Draft:

Agents, Games, and Evolution

A Society of Ideas

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Part I, chapters 1–3, Part II, chapter 4.

$Id: abookwrapper.tex,v 2.7 2003/09/30 20:20:58 sok Exp $

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File: abookwrapper.tex. Created: 2003-09-01, from age-master.tex.

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Part I

Starters

Chapter 1

Contexts of Strategic Interaction

Ideas have lives of their own. They arise at surprising times and come from surprising places. Ideas interact with other ideas. They become associated and these associations—or societies—lead to new social structures and to new ideas. These in turn may calve off in groups and form new societies. Of course, ideas live and have their being in human minds and cultural artifacts. Without us they wouldn't exist. Nor would most of us exist without the benefit of ideas that have created and sustained our civilization. Nor would there be animals without photosynthesizing plants, or photosynthesizing plants without photosynthesizing bacteria. Interdependence pervades.

The subject of this book is a certain society of ideas. I shall call it the AGE society (Agents, Games, and Evolution), without making any claim that the name ideally describes its subject. What name does? Like any interesting and reasonably complex society, AGE is a product, and continues to be a producer, of history. Its structure, its dynamics, even its constituents are often opaque and puzzling, and everything is in flux. Hence the need for study.

I shall not attempt to define the subject at hand. It likely can't be done and wouldn't be of much value even if it could. Hyperprecision would only be a distraction at this point. Rather, I will plunge in, immerse us at the center of AGE society, and explore from where we find ourselves. Better to start somewhere reasonable, then ask questions, attend to relevant problems and data, refine and recombine concepts and hypotheses, and build models and conduct experiments. All this is to be undertaken in an iterative, exploring, probing, nondeterministic search for sharper clarity, deeper understanding, and useful results. This shall be our mode throughout. If the process seems to be a sort of groping, adaptive muddling, so be it. The means are informed by the main results.

Let us begin, then, by discussing what I call *contexts of strategic interaction* (CSIs), also known as *games*. Here is a—perhaps *the*—main theme in our AGE society of ideas.

1.1 Games in the Wild

Games, or more descriptively *contexts of strategic interaction* (CSIs), are everywhere.[1] They pervade social situations and occur quite naturally (or appear "in the wild" as geneticists say of certain alleles). Two people play backgammon. They are in a game, or context of strategic interaction (CSI), because the reward (winning or losing) for each player depends at least in part on decisions made by the other player. One player cannot make a series of decisions that result in winning or losing, *independently of what the other player does.* The other player has to make decisions, too, and they matter. The context is interactive—two or more players are involved—and it is strategic because both players have interests, which they take into account in making their decisions.

Backgammon is representative of many games in that it is purely competitive.[2] One player's win is the other player's loss. The interests of the players are, we may assume, entirely opposed. In other CSIs (or games) the players' interests are entirely coincident. These are what we call *games of pure coordination.* Two people are conversing by telephone when the connection is suddenly dropped. How should they attempt to resume the conversation? If both call back simultaneously both will get a busy signal or perhaps voice mail. They share a joint interest in mutually divining a decision that results in prompt and unfrustrated resumption of their conversation. Here we may assume the interests of the two agents are identical. Neither really cares who makes the new call, so long as it results in immediate resumption.

Lying between games of pure competition (e.g., backgammon) and games of pure coordination (e.g., resuming a broken phone call) are *mixed motive* games (or

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[1] The term game is perhaps an unfortunate one for a number of reasons. It suggests a certain frivolity, also that only contexts of pure competition are of interest. More importantly, we need a distinction between a situation involving strategic interaction and a model of such a situation. *Game* gets used for both. When necessary to differentiate, I'll use game$_S$ for the situation, not always well defined with all vagueness left out, and game$_M$ for a model, presumably specified with great precision, of a game$_S$. Or, CSI for game$_S$ and game for game$_M$.

[2] At least approximately or often. Consider playing with a tyro and playing to lose for purposes of instruction.

CSIs). A small group negotiates where to have dinner. No two people have identical preferences, but everyone agrees that failing to come to a congenial decision quickly is the worst outcome. Remarkably subtle moves will typically attend this familiar situation. Bluff, bluster, threat, compromise, accommodation, probing, retreating, appeal to norms, humor, and much else are routinely employed with facile skill by everyone who participates in such groups.

How are we to understand games? In particular, how are we to predict and explain both behavior and outcomes in games? This is a large and important question. I remind the reader that our mode here is to make some progress through an "iterative, probing, nondeterministic search for sharper clarity and deeper understanding." To this end, a rough characterization of our topic will be helpful:

> Games, or CSIs, essentially involve at least two *agents* (or players) who make *choices* and receive *rewards* (or payoffs).[3] The reward to an individual agent is based in part on its choices *and the choices made by the other agent(s)*, as well as the underlying structure of the situation.

Now consider a few representative, idiosyncratically-chosen examples of contexts of strategic interaction.

1.1.1 War

Much more than a pure, brutal contest of strength, war has been recognized from the earliest writings as a field of interactive decision making. Deception especially has been and remains a primary theme; it is inherently a strategic concept. Think of the Trojan horse incident told in the *Iliad* and the story of the Cyclops in the *Odyssey.* Think of the elaborate obfuscations undertaken by the Allies in World War II concerning the time and place of D-Day. Sun Tzu, in *The Art of War* (http://www.chinapage.com/sunzi-e.html), the oldest known military treatise, wrote this:

[3] Later, it will be useful to distinguish *rewards*, which are received after each move in a strategic situation, and *returns*, which are the net of the rewards obtained in a multi-move strategic context. Unless otherwise noted, what I say about rewards applies to returns and *vice versa.*

18. All warfare is based on deception.
19. Hence, when able to attack, we must seem unable; when using our forces, we must seem inactive; when we are near, we must make the enemy believe we are far away; when far away, we must make him believe we are near.
20. Hold out baits to entice the enemy. Feign disorder, and crush him.
21. If he is secure at all points, be prepared for him. If he is in superior strength, evade him.
22. If your opponent is of choleric temper, seek to irritate him. Pretend to be weak, that he may grow arrogant.
23. If he is taking his ease, give him no rest. If his forces are united, separate them.
24. Attack him where he is unprepared, appear where you are not expected.
25. These military devices, leading to victory, must not be divulged beforehand.

