as the shaded lens-shaped region in Figure 4.2. The boundary of this shaded area is formed by the two isoprofit
curves passing through the Nash equilibrium. Within this shaded area, therefore, both firms move on to isoprofit curves closer to their respective monopoly outcomes, and so both firms are better off. On the boundary itself, apart from where the isoprofit lines intersect, one firm is strictly better off while the other firm receives the same level of profit as in Nash equilibrium.

In order for an outcome to be Pareto efficient the isoprofit curves must be tangential. If this is the case then one firm can only be made better off at the expense of the other firm’s profit level. Also drawn in Figure 4.2 is the contract curve. The contract curve shows the set of Pareto efficient outcomes, where the firms isoprofit lines are tangential.

Exercise 4.2 demonstrates that along the contract curve joint supply is equal to $\frac{(a - c)}{2}$, which is the quantity a monopolist would supply to the market. Aggregate profits along the contract curve are therefore, in this model, equal to the monopoly profit level. Distribution of profits along this curve is, therefore, called a constant-sum game. If the two firms could fully coordinate their supply decisions they will maximise joint profit by being on the contract curve. From this contract curve it can be seen, however, that there are an infinite number of Pareto efficient outcomes. This raises the question of which of these many output combinations firms might attempt to coordinate upon. In answer to this question, it seems reasonable to exclude points on the contract curve which are outside the shaded area. These outcomes can be rejected because one of the two firms strictly prefers the Nash equilibrium, and so will not maintain the collusive outcome. This leaves all the points on the contract curve within the shaded area. If the two firms are identical one focal point for collusion is where both firms produce half the monopoly level of supply, and so each receive half the monopoly profit level. This focal point is labeled as point C in Figure 4.2. Although this outcome might seem entirely reasonable, there is good reason to believe that in the game we are considering it is unsustainable. This is because point C is not a Nash equilibrium, and hence firms have the incentive to unilaterally deviate from it by changing their level of supply. The reason for this is that the cartel maximizes joint profits by acting like a monopolist, i.e. it restricts output and increases price. At this output level both firms, in general, will want to increase their own production, thereby increasing their own profits. Diagrammatically at point C both firms are off their reaction curves. In Cournot equilibrium the total output is higher and the price is lower but at this point neither firm has an incentive to change their level of output. This is an example of a prisoners’ dilemma game considered in chapter 2. Each player has an incentive to deviate from the collusive outcome, with the result that the final equilibrium is Pareto inefficient.

This is the classic argument for why we might expect a cartel to be unstable. Each firm has a unilateral incentive to deviate from the collusive arrangement. We return to this issue of collusion, and specifically how a noncooperative collusive outcome might be sustained, in section 4.2.

**Exercise 4.1.**

Assume that there are $i=1,...,n$ identical firms in an industry, each with constant marginal costs of $c$ and no fixed costs. If the market price, $P$, is determined by the equation $P = a - Q$, where “$Q$” is total industry output and “$a$” is a constant, determine the Cournot-Nash equilibrium output level for each firm. What happens as $n \to \infty$?

**Exercise 4.2.**

Find an equation for the contract curve for the model presented in Exercise 4.1 when $n=2$. Show that along this contract curve aggregate supply of the good is equal to the quantity that a profit maximising monopolist would supply.

**b. Stackelberg Competition.**

With Cournot competition each firm chooses their desired level of supply simultaneously. In Stackelberg competition it is assumed that at least one of the firms in the market is able to precommit itself to a particular level of supply before other firms have fixed
their level of supply. These other firms observe the leader’s supply and then respond with their output decision. The firms able to initially precommit their level of output are called the market leaders and the other firms are the followers. Again to aid understanding we examine the case of duopoly, where there is one leader and one follower. The extensive form game for this type of competition is shown in Figure 4.4.

![Extensive Form Game](image)

**Figure 4.4. Stackelberg Duopoly: Extensive Form Game.**

In Figure 4.4 firm A is the leader and firm B the follower. The only difference between this diagram and the extensive form game drawn for Cournot duopoly, is that firm B’s decision nodes are now in separate information sets, rather than being in the same information set. This corresponds to the assumption that the follower now observes the leader’s supply decision before choosing its own optimal response. This change in market structure significantly alters the predicted behaviour of the two firms.

In predicting the outcome of this extensive form game we need to realize that the game is no longer static but inherently dynamic. This raises the possibility of threats and promises being made and possibly acted upon. For example, firm B may threaten to produce such a large amount of the good that if believed the leader would set output equal to zero, thus leaving firm B the sole supplier. This represents one possible Nash equilibrium. There will, with a continuum of such threats and promises, be an infinite number of such Nash equilibria. The problem with this reasoning is that most of these equilibria involve the leader believing incredible threats or promises. They are incredible because they are not in the interest of the follower to actually carry them out if required to do so. In order to rule out such incredible threats and promises we require that the predicted outcome of the game be subgame perfect. To find the subgame perfect Nash equilibrium for this game we apply the principle of backward induction.

To see how the principle of backward induction may be applied in this situation we make the same assumptions previously used to analyse Cournot competition, except now we let firm A be the leader and firm B the follower. Using backward induction we start with the last period first, and initially determine the follower’s output decision. Given that the follower is rational it will attempt to maximize its payoff, here given in terms of its profit level, subject to the leader’s known level of supply. The follower’s profit function is given as
Differentiating this with respect to \( q_B \) and setting this equal to zero gives us the first order condition for a maximum.

\[
\frac{d\Pi_B}{dq_B} = a - q_A - 2q_B - c = 0
\]

\[
\therefore q_B = \frac{a - c - q_A}{2}
\]

This equation is the followers reaction function. (Note that the term reaction function is no longer a misnomer in this dynamic game.) This function shows the followers optimal response for any level of supply chosen by the leader. Therefore the only credible threat/promise that the follower can make is that it will be on its own reaction function.