Other themes abound, but deception and surprise remain keystones to military strategy. Other works on the short list of classics in military strategy include: *On War,* by Karl von Clausewitz, *The Prince,* by Niccola Machiavelli, and *A Book of Five Rings,* Miyamoto Musashi.[4] Liddell Hart, e.g., [60] is an especially persuasive spokesman for the importance of military deception and surprise. Thomas Schelling is uniformly insightful and a joy to read, e.g., [84, 82, 83, 85]. The *Memoirs* of Ulysses S. Grant are chock full of material to stimulate reflection on war and on interactive decision making in general. Here is my favorite passage. Grant is describing his first field command in the American Civil War.

> My sensations as we approached what I supposed might be a 'field of battle' were anything but agreeable. I had been in all the engagements in Mexico that it was possible for one person to be in; but not in command. If someone else had been colonel and I had been lieutenant-colonel I do not think I would have felt any trepidation. . . . As we approached the brow of the hill from which it was expected we would

[4] You may find these on-web at http://www.gametheory.net/html/books.html#classics.

> see the enemy. . . my heart kept getting higher and higher until it felt as though it was in my throat. I would have given anything to have been back in Illinois, but I had not the moral courage to halt and consider what to do; I kept right on. When we reached a point from which the valley below was in full view I halted. The place where the Confederates had been encamped was still there but the troops were gone. My heart resumed its place. It occurred to me at once that [Colonel Thomas] Harris had been as much afraid of me as I had been of him. This was a view of the question I had never taken before; but it was one I never forgot afterwards. From that event to the close of the war, I never experienced trepidation upon confronting an enemy, though I always felt more or less anxiety. I never forgot that he had as much reason to fear my forces as I had his. The lesson was valuable.
>
> —Ulysses S. Grant, *Memoirs*

1.1.2 Trading and Investing

"Buy low, sell high" is great advice if (and only if) you know what to do. As the song says about that special form of trade and investment called love, "Nice work if you can get it, And you can get it if you try."

Examples of buying low or selling high? This is from *The Reader's Digest,* June 2003, pages 76–7:

> Customers at The Home Depot who overestimate how much paint they need return the unopened cans, which are stocked in the "Oops Paint" section. The "remnant" paint—perfect for bathrooms and other small projects—sells for $5 a gallon and $1 a quart (regular gallon prices are $21 to $25). "And it's not all chartreuse," says The Home Depot spokesperson Mandy Holton. "There are usually a lot of great neutrals." Best time to buy: Sundays and Mondays, because folks return unwanted paint over the weekend.

More generally, traders and investors are in the business of finding assets that are either under-valued or over-valued in the market. In other words, they seek opportunities for *risky arbitrage.* Risky because—unlike the paint at The Home Depot—the values of the assets in question are typically not known with much certainty. Arbitrage because the traders are looking to buy assets that are under-priced (and then resell them at their proper prices) or looking to sell ("unload") assets that are over-priced. In any event, the trick is to have and exploit knowledge that is superior to what is represented in the market. The nature of this knowledge and the means of getting it vary greatly. An investor in equities may look deeply and carefully at the fundamentals of the companies. Which are and which are not well managed, well positioned, in possession of new products and alliances? An investor may look at the "technical" data, the trends and other movements in prices. In the extreme, so-called day traders do this in real time, attempting to out-guess the market, that is out-do the other traders in discerning what is over-valued or under-valued. Note that investing on analysis of fundamentals would seem to have less strategic content than investing on technical grounds.

On-line, Internet-based examples are readily available for those who wish to trade or just to study and learn. Tradesports (http://www.tradesports.com) is a real-time, on-line trading market that affords an excellent case for study. Academic analogs—but with real money if you want—are available for elections at Iowa Electronic Markets (IEM; http://www.biz.uiowa.edu/iem/; http://www.biz.uiowa.edu/iem/markets/) and the University of British Columbia Election Stock Market (http://esm.ubc.ca/). A *BusinessWeek* article, "The 'Election Futures' Market: More Accurate than Polls?"[5] presents the case in a popular format that these markets predict election outcomes better than opinion polls. The Bush administration even toyed with creating a similar market for the purpose gauging intelligence in the Middle East (see "Betting on Terror: What Markets Can Reveal" by Floyd Norris in *The New York Times*, August 3, 2003). The idea was dropped after being exposed to public ridicule. Is it ridiculous? Consider: What would it take to "game" (distort for ulterior purposes) these markets? When would anyone want to do this? What might be done to prevent manipulation? Does it make sense to have an SEC for markets in international affairs?

There are always the public equity markets. Consider this comment on the bond market by a Salomon trader in the 1980s.

> The men on the trading floor [Salomon's bond trading area] may not have been to school, but they have Ph.D.'s in man's ignorance. In any market, as in any poker game, there is a fool. The astute investor Warren Buffett is fond of saying that any player unaware of the fool in the market probably *is* the fool in the market. In 1980, when the bond market emerged from a long dormancy, many investors and even Wall Street banks did not have a clue who was the fool in the new game. Salomon bond traders knew about fools because that was their job.

[5] 1996; http://www.businessweek.com/1996/46/b3501116.htm.

> Knowling about markets is knowing about other people's weaknesses. And a fool, they would say, was a person who was willing to sell a bond for less or buy a bond for more than it was worth. A bond was worth only as much as the person who valued it properly was willing to pay. And Salomon, to complete the circle, was the form that valued the bonds properly.
>
> —*Liar's Poker*, Michael Lewis [58, page 35]

1.1.3 Athletic Contests

There are sports, called games in common parlance, that have little or no strategic content. They amount more or less to contests of skill. Among them are golf, bowling, darts, skiing, track and field events, and bob sledding. Still other competitive games, such as billiards, have strategic content only with fairly advanced play. These are not, for the most part, of interest as CSIs, contexts of strategic interaction, and will not concern us further.

Many other athletic contests most unambiguously count as CSIs. Baseball has given us a wonderful strategic slogan, entirely appropriate for war and other games: "Hit 'em where they ain't." Wee Willie Keeler hit .432 in 1897. Asked how a man of his diminutive size could put together such an average, Keeler responded: "Simple. I keep my eyes clear and I hit 'em where they ain't."[6] Deception—or the fake-out—plays as prominent a role in these athletic contests as it does in warfare. Think of the pitcher-batter duel in baseball, the fake-out moves in basketball, or the mixing of plays in American football.