Having made the prediction that the follower will locate on its reaction function in the last period of the game, we can now consider what the leader will do in the first period. From the arguments above the leader knows that the eventual outcome of the game must be on the follower’s reaction function. The leader will therefore maximize its own profits subject to this constraint. The first order condition for a maximum is derived as follows, where we substitute firm B’s reaction function in for its level of output.

\[
\Pi_A = Pq_A - cq_A
\]

\[
\therefore \Pi_A = (a - q_A - q_B)q_A - cq_A
\]

\[
\therefore \Pi_A = aq_A - q_A^2 - \frac{a - c - q_A}{2}q_A - cq_A
\]

\[
\therefore \Pi_A = \frac{a - c}{2}q_A - \frac{1}{2}q_A^2
\]

\[
\therefore \frac{d\Pi_A}{dq_A} = \frac{a - c}{2} - q_A = 0
\]

\[
\therefore q_A = \frac{a - c}{2}
\]

This is the subgame perfect Nash equilibrium level of supply for the leader. Substituting this in to the followers reaction function gives us that firm B’s optimal response is to produce

\[
q_B = \frac{a - c}{4}
\]

With one leader and one follower, the follower produces half the amount of the leader, and \( Q = \frac{3(a - c)}{4} \). These results are illustrated in Figure 4.5.
The leader knows that if the follower is rational it will produce on its reaction function. Given this constraint the leader will maximise profits where its isoprofit curve is tangential to the follower’s reaction curve. Firm A would ideally like to be the sole supplier in this market, so that it could earn monopoly profits. This corresponds to point A in Figure 4.5. However, the lowest isoprofit curve that it can reach, subject to the final output combination being on the follower’s reaction function, is the one through point S. This is the Stackelberg - (subgame perfect) Nash equilibrium. Interestingly with only one leader and one follower, the leader produces the monopoly output level. (This result does not generalize to the case of more than one follower, as Exercise 4.3 demonstrates.) The leader, however, does not earn monopoly profits. This is because the follower’s positive level of output drives down the market price that the leader receives. In the Stackelberg - Nash equilibrium the firms isoprofit curves continue to intersect and so, as with the Cournot - Nash equilibrium, there are still potential gains to be had from collusion.

Figure 4.5 also illustrates the Cournot - Nash equilibrium. Compared to this equilibrium the Stackelberg equilibrium entails higher profits for the leader and smaller profits for the follower. More information has actually made the follower worse off! In contrast the ability of the leader to precommit itself to a particular level of supply has made that firm better off. In this model there is said to be a first move advantage.

**Exercise 4.3.**

Assume there are \( m \) identical Stackelberg leaders in an industry, indexed \( j = 1, \ldots, m \), and \( n \) identical Stackelberg followers, indexed \( k = 1, \ldots, n \). All firms have a constant marginal cost of \( c \) and no fixed costs. The market price, \( Q \), is determined according to the equation \( P = a - Q \), where \( Q \) is total industry output, and “\( a \)” is a constant. Find the subgame perfect Nash equilibrium supply for the leaders and the followers. Confirm the duopoly results for both Cournot competition and Stackelberg competition, and the generalized Cournot result for \( n \) firms derived in Exercise 4.1.

c. Bertrand Competition.

In Cournot and Stackelberg competition the firms’ strategic variable is the quantity of the good they supply to the market. In Bertrand competition the strategic variable is the price firms charge their customers. In this model firms simultaneously announce the price that they are prepared to sell their product, and then consumers determine the amount they will buy.
The extensive form for this type of competition with just two firms is the same as for Cournot duopoly, except now firms decide on the price to sell their product. This is shown in Figure 4.6.

![Extensive Form Game Diagram](image)

**Figure 4.6. Bertrand Duopoly: Extensive Form Game.**

The nature of the equilibrium that results from Bertrand competition depends critically upon whether the firms sell identical or differentiated products. We will initially assume that the firms' products are identical and then consider the case of product differentiation.

**Undifferentiated Products.**

With undifferentiated products there is a unique Nash equilibrium, where the firms charge the same price, and just earn normal profits. To see that this is the case consider the following two stage argument.

If the firms sell identical goods then consumers will only buy from the firm offering the product at the lowest price. Therefore, if firms were to sell the product at different prices the firm with the lower price would capture the whole market. If in this situation this firm is making supernormal profits, then the other firm has an incentive to just slightly undercut its competitor's price. Doing this it will capture the whole market and begin earning positive profits. If, on the other hand, the initial firm is earning less than normal profits, then that firm has an incentive to raise its price. This will be to where it either earns at least normal profits, or has zero sales and leaves the industry. From these scenarios it is clear that firms charging different prices cannot be a Nash equilibrium.

A similar argument can be made concerning firms charging the same price but earning more or less than normal profits. In this situation both firms will have an incentive to either slightly increase or decrease the price they charge. The unique Nash equilibrium is, therefore, all firms charging the same price and earning normal profits. This is identical to the perfectly competitive outcome. This result, that as few as two firms will yield the competitive outcome, is referred to as the Bertrand paradox. It is a paradox because it seems implausible to believe that so few firms would not find some way of colluding so as to move away from the competitive outcome, in order to earn supernormal profits. One way of avoiding the Bertrand paradox is to allow firms to repeatedly interact. As argued in section 4.2, this allows the possibility of non-cooperative collusion, where firms can earn greater profits than those suggested by this one-off game. An alternative way of avoiding the paradox is to allow the firms to sell differentiated products. This situation is analysed below.

**Product Differentiation.**

With product differentiation the firms will no longer face all or nothing demand as experienced when they produce a homogenous good. Instead firms will now face a
downward sloping demand curve. The firms remain interdependent but this is not as extreme as when they produce identical products. Here we examine the case of Bertrand duopoly with differentiated products. Let two firms, A and B, set the prices $p_A$ and $p_B$ respectively. We assume that the quantity each firm sells is determined by the following equations

$$q_A = a - p_A + b p_B$$

$$q_B = a - p_B + b p_A$$

where $b > 0$ reflects that the two goods are substitutes for each other. As with previous models we assume that the firms have constant marginal costs equal to $c$ and no fixed costs. Assuming firms attempt to maximize profits, we can derive the Nash equilibrium as follows. First we derive each firm’s reaction function, which gives us their optimal price given the price the other firm sets.

**Firm A**

\begin{align*}
\Pi_A &= p_A q_A - c q_A \\
\therefore \Pi_A &= p_A (a - p_A + b p_B) - c (a - p_A + b p_B) \\
\therefore \frac{d\Pi_A}{dp_A} &= a + b p_B - 2p_A + c = 0 \\
\therefore p_A &= \frac{a + c + b p_B}{2}
\end{align*}

\begin{align*}
\frac{d^2\Pi_A}{dq_A^2} &= -2 < 0 \therefore \text{max.}
\end{align*}

**Firm B**

\begin{align*}
\Pi_B &= p_B q_B - c q_B \\
\therefore \Pi_B &= p_B (a - p_B + b p_A) - c (a - p_B + b p_A) \\
\therefore \frac{d\Pi_B}{dp_B} &= a + b p_A - 2p_B + c = 0 \\
\therefore p_B &= \frac{a + c + b p_A}{2}
\end{align*}

\begin{align*}
\frac{d^2\Pi_B}{dq_B^2} &= -2 < 0 \therefore \text{max.}
\end{align*}

These two reaction functions are plotted in Figure 4.7.
The reaction curves for Bertrand competition are seen to be upward sloping, and so the prices of the two firms are strategic complements. To find a Bertrand - Nash equilibrium both firms must be maximising their profits simultaneously given the expected behavior of the other firm. Again this means that both firms must be on their reaction curves. Setting the two reaction functions equal to each other we obtain the unique Nash equilibrium that each firm will set their price equal to \( \frac{a + c}{2 - b} \). As with the Nash equilibrium associated with Cournot and Stackelberg competition this outcome is Pareto inefficient. Both firms can be made better off if they set higher prices. These Pareto dominant outcomes are shown as the lens-shaped area in Figure 4.7. The problem with achieving one of these Pareto dominant outcomes is that, again, at least one firm has an incentive to deviate from it. This justifies the Nash equilibrium as being the expected outcome for this particular model.

4.2. Non - Cooperative Collusion.

In the previous section we illustrated the three classic models of oligopoly as one-off games. The predicted outcome of each model was either a Nash equilibrium or a subgame perfect Nash equilibrium. In all three models the equilibrium was shown to be Pareto inefficient. With effective collusion all firms could receive higher profits. The problem, however, is that without the use of legally enforceable contracts such collusion is not credible in these one off games. This is because at least a subset of the firms involved in the collusion has an incentive to unilaterally deviate from it. This section extends our analysis of oligopoly by discussing the possibility of non-cooperative collusion when firms repeatedly interact. This is closer to reality for many oligopolies, and provides us the opportunity to see how the issues of repeated games, as presented in Chapter 3, can be applied to oligopolistic competition.

First we examine the situation where firms experience infinite interaction with each other. We will then consider what happens when there are only a finite number of repetitions of the one-
off game. In this latter context we discuss the role of multiple equilibria, uncertainty about the future, and uncertainty about one’s competitors.

**a. Infinite Repetitions.**

With infinite interaction firms have the possibility of adopting punishment strategies that induce other firms to maintain the non-cooperative collusive outcome. This is because with repeated interaction there is the possibility of firms being punished if they break an explicit or tacit collusive agreement. For example, if one firm increases its output other firms may retaliate by increasing their output causing all firms to be worse off. If this effective punishment is sufficiently severe firms will voluntarily maintain the collusive outcome. This involves firms forgoing an increase in short run profits in order to avoid the costs associated with future punishment.

One particular form of punishment that firms might adopt is a *trigger strategy*. As discussed in Chapter 3 a trigger strategy is where a certain action by one player induces other players in the game to *permanently* change the way they act. With a trigger strategy, therefore, a player faces the prospect of an infinite punishment period if they deviate from the collusive outcome. It was J. Friedman (1971) who first showed that non-cooperative collusion could be maintained if oligopolies continually interact and adopt an appropriate trigger strategy.

To see how adoption of a trigger strategy can lead to non-cooperative collusion we apply a specific trigger strategy to our model of Cournot duopoly. In section 4.1 it was shown that any point within the shaded region of Figure 4.2 makes both firms better off compared to the Nash equilibrium. However, it was previously argued that given the symmetry of our model, and the assumption that firms will aim for a Pareto efficient outcome, a focal point for attempted collusion is where each firm produces half the monopoly output level. This corresponds to point C in Figure 4.2. If firms do seek to coordinate upon this output combination then they may adopt the following trigger strategy:

*Produce half the monopoly output level in the first period, and continue to do so if the other firm has always done so in the past. Otherwise produce the Cournot - Nash output level.*

With this trigger strategy firms, in effect, promise to collude as long as they perceive that the other firm has always done so. The threat, however, is that if a firm deviates from the collusive outcome it will be punished by the other firm producing the Cournot - Nash equilibrium output level forever. For the collusive outcome to be a subgame perfect Nash equilibrium both the promise and the threat implied by this strategy must be credible.

The threatened punishment is clearly credible because if one firm is seen to have deviated from the collusive outcome then it is always rational for the other firms to produce the output level associated with the Nash equilibrium. For the promise to be credible the present value of maintaining collusion must exceed the present value of deviating from it. This will be true provided that firms do not discount the future “too much”. This is shown as follows. If both firms continue to collude then they each earn half the monopoly level of profit, $\frac{\Pi_M}{2}$, in every time period. Discounting this by the relevant discount factor $\delta = \frac{1}{1 + r}$, 0 $\leq$ $\delta$ $\leq$ 1, we obtain the present value of always continuing to collude:

$$\frac{\Pi_M}{2} + \delta \frac{\Pi_M}{2} + \delta^2 \frac{\Pi_M}{2} + \delta^3 \frac{\Pi_M}{2} + \ldots = \frac{1}{1 - \delta} \frac{\Pi_M}{2}$$

If, alternatively, one firm deviates from the collusive outcome, then let the profit it earns in the first time period be equal to $\Pi_D$, (where the subscript D refers to deviation). In subsequent periods the most it can earn is equal to the Cournot - Nash equilibrium profit level $\Pi_C$. The present value of deviating is therefore:
Non-cooperative collusion will be maintained if

\[
\frac{1}{1 - \delta} \frac{\Pi_M}{2} \geq \Pi_D + \frac{\delta}{1 - \delta} \Pi_C
\]

\[\therefore \delta \geq \frac{\Pi_D - \frac{\Pi_M}{2}}{\Pi_D - \Pi_C}\]

As \(\Pi_D > \frac{\Pi_M}{2} > \Pi_C\) the right handside of the inequality will be between zero and one. This condition is therefore satisfied provided \(\delta\) is sufficiently close to one. This confirms that non-cooperative collusion will be maintained provided that the rate of discount is not too small. When this condition is satisfied the resulting non-cooperative collusion is self-enforcing. Each firm, given the other firm has adopted the above trigger strategy, willingly maintains the collusive outcome.