Management of sports teams is as much a matter of strategic interaction as the play itself. In *Moneyball: The Art of Winning an Unfair Game,* Michael Lewis describes how the Oakland A's baseball team, with consistently small amounts spent on player salaries, is consistently able to contend in major league baseball and reach the playoffs [59]. In two words: risky arbitrage. The A's, and in particular their general manager Billy Bean, have identified predictive measures superior to those used by other teams, for example using on-base percentage instead of batting average to evaluate the worth of a batter. Better measures of value allow the A's to 'buy' (hire) under-priced players. They may not have the best team in baseball, but they have one of the best. Their efficiency in the sense of what it costs them to win a game is tops and they operate at a profit in a media market dominated by the San Francisco Giants baseball team.

[6] From http://www.baseballtips.com/slang.html.

1.1.4 Gambling

Many forms of gambling do not involve strategy or even much skill. Examples include playing slot machines and playing roulette. Not so with poker at the professional level. Poker is prototypical. It is to competitive games of strategy what robins are to birds: a standard, familiar, readily available example, displaying in typical form many of the characterizing features of the subject. Everyone over time gets roughly the same quality of hands, yet there is an enormous difference among players in their success rates. Everyone can count cards and figure the odds. What actually matters is bluffing, reading your opponents (discerning their "tells," behavior such as slamming down chips that indicates what is in their hands), and preventing your opponents from reading you. The following fine passage from a master poker player is well worth quoting at length:

> Let's take a quick glimpse at the high-stakes poker world, an enterprise that yields several of my friends over a million dollars a year! At this level, too, luck is a factor on any given day, week, or month, but what's different is that if you play better poker than your opponents do, pretty consistently, you'll find that over almost any *two*-month period your winnings have exceeded your losses. Furthermore, if you play better poker than your opponents over a *six*-month period, your results will have moved very solidly in the winning direction. Making a few well-timed bluffs each day will add up to a lot of money each year!
>
> In fact, if an inexperienced poker player were to sit down for a few hours with a group of wolrld-class poker players, he would have virtually no chance to win over even an eight-hour period. This very fact is why five or six top pros might be willing to sit down in the same game with this fellow and each other: the money that even one amateur is likely to contribute makes it work their while to do battle with so many respected opponents.
>
> This is why so many of the top poker players today drive fine cars and live in palatial homes [the author of this passage lives with his family in Palo Alto]. Right now, as you're reading this book, there is a $600–$1,200-limit poker game at the Bellagio Casino in Las Vegas and a $400-$800-limit poker game at the Commerce Casino in Los Angeles. There is . . .
>
> If that's not enough action for you, four nights a week in Los Angeles,

> there is a $2,000-$4,000-limit Seven-Card Stud game a Larry Flynt's Hustler Club Casino, with Larry himself often playing. In the $400-$800-limit poker game it's easy to take a $25,000 swing in one hour. In the $2,000-$4,000-limit game, where movie stars, former governors, and billionaires play, it's not uncommon for someone to win or lose $250,000 in one night. In these "nosebleed" poker games (the term refers to the altitude of the stakes), strategy, discipline, calculation of the odds, and practiced observation contribute to a game that involves much more skill. Better play wins more hands in the long run.
>
> –*Play Poker Like the Pros* by Phil Hellmuth, Jr., 2003 [42, pages 4–5]

The society of poker players has given us an important concept—the *tell*—not only for poker but for CSIs generally. *Webster's Unabridged Dictionary* finds only two senses for *tell* as a noun. Quoting:

1. something that is told : TALK, TALE, ACCOUNT

   "have a tell with you —Eden Phillpott"

2. [Ar *tall*]

   : HILL, MOUND

   specif : an ancient mound in the Middle East composed of remains of successive settlements — compare TEPE

The *Oxford English Dictionary* is no more helpful. The Wikipedia gets it right. Quoting:

> Tell (poker)
>
> From Wikipedia, the free encyclopedia.
>
> In poker, a tell is a detectable change in a player's behavior that gives clues to that player's hand. Possible tells include leaning forward or back, placing chips with more or less force, fidgeting, changes in breathing or tone of voice, direction of gaze and actions with the cards, cigarettes, or drinks.
>
> For example, a player with a weak hand, hoping to bluff, may throw his chips into the pot forcefully and with a direct gaze at a player he hopes to discourage from calling.

> Tells may be common to a class of players or unique to a single player. A player gains an advantage if she observes another player's tell, particularly if that action is unconscious and reliable. However, better players may fake tells, hoping to lead their opponents into costly traps when they rely on the false information. So the observing, creating, and evaluating of tells can add another level to the play of poker.
>
> Mike Caro has published the most comprehensive information on tells; his *Book of Tells* (ISBN 0897461002) is now a standard reference on the subject.
>
> David Mamet's 1987 movie *House of Games* includes an interesting discussion and visual reference to tells as an essential part of the plot. The 1998 movie *Rounders* contains an even more subtle use of strategy: at one point, "Mike" discovers a tell in his opponent (that he eats cookies in a particular way after he has bet a very strong hand), and after using it once, he reveals to the opponent that he has this tell; although this eliminates the usefulness of the tell itself, it upsets his opponent so much that it affects his later play.
>
> –*Wikipedia*, http://www.wikipedia.org/wiki/Tell_(poker)

The Wikipedia is also better than the dictionaries on *tell* as a noun, not in the context of poker. Quoting:

> Tell
>
> From Wikipedia, the free encyclopedia. A tell (Arabic, or tel, Hebrew) is a mound site formed through successive human occupation over a very long timespan.
>
> The word is used as a term in archaeology, particularly Middle-Eastern archaeology. It is sometimes used in a toponym, that is, as part of a town or city name, the most well known example being the city of Tel Aviv. Often a modern city is located next to an ancient mound with a similar tell name, for example the city of Arad is a few kilometers (miles) away from an ancient mound called Tel Arad.
>
> This article is a stub. You can help Wikipedia by fixing it.
>
> External link
>
> * http://www.webref.org/anthropology/t/tell.htm

> Tell is an English verb meaning "to speak to" or "to talk to"; also "to give an order". For more information on what that is, see talking.
>
> Retrieved from "http://www.wikipedia.org/w/wiki.phtml?title=Tell" This page was last modified 01:47, 19 Jun 2003. All text is available under the terms of the GNU Free Documentation License.

Note the meaning associations and similarities between these two senses of *tell*.

Hellmuth's book has a great deal of information on Texas Hold 'em, which is generally the most popular form of poker in tournaments and is the variety of poker played at the World Series of Poker each year at Binion's Horseshoe Hotel & Casino in Las Vegas.[7] *Positively Fifth Street* [66] by Jim McManus, a published poet, novelist, and professor, describes the 2001 World Series of Poker and the world around it. Strategic insight abounds in both works.

1.1.5 Business Strategy

The gambit, a term from chess, is a favorable trade, which the opponent may or may not realize is happening. The player offering the gambit offers a comparatively small loss in exchange for a larger gain in position or other form of resource. Here is something very like a gambit played big time in business.

> Analysts called it "Marlboro Friday"—Philip Morris announced on April 2, 1993 that it would reduce the U.S. price of its premium brand of cigarettes by 20%. The tobacco manufacturer also said it would increase the budget for its domestic advertising by a substantial amount. R.J. Reynolds, Philip Morris's biggest competitor, responded by matching the price cut on its own premium brands (Camel and Winston among them) and by pouring more money into its own domestic advertising.
>
> The pricing war that ensued cost both companies tens of millions of dollars. But was the domestic market share the real reason Philip

[7] See http://conjelco.com/wsop.html and http://www.binions.com/home.asp. The Wikipedia has an introduction to Texas Hold 'em: http://www.wikipedia.org/wiki/Poker/Texas_holdem. At the World Series of Poker, No Limit Texas Hold 'em is the game. "No Limit" means that the largest bet permitted is the size of the current wealth outside the pot of the poorest player still in the hand. Once a player has bet all of his or her chips, the player is said to be "all in," since the player's wealth is all in the pot. Once a player is all in for a particular hand, other players may call but may not raise.

> Morris lowered the price of Marlboro cigarettes? Consider that just as R.J. Reynolds had depleted its cash resources trying to keep up with its chief opponent, Philip Morris was expanding aggressively into the Eastern European market, investing $800 million in Russia and other regions that were formerly part of the Soviet Union. R.J. Reynolds was in no position to fight back, having spent so much money to maintain its market share in the United States, and Philip Morris won the battle for Eastern European market share, hands down.
>
> –"Global Gamesmanship" by Ian C. MacMillan et al., *Harvard Business Review*, May 2003 [63]

Sometimes you can even make a profit on a gambit:

> One day earlier in his career [Robert] Dall was in the market to buy (borrow) fifty million dollars. He checked around and found the money market was 4 to 4.25 percent, which meant he could buy (borrow) at 4.25 percent or sell (lend) at 4 percent. When he actually tried to buy fifty million dollars at 4.25 percent, however, the market moved to 4.25 to 4.5 percent. The sellers were scared off by a large buyer. Dall bid 4.5. The market moved again, to 4.5 to 4.75 percent. He raised his bid several more times with the same result, then went to Bill Simon's office to tell him he couldn't buy money. All the sellers were running like chickens.
>
> "Then you be the seller," said Simon.
>
> So Dall became the seller, although he actually needed to buy. He sold fifty million dollars at 5.5 percent. He sold another fifty million dollars at 5.5 percent. Then, as Simon had guessed, the market collapsed. Everyone wanted to sell. There were no buyers. "Buy them back now," said Simon when the market reached 4 percent. So Dall not only got his fifty million dollars at 4 percent but took a profit on the money he had sold at higher rates. *That* was how a Salomon bond trader thought: He forgot whatever it was that he wanted to do for a minute and put his finger on the pulse of the market. If the market felt fidgety, if people were scared or desperate, he herded them like sheep into a corner, then made them pay for their uncertainty. He sat on the market until it puked gold coins. *Then* he worried about what he wanted to do.

—*Liar's Poker*, Michael Lewis [58, page 88]

1.1.6 Negotiation

Negotiation exemplifies strategic interaction *par excellence.* After all, there is no point in negotiating if your counter-party's actions don't matter to you. Familiar as negotiation is to everyone, it is useful to be reminded that often negotiation is not explicit, at least not at first. Here is a description of this sort of encounter "in the wild."

> To begin to negotiate the environment does not, of course, mean that you enter the negotiation with a clear-cut goal in mind. A clear-cut goal is not needed even in purely human negotiations. Suppose you pass a stall in a market every week and notice an antique ornament for sale. At first it seems ugly, but as it grows familiar, you catch yourself wondering how it would look on your shelf. One day it rains while you are crossing the market and you take shelter in the stall. the ornament is still there; for something to do you ask its price. Even when a low price is mentioned you automatically snort in contempt, for you have no intention of buying. . . or have you? During the week that follows you decide that the price really was low and think of a friend who has a birthday soon and might like it. Next week you stop and begin to bargain.
>
> When did the negotiation begin? When you started to bargain? Or earlier, when you asked the price? Or earlier still, when you first noticed the ornament among an anonymous heap of others? Pointless to say, as pointless as to say where mind began.
>
> –*Language and Species* by Derek Bickerton [9, pages 234–5]

1.1.7 Coordination, Symbiosis, Mutualism, Cooperation

Contexts of strategic interaction are not all adversarial in the sense that one agent's gain is another's loss (so-called *constant-sum* or equivalently *zero-sum* games). In *coordination games* all players gain if they can arrive at a common outcome and lose if they fail. Think of the game of finding someone you have separated from during a shopping trip. You both wish to meet up again, but did not plan for the separation and have no easy means of communication. Schelling's early treatment of such games is masterful and well worth reading today [84].

Biologists have named and studied several kinds of interactive decision making that—in terms of game theory lingo—is not constant-sum. Symbiosis and mutualism are two of the most important for our purposes.

> Symbiosis (pl. symbioses) is an interaction between two organisms living together in more or less intimate association or even the merging of two dissimilar organisms.
>
> The term host is used for the larger of the two members of a symbiosis. The smaller member is called the symbiont.
>
> Symbiosis may be divided into two distinct categories: ectosymbiosis and endosymbiosis. In ectosymbiosis, the symbiont lives on the body surface of the host, including the inner surface of the digestive tract or the ducts of exocrine glands. In endosymbiosis, the symbiont lives in the intracellular space of the host.
>
> An example of mutual symbiosis is the relationship between anemonefishes of the genus Amphiprion (family, Pomacentridae) that dwell among the tentacles of tropical sea anemones. The territorial fish protects the anemone from anemone-eating fish, and in turn the stinging tentacles of the anemone protects the anemone fish from its predators (a special mucous on the anemone fish protects it from the stinging tentacles).
>
> The biologist Lynn Margulis, famous for the work on endosymbiosis, contends that symbiosis is a major driving force behind evolution. She considers Darwin's notion of evolution, driven by competition is incomplete, and claims evolution is strongly based on co-operation, interaction, and mutual dependence among organisms. According to Margulis and Sagan (1986), *Life did not take over the globe by combat, but by networking.*
>
> (From: http://www.wikipedia.org/wiki/Symbiosis.)