One natural question that arises from the above analysis is what happens if the rate of discount is smaller than that necessary to maintain the collusive outcome? Here we discuss two possibilities.

One possibility is that the firms continue to adopt the previous trigger strategy where punishment is equivalent to the Cournot-Nash equilibrium being played forever. Although this punishment strategy cannot sustain joint output equal to the monopoly outcome, because firms are assumed to discount future profits too much, other less profitable collusive outcomes may be supported. As long as the rate of discount is not zero there always exist other sustainable collusive outcomes that yield greater present value profits compared to the Cournot-Nash equilibrium. From these possible outcomes it seems reasonable to assume that firms will coordinate on the symmetric outcome which yields highest present value profits given the firm’s actual rate of time preference. If the rate of discount is zero, i.e. firms are only interested in present profits, then we return, in effect, to the one-off game, and the only subgame perfect equilibrium is that the Cournot-Nash solution is played every period.

An alternative possibility in trying to maintain an otherwise unsustainable collusive outcome, is to make the threat of punishment more severe. Instead of seeking to collude on an alternative outcome that is sustainable with the proposed trigger strategy, the firms might try to adopt an alternative trigger strategy. In order for this alternative trigger strategy to sustain the desired collusive outcome it must credibly threaten a more severe punishment for deviation. The problem with this proposal is that in this model any permanent punishment which is more severe, implies that the adopted trigger strategy is not credible. This can be illustrated by the following argument.

The most severe trigger strategy possible is to punish a deviant with its so called minimax punishment forever. The minimax punishment is the worst outcome players can inflict upon another, given that the other player will be seeking to maximise their payoff. In our Cournot duopoly model this corresponds to one firm seeking to minimise the other firm’s profits given that that firm will always seek to be on its reaction function. This occurs when the punishing firm produces the perfectly competitive output \(a - c\). If the other firm expects to observe this level of output then, from its reaction function it will voluntarily cease production and earn zero profit. This confirms that this is the minimax punishment. (Note that a firm cannot be forced to earn negative profits because it always has the option of leaving the market and just breaking even.) If a firm believes it will face this extreme punishment if it deviates, it clearly has a stronger incentive to maintain the collusive outcome. In this way the desired collusive outcome can be part of a Nash equilibrium. The problem with this punishment strategy, however, is that it is not credible, and so the collusive outcome supported by it cannot be subgame perfect. The reason the minimax threat in this model is not credible is because it is off the punishing firm’s reaction function. If it believed the other firm were going to produce zero output, then its optimal output is the monopoly output level.
\[ \frac{a - c}{2} \], and not the competitive output level. For a trigger strategy to be credible the punishment to be inflicted must be on the punishing firm’s reaction function. However, as the other firm will always seek to maximise its own profit level it must also be on that firm’s reaction function as well. The only credible punishment, given that the firm’s adopt a trigger strategy, must be a Nash equilibrium in the stage game. The only credible trigger strategy, within our duopoly model, is the one considered previously where punishment corresponds to the Cournot - Nash equilibrium.

The above argument seems to imply that a more severe punishment strategy will not be credible. This however is not the case. The above argument rules out more severe trigger strategies, but not more severe punishment strategies in general. Indeed Abreu (1986) has proposed a way in which firms can threaten more severe punishment for observed deviation, and yet for it to still remain credible. The way to make punishment more severe and remain credible is to move away from a trigger strategy and adopt a carrot and stick approach. Abreu’s suggestion is that firms need not threaten an infinite punishment period, as suggested by the use of trigger strategies, but instead threaten only a temporary period of punishment. The reason a more severe punishment can now be credible is that firms are punished themselves if they do not punish other firms.

The strategy suggested by Abreu is as follows. If all firms adopt the punishment strategy within the punishment period firms revert back to the collusive outcome. This is the carrot. If, however, firms deviate from the prescribed punishment then punishment is continued. This is the stick. In this way firms are punished if they do not punish other deviant firms. This strategy, therefore, gives firms a greater incentive to punish other firms. As a result the punishment itself need no longer be the Cournot - Nash outcome. More severe punishment can become credible because all firms have an incentive to carry it out so as to avoid being punished themselves. With more severe punishment being credible the desired collusive outcome can be sustained for even smaller values of the discount factor. This possibility is demonstrated for our Cournot duopoly model in Exercise 4.4.

The above result is an application of the Folk Theorem as presented by Fudenberg and Maskin (1986). In this context the theorem states that all possible output combinations that Pareto dominate the minimax outcome can form part of a subgame perfect Nash equilibrium provided firms do not discount the future “too much”, and that they adopt appropriate punishment strategies. The minimax outcome is where each player is trying to minimise other player’s payoffs. In Cournot duopoly this is when both firms receive zero profits. This theorem states, therefore, that all feasible profit allocations can be subgame perfect equilibria provided both firms earn at least zero profits and do not overly discount future profits. This is a strong result. Due to the resulting multiple equilibria implied by this theorem firms must somehow coordinate their output on one particular equilibrium. The selection process often used in the literature, and which has been applied above, is to suppose that the equilibrium will be symmetric and on the contract curve provided the discount factor is not too small.