(See [64] for a recent treatment of this theme by Margulis and Sagan.)

> Mutualism is a interaction in which both organisms in a close relationship derive some degree of benefit. Mutualism is usually temporary or not obligatory.
>
> (From: http://www.wikipedia.org/wiki/Mutualism.)

Lichens—those familiar greenish splotches on trees and rocks—present a most striking example of symbiosis.

> Lichens have been described as “dual organisms” because they are symbiotic associations between two (or sometimes more) entirely different types of microorganism -
>
> - a fungus (termed the mycobiont)
> - a green alga or a cyanobacterium (termed the photobiont).[8]
>
> There are many examples of symbiosis in nature, but lichens are unique because they look and behave quite differently from their component organisms. So, lichens are regarded as organisms in their own right and are given generic and species names. However, for taxonomic purposes the names are actually fungal names: lichens are regarded as a special group of fungi - the lichenised fungi.
>
> There are an estimated 13,500 to 17,000 species of lichens, extending from the tropics to the polar regions. Some of them grow on the bark of temperate trees or as epiphytes on the leaves of trees in tropical rain forests. Others occupy some of the most inhospitable environments on earth, growing on cooled lava flows and bare rock surfaces, where they help in the process of soil formation, and on desert sands where they help to stabilise the surface and enrich it with nutrients (see Cyanobacteria [cf., footnote 8, page 17]). Some other types of lichen grow abundantly on tundra soils, providing a vital winter food source for animals (including reindeer and caribou) in arctic and subarctic regions.Yet other lichens grow on or in the perennial leaves of some economically important tropical crop plants such as coffee, cacao and rubber, where they are regarded as parasites.

[8]Author’s note: From the Wikipedia, www.wikipedia.org/wiki/Cyanobacteria: “Cyanobacteria or blue-green bacteria are a group of aquatic bacteria that obtain their energy through photosynthesis. They are often referred to as blue-green algae, even though it is now known that they are not related to any of the other algal groups, which are all eukaryotes. Nonetheless, the description is still sometimes used to reflect their appearance and ecological role. Fossil traces of cyanobacteria have been found from around 3800 million years ago, making cyanobacteria some of the earliest living things known.” Implied but not said, the cyanobacteria are prokaryotes, of ancient origin and lacking a cell membrane. It is thought by some that photosynthesizing plants acquired or incorporated the genomes of photosynthesizing bacteria.

> All these features make lichens interesting and significant in environmental terms. But lichens also pose challenging scientific problems - how do two or more microorganisms interact at the cellular, genetical and biochemical levels to produce a unique, hybrid organism?
>
> (From: http://helios.bto.ed.ac.uk/bto/microbes/lichen.htm.)

A form of emergence occurs with lichens. Surprisingly, what appears to be, and in many ways is, a single individual is actually composed of, arises through the interactions of, individuals from two distinct biological kingdoms. It is even surprising who first noticed this underlying, symbiotic structure.

> Lichens are unusual creatures. A lichen is not a single organism the way most other living things are, but rather it is a combination of two organisms which live together intimately. Most of the lichen is composed of fungal filaments, but living among the filaments are algal cells, usually from a green alga or a cyanobacterium.
>
> In many cases the fungus and the alga which together make the lichen may each be found living in nature without its partner, but many other lichens include a fungus which cannot survive on its own – it has become dependent on its algal partner for survival. In all cases though, the appearance of the fungus in the lichen is quite different from its morphology as a separately growing individual.
>
> The true identity of lichens as symbiotic associations of two different organisms was first proposed by Beatrix Potter, who is best remembered for her children's books about Peter Rabbit. In addition to her books, she spent time studying and drawing lichens. Her illustrations are still appreciated for their detailed and accurate portrayal of the delicate beauty of these bizarre organisms.
>
> (From http://www.ucmp.berkeley.edu/fungi/lichens/lichens.html.)

(Searching Google's image base on "lichens" turns up an excellent collection of images.)

Next, cooperation is—in its prototypical sense—a human social phenomenon, one that has been much noticed and remarked upon by social scientists, including game theorists. Cooperation, or roughly non-greedy behavior, has been called "the cement of society" [26] (by analogy with causation, which Hume called "the cement of the universe"). Without it, in the pungent phrasing of Thomas Hobbes, there would be

> no place for industry, because the fruit thereof is uncertain; and consequently no culture of the earth; no navigation, nor use of the commodities that may be imported by Sea; no commodious Building; no Instruments of moving and removing such things as require much force; no Knowledge of the face of the Earth; no account of Time; no Arts; no Letters; and which is worst of all, continuall feare, and danger of violent death; And the life of man, solitary, poore, nasty, brutish, and short. (Hobbes, *Leviathon*)

(See, e.g., http://plato.stanford.edu/entries/hobbes-moral/, http://www.philosophypages.com/ph/hobb.htm.) Without cooperation we are lost. How, then, does it arise and how might it be sustained? Hobbes thought that realistically it was necessary to turn power over to a sovereign (king or powerful government)—a leviathon—who would enforce cooperation on society. Others have thought that perhaps cooperation could emerge and be sustained naturally, without a central authority, much as, say, lichens emerge and are sustained naturally. Is this possible? If so, what is required of the games and the players?

1.1.8 Conversation

When we speak we have in mind how others will react to what we say and what we do not say. In this regard, a representative news story—"Official's comments set off euro's surge. U.S. Treasury's Snow said a weaker dollar would help U.S. exports. The dollar fell against the euro." by David McHugh—appeared in *The Philadelphia Inquirer* on May 13, 2003. The first sentence says it all: "The U.S. dollar fell to another four-year low against the euro yesterday, inching closer to its all-time low, after U.S. Treasury Secretary John Snow said a weaker dollar would help U.S. exports." Secretary Snow never said he favored letting the dollar fall, but what he did say, as he no doubt understood, led the markets to infer that he favored a decline in the dollar. This form of strategic interaction is rife in linguistic communication and even has a special name: conversational implicature. Examples abound. A sign at Big Sur Lodge, Pfeiffer State Park, near a food counter:

Stressed?
Spelled
Backwards
Is
Desserts

Translation: Buy a dessert from us; it'll make you feel good. Or the concluding line in Hitchcock's movie, "Frenzy": "Mr. Rusk, you're not wearing your tie." Translation: You're the necktie murderer and I'm placing you under arrest. Or the use of irony, as in "Rick, Major Strasse is one of the reasons the German Reich enjoys the reputation it has today," from the movie "Casablanca." Translations: (to Strasse) The Reich is an impressive accomplishment and you are a big part of it; (to Rick) Watch out, this guy Strasse is a very bad man. Or the dialog-less eating scene in the movie "Tom Jones" with Albert Finney. Translation: This is just foreplay foreplay; the best is yet to come. See Paul Grice's "Logic and Conversation" [40] for the original treatment, still worth reading.

1.1.9 Games against Yourself

The long and justly celebrated story from the *Odyssey* of Ulysses and the Sirens continues to enchant and inform us. (Jon Elster has even written an entire book relating the story to modern social science [25].) From the perspective of strategic interactions, the story may be interpreted as a game played by Ulysses at one time against Ulysses at another time. At t_0, before approaching within earshot of the Sirens, Ulysses foresees that Ulysses at t_1, within earshot, will have preferences and inclinations quite at variance from Ulysses at t_0 and from Ulysses at t_2, post the encounter with the Sirens (if he should live that long). So Ulysses at t_0 cleverly prevents Ulysses at t_1 from acting as Ulysses at t_1 would prefer. He hears the Sirens and lives to tell the tale.

Robert Louis Louis Stevenson's familiar story, *Dr. Jekyll and Mr. Hyde,* carries a similar theme. Thomas Schelling tells a fable about a man who is struggling to quit smoking. A friend who smokes arrives at his house, converses, and leaves without incident. The friend, however, forgets his jacket and our protagonist notices the jacket contains a package of cigarettes. Not having an immediate compulsion to smoke and knowing the friend will return tomorrow, he puts the jacket away. Later, upon reflection, he recovers the jacket, removes the cigarettes, and destroys them.

1.1.10 Confidence Games

The con man (or woman) first gets your trust, your confidence, and then abuses it for profit. "Take the money and run" is the operating creed. Confidence rackets are celebrated in literature, theater, and film. Examples include Herman Melville's novel *The Confidence Man,* Sinclair Lewis's novel *Elmer Gantry* (movie with Burt

Lancaster and Jean Simmons), Jim Thompson's novella *The Grifters* (movie with Anjelica Huston, John Cusack, and Annette Bening), Guy Owen's short story "The Flim-Flam Man" (movie with George C. Scott and Sue Lyon), N. Richard Nash's play *The Rainmaker* (movie with Burt Lancaster and Katherine Hepburn), David Mamet's movie "House of Games" (with Lindsay Crouse and Joe Mantegna), and Meredith Wilson's Broadway musical *The Music Man* (movie with Robert Preston and Shirley Jones). This is from a Penn Web site, August 2003:[9]

```
6:30 pm - 8:30 pm     Confidence Games at the GSC

The GSC shows films about con artists:
Catch Me If You Can on 7/31;
The Thomas Crown Affair on 8/7;
The Spanish Prisoner on 8/14; and
The Grifters on 8/21.

Location:  Graduate Student Center, 3615 Locust Walk
Category:  Film
More info:
http://www.upenn.edu/gsc/programs/film.htm#con
```

Con games lie at the core of much detective fiction and fact, as well as recently popular email scams. There is a confidence business, indeed an industry, with its own lessons and skills. (This takes us beyond the scope of the book. Those wishing to go further might consult such works as *How to Become a Professional Con Artist*, by Dennis M. Marlock.)

1.1.11 Statesmanship

Ending this list on a less cynical note, George Washington is understood to have been a politically ambitious man throughout his life. He actively, deliberately sought and schemed for the power, influence, and adulation he ultimately received. Washington notoriously wore his military uniform during the deliberations on the Declaration of Independence, just to remind the other delegates of his availability for command. In pursuing his ambitions Washington consistently and consciously

[9] See http://www.gametheory.net/ for yet another list of game-related movies, as well as lots of useful material on game theory.

followed a strategy of seeking rewards by actually deserving to get them. Resigning from the army at the end of the Revolution, an unexampled act, was a move calculated to make him fit for political leadership in a democracy. Declining to run for a third term as president was a move calculated to secure the success of the new country and of Washington's legacy.

Napoleon on his deathbed and in prison lamented that "They expected me to be another Washington."

1.2 Why Study Games?

Why are we interested in contexts of strategic interaction (CSIs)? The examples just discussed will, I hope, make it plain that games are interesting because war, diplomacy, poker, business strategy, and so on are each interesting in themselves and each are examples from a larger pool of important phenomena meriting attention. Besides interdependent, interactive choice—the characterizing feature of games—we see the interplay of reasoning, calculation and reckoning, deception, skill, bluffing, power, adaptation, flexibility, cooperation, learning, arbitrage, coordination, norms, communication, markets, social organization, and much else that is pervasive in, and fundamental to our understanding of, the social order. (And belonging to the AGE society of ideas.) Games are interesting because they are vortices of many interesting phenomena. Social phenomena manifest themselves—play themselves out—in games.

What would we like to know about games? As in any field of science we seek to describe, explain, predict, and intervene. We wish to describe and classify games in the wild systematically. The previous section is merely a hint at a much-needed natural history of games. We wish to explain and predict game outcomes. This often called *solving the game* in the classical literature. We also wish to understand—explain and predict—how it is outcomes are reached. How do agents of various sorts (experienced humans, naïve humans, monkeys, rats, lichens, organizations, artificial agents) find and implement their strategies of play? How does play unfold over time (and over space when geography is relevant)? Finally, we seek understanding of games in order to intervene in the world. We might hope to improve our own play in strategic contexts, or to design better social institutions (such as markets for electrical power that resist manipulation—"gaming"—as in the Enron affair [90]), or to field artificial agents that labor on our behalves (perhaps for negotiation or purchasing over the Internet) . The scope of potential investigation is both magnificent and beyond our means. We should be content with modest progress, while keeping ourselves reminded of the larger issues. That, at least, describes my aims in this essay.

1.3 Methods of Study

Games in the wild, we must always remind ourselves, are the primary phenomena that motivate study of contexts of strategic interaction. The games we make up or develop as abstract models are ultimately interesting only because they contribute to understanding games in the wild. How, in particular then, can we study strategic interaction? Various ways are open to us:[10]

1. *a priori.* CSIs or games in the wild may be abstracted and reduced to formal models, then studied mathematically, typically upon assumption of axioms of rationality. From this perspective, the theory of games is a branch of mathematics. Much of classical game theory proceeds in this mode. Standard textbooks and reference works include [10, 34, 55, 61, 86].