To conclude our discussion on infinite competition between firms it should be noted that the above results strongly contrast with those derived when examining one-off competition between firms. There it was argued that non-cooperative collusion is not possible. By modelling infinite repetitions of these one off games we have derived the opposite result. Provided firms are sufficiently interested in future profits non-cooperative collusion is now possible. In these infinitely repeated games collusion is sustained by the credible threat of future punishment. Given the stark contrast of these results it is important that we go on to explore the predicted outcome of competition between firms when the one-off game is only finitely repeated.
Exercise 4.4. *

In the text we derived that \( \delta \geq \frac{\Pi_b - \Pi_m}{2 (\Pi_b - \Pi_c)} \) is a necessary and sufficient condition for each firm in a Cournot competing duopoly to produce half the monopoly output level if they face continual interaction and adopt the following trigger strategy:

*Produce half the monopoly output level in the first period, and continue to do so if the other firm has always done so in the past. Otherwise produce the Cournot - Nash output level.*

(i). Determine the necessary and sufficient conditions for this collusive outcome to be maintained if the firms adopt the following alternative punishment strategies.

| a. | Each firm produces the collusive output level initially, and if this outcome was maintained in the previous period. If, however, one firm deviates from the collusive output level then it will be punished for one period by the other firm producing the Cournot - Nash output level for only one period, and then reverts back to producing the collusive output level. |
| b. | Each firm produces the collusive output level initially, and if this outcome was maintained in the previous period. If, however, one firm deviates from the collusive output level then the other firm produces the Cournot - Nash output level for only one period, and then reverts back to producing the collusive output level. |

(ii). Using the necessary and sufficient conditions just derived to maintain the collusive outcome for each of the above three punishment strategies calculate the specific conditions for \( \delta \) in the following model. Price is determined by the equation \( P(Q) = 65 - Q \), where \( Q \) is the joint output of the two firms. Each firm has constant marginal costs equal to £5, and no fixed costs. For the carrot and stick punishment strategy given in part i. b. calculate the necessary conditions when \( q_p \) is 25, 30 and 35. Identify which of these punishment strategies supports the collusive outcome with the lowest value of \( \delta \).

b. Finite Repetitions.

In Chapter 3 it was argued that there is a fundamental difference between infinitely repeated and finitely repeated games. This is demonstrated by the paradox of backward induction. The relevance of this paradox to finite interaction between oligopolies is that with a unique Nash equilibrium in the one-off game non-cooperative collusive outcomes are unsustainable. This can be applied to all three models analyzed in section 4.1. Here we consider its application to Cournot competition.

Assume that a finite number of oligopolies Cournot compete with each other over a known finite number of time periods. In applying the paradox of backward induction we consider the last period first. In this last period there can be no subsequent punishment for deviation from a collusive outcome. The only credible outcome in this period, therefore, is the Cournot - Nash equilibrium. We now consider the period before last. As both firms know that the Cournot - Nash solution will be played in the next time period there is again no effective punishment for deviation in this penultimate period. The Cournot - Nash equilibrium will, therefore, be played in this last but one period. This argument can be repeated for all
successive time periods until we reach the beginning of the game. The only subgame perfect Nash equilibrium for this finitely repeated game is that the Cournot - Nash equilibrium is played in every time period. The paradox of backward induction rules out the possibility of non-cooperative collusion between oligopolies in finitely repeated games of complete information, where there is a unique Nash equilibrium in the stage game. As many economists believe that oligopolies do collude so as to increase profit a number of ways of avoiding the paradox have been applied to oligopolistic competition. Here we discuss three suggested ways oligopolies might succeed in colluding without assuming infinite interaction between firms.

**Multiple Nash equilibria.**

In the one-off models we examined in section 4.1 there was shown to be a unique Nash equilibrium. This however need not always be the case. For example, with more complex demand and/or cost functions it is quite possible for there to be multiple Nash equilibria in these one-off games. Figure 4.8 illustrates how non-linear reaction functions can generate multiple Nash equilibria when two firms Cournot compete. In this diagram there are three Nash equilibria at points A, B, and C. The existence of multiple Nash equilibria means that the outcome in the final period of play is no longer uniquely determined. This in turn means there is no unique subgame perfect equilibrium for the whole game. In particular, if the multiple Nash equilibria in the one-off game are associated with different levels of profit for the competing firms, a collusive outcome can be self supporting in the early periods of the game.

![Figure 4.8. Multiple Cournot Equilibria.](image)

Non-cooperative collusion is now possible because multiple Nash equilibria allows firms, to be effectively punished if they deviate from the collusive outcome. If the punishment is sufficiently severe then it will be in firms self interest to maintain the collusive outcome. As the final period is approached the non-cooperative cartel will break down. This is demonstrated by the fact that the outcome in the final period must be a Nash equilibrium. Benoit and Krishna (1987) demonstrate that by combining multiple Nash equilibria in the stage game with a suitable carrot and stick punishment strategy the set of possible non-cooperative collusive outcomes is almost identical to that derived under infinite repetition, provided the game is repeated a sufficiently large number of times.

**Uncertainty about the future.**

As discussed in Chapter 3 an alternative way of avoiding the paradox of backward induction is to introduce uncertainty about when the game might end. Without a known last period of interaction between competing firms the process of backward induction cannot be initiated. In this situation firms can credibly threaten future punishment if other firms deviate from the collusive outcome. Given that this future punishment may not be forthcoming, as
interaction may have ceased before this is possible, this threat is not as severe as when
made in an infinitely repeated game. However, similar results derived under infinite repetition
can be reproduced when there is uncertainty about when finite interaction will end. Once
again the non-cooperative collusive outcome can be sustained given that the firms do not
overly discount future returns.