2. *in vivo.* Games, or strategic situations, may be studied *in situ,* as they (more or less) naturally occur. This is an historical—"natural history"—mode of investigation, but of course the history may be contemporary and the means of study may use techniques from anthropology, sociology, or journalism. Pioneers of this approach include Thomas Schelling (e.g., [84, 82, 83, 85]), and Jon Elster (e.g., [25, 26]).

3. *in vitro.* We can study games by doing experiments with real ("wet") agents, including humans (e.g., [29, 46]), monkeys (e.g., [31]), even blue jays (e.g., [89]). And why not lichens and bacteria? The literature uses such names as *behavioral game theory* and *experimental economics* to refer to these kinds of investigations.

4. Algorithmic or *in silico.* There is much to be learned about games by representing agents as decision algorithms that choose their plays, and then studying the behavior of the resulting system. Fundamentally, this method of simulation or experimental mathematics is a variant of the *a priori* method. Let us call it *algorithmic game theory.* By allowing ourselves to use computational methods (instead of purely analytic mathematics) we may greatly

[10] I am grateful for the discussion in [72].

extend the range, scope, and realism of models addressed, and concommitantly reduce the stringency of the assumptions required.

In what follows, we shall draw upon each of these methods. Our main focus methodologically, however, will be on algorithmic (or *in silico*) studies of artificial agents. Such studies may be, and have been, conducted from a variety of perspectives. Agents may be modeled as naked strategies (what we call *identity-centric* agents), possibly reactive or adaptive strategies, that play in tournaments (e.g., Axelrod's original and seminal study [2]) or that play in a populated ecology which evolves under the replicator dynamic (e.g., [2, 35, 87]) or that play in a differentiated geography (aka: spatial games; e.g. [28, 41]).

Again, we shall draw upon these and related studies but focus our efforts elsewhere. That focus has four main aspects:

1. *Finite, non-ideal* contexts of strategic interaction. Agents are finite beings. Their rationality, their abilities to reckon and foresee, are limited. The algorithms with which we model these agents must be computable[11] and computable without exorbitant use of resources. Play unfolds in a finite population, for a finite time, and in a finite space. A major theme will be to compare and contrast results under finite, non-ideal and infinite, ideal regimes of play. Classical game theory employs what philosophers call an *externalist* theory of rationality. Here we are asking different questions and shall be focusing on *internalist* notions of rationality. The upshot of this point will emerge as we proceed.

2. *Identity-centric* more so than strategy-centric agents. Humans, and indeed monkeys and blue jays, in contexts of strategic interaction may be said to *have* strategies (rather than to *be* strategies), and to be capable of changing them in response to experience. These players, and most of the agents we shall consider, may meaningfully be said to have identities distinct from the strategies they employ at any given time. They are more than naked strategies. In particular, they are

3. *Exploring, probing* agents, not merely reactive agents. Humans, monkeys, blue jays, and most of our agents face the exploration–exploitation dilemma/-tradeoff, addressed throughout the machine learning literature.

   Finally, the strategic contexts we will focus on will be

[11] Technically, effectively computable, which I assume to be coterminus with the partial recursive functions. We are only interested in algorithms that implement partially recursive functions.

4. *Chronic* and *social* more so than acute and singular, *ongoing* and *widespread* more so than unique. The games may be *repeated* or *iterated* or be like other games that will be played, rather than being unique, non-repeatable events.

A word of justification for this last aspect of our focus. We often *are* inclined to think of games in terms of acute, dramatic points of decision. This is captured in the penultimate stanza of “Casey at the Bat” (Ernest L. Thayer, alias Phin, page 4 of the San Francisco *Daily Examiner,* June 3, 1888).

> The sneer has fled from Casey’s lip, the teeth are clenched in hate.
> He pounds, with cruel violence, his bat upon the plate.
> And now the pitcher holds the ball, and now he lets it go,
> and now the air is shattered by the force of Casey’s blow.

Of course the final stanza is

> Oh, somewhere in this favored land the sun is shining bright.
> The band is playing somewhere, and somewhere hearts are light.
> And, somewhere men are laughing, and little children shout,
> but there is no joy in Mudville – mighty Casey has struck out.

But first, many games, many contexts of strategic interaction, *are* repeated, or approximately so. Agents do business with a particular merchant, doctor, lawyer, restauranteur repeatedly. Agents have friends, partners, lovers, spouses, colleagues they encounter more than once. Agents have competitors in the market for more than a day. Agents are embedded in societies. Very often indeed, strategic contexts cannot be separated from the future or the past.

And second, is Casey’s situation really unique, even for Casey? True enough, Casey is in a zero-sum game in the sense that only one team can win. It is also true that in any given at-bat the pitcher in baseball has the advantage; anyone can strike out. Most likely, however, there will be another game tomorrow or the next day. Casey’s interest lies in maximizing the expectation of his future contributions to the team. Getting angry, pounding the bat, focusing exclusively on this game and this moment is, perhaps, not the wisest of moves on Casey’s part. Better to take the long view. Better to have the pitcher strike you out than for you to strike yourself out. Perhaps the long view can inform the acute.

1.4 Looking Forward

Description, explanation, prediction, and support for intervention may, for our purposes, be considered the governing goals of science. We seek to describe and explain CSIs (and more broadly, agents, games, and evolution thereof); we seek to predict what will happen in them; and we seek to intervene in ways that will serve our interests. In particular, we want to study and understand CSIs that display or involve social phenomena such as cooperation (or its failure), trust (or its absence), commitment, norms, conventions, markets, emergence, and signaling and communication.

There is a lot to do, certainly too much for one book. Some things can be done, even if everything cannot. To repeat: my principle focus is study of algorithmic agents in strategic contexts. How, for important games, do societies of agents–strategies behave? Who wins, who loses, what happens under varying conditions? What does the behavior tell us about social phenomena of interest? We explore these questions (and others) in Part II. What happens when agents–players are not merely strategies, even complex, reactive, adaptive strategies, but true learners, who probe their environments and are capable of discovering their own strategies? We explore these questions (and others) in Part III. Part IV discusses several extensions and applications, or more specific models, of the sorts of agents explored in Parts II and III. Finally, in Part V we take stock, sum up, interpret, and look to future investigations and applications.