**Uncertainty about competitors.**

A final way of avoiding the paradox of backward induction is by introducing
incomplete information. In the context of oligopoly this involves competition between firms
who are unsure about some aspect of their rivals’ payoff function. This may either be because
firms are unsure about the parameters of their competitors’ profit function, or the objectives
of other firms. For example, firms may be uncertain about the demand or costs facing their
competitors, which determine their rivals’ profit function. Alternatively, it may not be clear
whether other firms are interested in maximising profits or have some other objective such as
maximising total revenue. Due to mutual interdependence firms will try and estimate the
payoff function of their competitors. This is necessary so that firms can attempt to predict the
behaviour of other firms. Firms will then attempt to maximise their payoffs based on these
estimates. One important way firms seek to learn what their competitors are like, or the
constraints they face, is by observing their current and past behaviour. Other firms realising
this may then seek to manipulate their own behaviour in order to influence the expectations of
other firms. This, in turn, will be taken into account when firms interpret the actions of their
rivals. Clearly solving such models can be quite complicated. As discussed in Chapter 3 a
relevant equilibrium concept to use in models of incomplete information is that of Bayesian
subgame perfect Nash equilibrium. With this type of equilibrium no incredible threats or
promises are made or believed, and firms update their expectations rationally according to
Bayes’ Theorem. Typically the equilibrium need not involve firms playing the Nash equilibrium
for the stage game in each and every period of the game. To illustrate the intuition behind this
result consider a number of firms Bertrand competing with incomplete information about each
other’s marginal costs of production.

As shown in section 4.1 Bertrand competing firms can be made better off if they all
set higher prices. Furthermore, as prices in the one-off game are strategic compliments, if a
firm believes other firms are going to raise their price it will also increase its price. Prices are,
in turn, positively related to marginal costs of production. With uncertainty about other firms’
marginal costs of production, each firm has an incentive to persuade its competitors that its
marginal costs are high. In this way firms have an incentive to increase the price they set now
so as to try and develop a reputation for having high marginal costs, and for setting high
prices. In equilibrium it is possible for all firms to set high prices and a non-cooperative
collusive outcome to be maintained in the initial periods of the repeated game.

The above argument illustrates how uncertainty can enhance the probability of non-
cooperative collusion being sustained. This, however, need not always be the case. For
e.xample, most of the proposed ways in which collusion outcomes have been demonstrated to
be self supporting have relied upon firms being punished if and when they deviate from the
collusive outcome. This mechanism presupposes that such deviation can be detected. If
deviation from the collusive outcome can go undetected then a firm may be able to cheat on a
tacit agreement without fear of being punished. In this situation firms will have to find other
ways in which to support the collusive outcome. This may involve the sharing of information
on prices charged and quantities produced, or the development of punishment strategies
conditional upon the observed market variables. Green and Porter (1984) and Porter (1983),
for example, develop models where Cournot competing oligopolies do not observe each
others output levels but only the derived market price. In these models the firms adopt a
**trigger price strategy,** where each firm produces the collusive output level as long as the
market price remains above the trigger price. If the market price falls below the trigger price
then a period of punishment where the Cournot - Nash equilibrium quantities are produced
ensues. In this way collusion can be maintained as long as demand shocks do not cause the
market price to fall below the trigger price. The typical equilibrium in models such as these
involve alternating periods of collusive behaviour followed by price wars when demand for the
good is sufficiently low.
4.3. Conclusions.

This chapter has illustrated how game theory can be used to model the strategic interaction faced by oligopolistic firms. Initially one-off games were used to present the three classic models of Cournot, Bertrand and Stackelberg competition. Each was shown to have a different underlying structure that greatly influences the way firms are predicted to behave. Nonetheless each of the Nash equilibria associated with these models are Pareto inefficient. With different output combinations all firms could be made better off. This raises the question of whether firms are able to coordinate upon a more profitable outcome. If firms face a period of one-off competition this seems unlikely. If, however, firms face continual interaction non-cooperative collusion would seem possible. This derives from the possibility of firms being credibly threatened with future punishment if they deviate from the collusive outcome. In this way tacit collusion can be self enforcing. The only exception to this is when the paradox of backward induction is applicable. However, as demonstrated in the text, this result depends on the extreme assumptions of a known end to competition, and common knowledge about all competing firms. In reality neither of these assumptions are likely to be satisfied.

Demonstrating the possibility of non-cooperative collusion is undoubtedly a major achievement of game theoretical analysis. In one sense, however, such analysis is too successful, in that often there are a large number of feasible outcomes upon which firms may coordinate. In the text we argued that with such multiple equilibria it seems reasonable to assume that firms will coordinate on a focal point. This raises the question of what features of the underlying competition make one equilibrium more salient than another. Without greater attention being given to these factors we will be unable to confidently predict the outcome of oligopolistic competition in any specific market. These considerations are the subject of on-going research, and some of the general issues involved are discussed in Chapter 12.
4.4. Solutions to Exercises.

Exercise 4.1.

The profit function for the n'th firm is equal to :

\[ \Pi_n = Pq_n - cq_n \]

\[ \therefore \Pi_n = (a - q_n - \sum_{i=1}^{n-1} q_i) \cdot q_n - cq_n \]

\[ \therefore \Pi_n = aq_n - q_n^2 - (\sum_{i=1}^{n-1} q_i)q_n - cq_n \]

where subscripts refer to a particular firm.

Differentiating with respect to \( q_n \) and setting this equal to zero gives us the first order condition for a maximum.

\[ \frac{dq_n}{d\Pi_n} = a - 2q_n - \sum_{i=1}^{n-1} q_i - c = 0 \]

\[ \therefore q_n = \frac{a - \sum_{i=1}^{n-1} q_i - c}{2} \]

As all firms are identical, in Nash equilibrium they will all produce the same level of output, and so \( \sum_{i=1}^{n-1} q_i = (n - 1) q_n \). Substituting this into the above equation gives us :

\[ \therefore q_n = \frac{a - c}{n + 1} \]

This corresponds to the Nash equilibrium output level of each firm in the industry. With \( n = 2 \) we obtain the previous result that \( q_n = \frac{a - c}{3} \). As \( n \rightarrow \infty \) so industry output approaches the perfectly competitive outcome with price equal marginal cost, and \( Q = a - c \).

Exercise 4.2.

There are two possible ways in which the equation for the contract curve can be found. The first method is to select a fixed level of profit for one of the firms and then maximise profit for the other firm. The second method is to maximise the weighted sum of profits of the two firms. Here we demonstrate the second method.

Define the weighted sum of the two firms profits as \( W \) so that

\[ W = k\Pi_A + (1 - k)\Pi_B ; \quad 0 \leq k \leq 1 \]

\[ \therefore W = k (a - q_A - q_B - c)q_A + (1 - k) (a - q_A - q_B - c)q_B \]

The first order condition for a maximum is that the two partial derivatives \( \frac{\partial W}{\partial q_A} \) and \( \frac{\partial W}{\partial q_B} \)
\( \frac{\partial W}{\partial q_B} \) equal zero. From these two requirements the following quadratic equation can be obtained

\[
(q_A + q_B)^2 - \frac{3(a-c)}{2}(q_A + q_B) + \frac{(a-c)^2}{2} = 0
\]

This equation has two solutions where joint supply, \( Q = q_A + q_B \), equals \((a-c)\) or \(\frac{(a-c)}{2} \).

The second solution corresponds to the output a monopolist would supply to the market, and so this is the solution that maximises the sum of weighted profits. The equation for the contract curve is therefore written as

\[
q_B = \frac{a - c}{2} - q_A
\]

As stated in the text a Pareto efficient outcome results when the two firms agree to divide the monopoly output level between them.

**Exercise 4.3.**

To find the subgame perfect Nash equilibrium of this inherently dynamic game we use the principle of backward induction. We first find the reaction function of a typical follower. Throughout the following derivation we make use of the fact that in Nash equilibrium all leaders will have the same level of supply, as they are assumed to be identical. Similarly all followers will supply the same quantity. The profit function for any one follower is given as:

\[
\Pi_k = \Pi(q_k - c q_k)
\]

\[
\therefore \Pi_k = [a - q_k - (n - 1)q_{-k} - m q_j]q_k - c q_k
\]

where the subscript -k refers to all firms in the vector k other than the k'th firm. Differentiating this with respect to \( q_k \) and setting this equal to zero, gives us the first order condition for maximizing profits.

\[
\frac{d\Pi_k}{dq_k} = a - 2q_k - (n - 1)q_{-k} - m q_j - c = 0
\]

\[
\therefore q_k = \frac{a - c - (n - 1)q_{-k} - m q_j}{2}
\]

As in Nash equilibrium all followers will have the same supply, we can rewrite this as:

\[
q_k = \frac{a - c - m q_j}{n + 1}
\]

In determining the optimal strategy for the leaders, we need to substitute this reaction function into their profit function. Again taking a representative leader j and letting the subscript -j correspond to all other leaders we have:
\[ \Pi_j = Pq_j - cq_j \]

\[ \therefore \Pi_j = \left[ a - q_j - (m - 1)q_{-j} - nq_k \right]q_j - cq_j \]

\[ \therefore \Pi_j = aq_j - q_j^2 - (m - 1)q_{-j}q_j - nq_k q_j - cq_j \]

\[ \therefore \Pi_j = aq_j - q_j^2 - (m - 1)q_{-j}q_j - n\left( \frac{a - c - mq_j}{n + 1} \right)q_j - cq_j \]

\[ \therefore \Pi_j = \left( 1 - \frac{n}{n + 1} \right)(a - c)q_j - \left( 1 - \frac{nm}{n + 1} \right)q_j^2 - (m - 1)q_{-j}q_j \]

Differentiating with respect to this firm's output, and setting this equal to zero yields the first order condition for a maximum

\[ \frac{d\Pi_j}{dq_j} = \frac{a - c}{1 + n} - 2\left( \frac{n + 1 - mn}{n + 1} \right)q_j - (m - 1)q_{-j} = 0 \]

Once more in Nash equilibrium all leader's supply will be equal to each other, and so :

\[ \frac{d\Pi_j}{dq_j} = \frac{a - c}{1 + n} - \left[ 2\left( \frac{n + 1 - mn}{n + 1} \right) - \frac{(1 - m)(n + 1)}{n + 1} \right]q_j = 0 \]

\[ \therefore (n + 1 - mn + m)q_j = a - c \]

\[ \therefore q_j = \frac{a - c}{1 + n + m - mn} \]

This is the Nash equilibrium supply for any leader. Substituting this into the follower's reaction function gives us their equilibrium level of supply.

\[ q_k = \frac{a - c}{n + 1} - \frac{m}{n + 1}\left( \frac{a - c}{1 + n + m - mn} \right) \]

\[ \therefore q_k = \frac{1}{1 + n}\left( 1 - \frac{m}{1 + n + m - mn} \right)(a - c) \]

In Cournot duopoly we have either two “leaders” and no “followers”, or no “leaders” and two “followers”. In each case we get the result that both firms supply \( \frac{a - c}{3} \). For Stackelberg duopoly we get that the leader supplies \( \frac{a - c}{2} \) and the follower supplies \( \frac{a - c}{4} \). With n
Cournot competitors, either all “leaders” or all “followers”, each firm supplies $\frac{a-c}{n+1}$. These findings confirm previous results.

**Exercise 4.4.**

(i). a. Given the proposed punishment strategy, if a firm continually produces the collusive output level then its present value profits is:

$$\frac{\Pi_M}{2} + \delta \frac{\Pi_M}{2} + \delta^2 \frac{\Pi_M}{2} + \delta^3 \frac{\Pi_M}{2} + \ldots = \frac{(1 + \delta) \Pi_M}{1 - \delta^2}$$

Alternatively if the firm deviates from the collusive outcome in the first period then it will rationally deviate in every alternate period, and so its present value profit is, at most:

$$\Pi_D + \delta \Pi_C + \delta^2 \Pi_D + \delta^3 \Pi_C + \ldots = \frac{\Pi_D - \delta \Pi_C}{1 - \delta^2}$$

The firms will maintain the collusive outcome if

$$\frac{(1 + \delta) \Pi_M}{1 - \delta^2} \geq \frac{\Pi_D - \delta \Pi_C}{1 - \delta^2}$$

$$\therefore \delta \geq \frac{\Pi_D - \frac{\Pi_M}{2}}{\frac{\Pi_M}{2} - \Pi_C}$$

Comparing this condition with that derived in the text when the punishment period is infinite, we can note that the set of discount values for which collusion on the monopoly outcome is possible is smaller. This is because the threatened punishment is less severe, and so firms are less likely to be deterred from deviating from the cartel.

b. In order for the collusive outcome to be maintained with this punishment strategy two conditions will need to be simultaneously satisfied. The first condition is that neither firm has an incentive to deviate from the collusive output level, given that the proposed future punishment is credible. The second condition is that neither firm has the incentive to deviate from producing the punishment level of output $q_p$, when required, given that the first condition is satisfied.

The first necessary condition is derived as follows.

If a firm maintains the collusive outcome then present value profits are:

$$\frac{\Pi_M}{2} + \delta \frac{\Pi_M}{2} + \delta^2 \frac{\Pi_M}{2} + \delta^3 \frac{\Pi_M}{2} + \ldots = \frac{(1 + \delta) \Pi_M}{1 - \delta^2}$$

If a firm deviates from the collusive output then present value profits are at most:

$$\Pi_{Dev.\,collusion} + \delta \Pi_{Punished} + \delta^2 \Pi_{Dev.\,collusion} + \delta^3 \Pi_{Punished} + \ldots = \frac{\Pi_{Dev.\,collusion} - \delta \Pi_{Punished}}{1 - \delta^2}$$

where $\Pi_{Dev.\,collusion}$ and $\Pi_{Punished}$ are the period profits associated with deviating from the collusive outcome and being punished for so deviating respectively.

The firms will continue to produce the collusive outcome if the proposed punishment is credible and

$$\frac{(1 + \delta) \Pi_M}{1 - \delta^2} \geq \frac{\Pi_{Dev.\,collusion} - \delta \Pi_{Punished}}{1 - \delta^2}$$
Similarly the second necessary condition is derived as follows.
If a firm produces $q_p$ following deviation by the other firm then, if the first condition is satisfied, its present value profits equal:

$$
\Pi_{\text{Punishing}} + \delta \Pi_{\text{Punished}} + \delta^2 \frac{\Pi_M}{2} + \delta^3 \frac{\Pi_M}{2} + \ldots
$$

where $\Pi_{\text{Punishing}}$ is the expected level of profit when punishing the other firm. Alternatively if it deviates from the proposed punishment level of output then present value profits will be at most:

$$
\Pi_{\text{Dev.punishing}} + \delta \Pi_{\text{Punished}} + \delta^2 \frac{\Pi_M}{2} + \delta^3 \frac{\Pi_M}{2} + \ldots
$$

where $\Pi_{\text{Dev.punishing}}$ is the expected profit level associated with deviating from the proposed level of punishment output. A firm will therefore maintain the proposed punishment strategy if the first condition is satisfied and

$$
\Pi_{\text{Punishing}} + \delta \frac{\Pi_M}{2} \geq \Pi_{\text{Dev.punishing}} + \delta \Pi_{\text{Punished}}
$$

$$
\therefore \delta \geq \frac{\Pi_{\text{Dev.punishing}} - \Pi_{\text{Punished}}}{\Pi_M/2 - \Pi_{\text{Punished}}}
$$

(ii). Under the trigger strategy where deviation from the collusive outcome causes firms to permanently switch to the Cournot-Nash equilibrium the collusive outcome is self-supporting if

$$
\delta \geq \frac{\Pi_D - \Pi_M/2}{\Pi_D - \Pi_C} = \frac{506.25 - 450}{506.25 - 400} = 0.529
$$

With the punishment strategy given in part i. a. the condition becomes

$$
\delta \geq \frac{\Pi_D - \Pi_M/2}{\Pi_M/2 - \Pi_C} = \frac{506.25 - 450}{450 - 400} = 1.125
$$

Finally the results derived assuming the carrot and stick punishment strategy are shown in Table 4.1, where the binding constraint is underlined.
The results in the above table indicate that as the punishment level of output $q_p$ increases the minimum rate of discount needed to maintain the collusive outcome initially falls and then rises. This is confirmed in Figure 4.9 where the two conditions over $\delta$ are plotted for values of $q_p$ between 20 (the Cournot-Nash equilibrium output level) and 60 (the minimax output level). From these results the most effective punishment strategy in maintaining collusion in this model is the carrot and stick punishment strategy when $q_p$ is set equal to 30. With this punishment strategy collusion on the monopoly outcome can be sustained provided $\delta \geq 0.25$.
Further Reading:
Figure 4.1. Cournot Duopoly: Extensive Form Game.

Figure 4.2. Cournot - Nash Equilibrium.

Figure 4.3. Cournot Disequilibrium Dynamics.
Figure 4.4. Stackelberg Duopoly: Extensive Form Game.

Figure 4.4. Stackelberg Duopoly: Extensive Form Game.
Figure 4.5. Stackelberg - Subgame Perfect Nash Equilibrium

Figure 4.6. Bertrand Duopoly : Extensive Form Game.
Figure 4.7. Bertrand - Nash Equilibrium.

Figure 4.8. Multiple Cournot Equilibria.
Table 4.1.

<table>
<thead>
<tr>
<th>q_p</th>
<th>$\delta \geq \frac{\Pi_{Dev\text{-}collusion} - \Pi_M}{\Pi_M} - \Pi_{Punished}$</th>
<th>$\delta \geq \frac{\Pi_{Dev\text{-}punishing} - \Pi_{Punishing}}{\Pi_M} - \Pi_{Punished}$</th>
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</thead>
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<tr>
<td>25</td>
<td>$\delta \geq 0.391$</td>
<td>$\delta \geq 0.098$</td>
</tr>
<tr>
<td>30</td>
<td>$\delta \geq 0.250$</td>
<td>$\delta \geq 0.250$</td>
</tr>
<tr>
<td>35</td>
<td>$\delta \geq 0.191$</td>
<td>$\delta \geq 0.431$</td>
</tr>
</tbody>
</table>

Figure 4.9.

\[ \delta \geq \frac{\Pi_{Dev\text{-}collusion} - \Pi_M}{\Pi_M} - \Pi_{Punished} \]
\[ \delta \geq \frac{\Pi_{Dev\text{-}punishing} - \Pi_{Punishing}}{\Pi_M} - \Pi_{Punished} \]