Two chapters remain in Part I. Chapter 2 "Games in the Abstract," rehearses essential concepts and terminology from game theory, and presents frameworks for understanding the abstracted forms of games amenable to the analytic or experimental treatments that follow. "Computational Explanation," Chapter 3 is philosophical and a bit of a digression. The interested reader may consult the chapter for development of the idea that computational models—including but not only those encountered in the present work—may often be superior instruments for scientific theorizing. Those in a hurry and confident of having an introductory level of knowledge of game theory may want to skip these chapters and proceed directly to Part II, in which we discuss experiments and the data that have arisen.

Chapter 2

Games in the Abstract

2.1 Interrogation

The scene is Sam Spade's apartment. Gutman and Cairo are partners in search of the Maltese Falcon. They have the guns. Spade has the information. Spade is being difficult.

> Cairo, his face and body twitching with excitement, exclaimed: "You seem to forget that you are not in a position to insist on anything."
>
> Spade laughed, a harsh derisive snort.
>
> Gutman said, in a voice that tried to make firmness ingratiating: "Come now, gentlemen, let's keep our discussion on a friendly basis; but there certainly is"—he was addressing Spade—"something in what Mr. Cairo says. You must take into consideration the—"
>
> "Like hell I must." Spade flung his words out with a brutal sort of carelessness that gave them more weight than they could have got from dramatic emphasis or from loudness. "If you kill me, how are you going to get the bird? If I know you can't afford to kill me till you have it, how are you going to scare me into giving it to you?"
>
> Gutman cocked his head to the left and considered these questions. His eyes twinkled between puckered lids. Presently he gave his genial answer: "Well, sir, there are other means of persuasion besides killing and threatening to kill."
>
> "Sure," Spade agreed, "but they're not much good unless the threat of death is behind them to hold the victim down. See what I mean? If

> you try anything I don't like I won't stand for it. I'll make it a matter of your having to call it off or kill me, knowing you can't afford to kill me."
>
> "I see what you mean." Gutman chuckled. "That is an attitude, sir, that calls for the most delicate judgment on both sides, because, as you know, sir, men are likely to forget in the heat of action where their best interest lies and let their emotions carry them away."
>
> Spade too was all smiling blandness. "That's the trick, from my side," he said, "to make my play strong enough that it ties you up, but yet not make you mad enough to bump me off against your better judgment."
>
> Gutman said fondly: "By Gad, sir, you are a character!"
>
> –From "The Fall-Guy" in *The Maltese Falcon* by Dashiell Hammett

The passage describes a context of strategic interaction. How might we model this game in the wild? A number of *game forms* are available. For the present we will use just one, the *strategic form*, which is well-suited to games with two players. See Figure 2.1 for the general, canonical strategic form for two players each having two strategies.

	C_1	C_2
R_1	r_1, c_1	r_2, c_2
R_2	r_3, c_3	r_4, c_4

Figure 2.1: Canonical game matrix for the 2×2 game in strategic form

The interpretation is straightforward. There are two players: Row, who choses R_1 or R_2, and Column, who choses C_1 or C_2. We say Row has available the *strategies* R_1 and R_2, and similarly Column has strategies C_1 and C_2. The form is called *strategic form* because the players' strategies are laid out so plainly. If players have more than two strategies, we add rows or columns to the *game matrix* as necessary. If there are more than two players we can use a game cube or hypercube (for more than 3 players) if needed.

Choosing simultaneously, or at least in ignorance of each other's choices, there are four possible outcomes and associated *rewards* for the players. In terms of the canonical game matrix, Figure 2.1:

Outcome	R's reward	C's reward
R_1C_1	r_1	c_1
R_1C_2	r_2	c_2
R_2C_1	r_3	c_3
R_2C_2	r_4	c_4

Returning now to our scene of interrogation in Sam Spade's apartment, the players are Spade (let us say Row) and Gutman (Column). Spade may either Blab (B) regarding the whereabouts of the bird, or Keep Silent (K). Gutman will interrogate. He will either Press (P) or Extreme Press (E), the latter killing Spade in the process, or at least severely disabling him. Spade's preference ordering is $KP > BP > KE > BE$. Gutman's is $BP > BE > KP > KE$. That is, Spade prefers most that Row play K and Column play P; Gutman prefers most that Row play B and Column play P. Converting these rankings to convenient numbers gives us Figure 2.2. The numbers range from 1 to 4, more being better. Here they simply record the ranking of the rewards or outcomes for the players in question.

	Press (P)	Extreme Press (E)
Blab (B)	3, 4	1, 3
Keep Silent (K)	4, 2	2, 1

Figure 2.2: Interrogation: Spade is Row, Gutman is Column

What will happen? Gutman is surely right that the situation "calls for the most delicate judgment on both sides." Putting that aside for the present and looking at the game as abstracted in Figure 2.2, we can see that no matter which strategy Gutman pursues, Press or Extreme Press, Spade will prefer—is rewarded more by—Keeping Silent. If Gutman Presses, Spade gets 4 for Keeping Silent and 3 for Blabbing. If Gutman Extreme Presses, Spade gets 2 from Keeping Silent and 1 from Blabbing. Either way, Spade does better by Keeping Silent. Notice that the actual values of the rewards to Spade matter little. We could replace 4 by A, 3 by B, 2 by C and 1 by D and this conclusion would follow for any numbers assigned to the letters, so long as higher numbers represent preferred rewards and $A > B > C > D$.

So we predict Spade will Keep Silent. What will Gutman do? Gutman is a smart fellow and he will presumably see the reasoning in the situation—as he appears to in the dialog—and conclude that Spade will choose to Keep Silent. Given that, Gutman's best strategy is to Press, since choosing Extreme Press won't get him the Maltese Falcon and burdens him with Spade's demise. Gutman's "By Gad, sir, you are a character!" is an admission of defeat.

Some terminology and associated concepts that we will need throughout: a dominant strategy, a Nash equilibrium, and a Pareto optimal outcome. We say Spade's (Row's) Keep Silent strategy *dominates* his Blab strategy, because on every alternative—here, either Press or Extreme Press—Spade's Keep Silent yields a higher reward to Spade than does his Blab. The typographic conventions we are using for game matrices were designed to make it easy to spot dominating or dominated strategies. See Figure 2.2. Gutman (Column) also has an absolutely dominate strategy: Press beats Extreme Press.

The *principle of dominance* enjoins us—or at least predicts of all rational players—never to choose a dominated strategy. If Spade and Gutman are rational in this rather minimal sense, KP (Gutman Presses and Spade Keeps Silent) will be the outcome. In this game the dominance principle is sufficient to predict a unique outcome. Is this in fact what will always happen? Gutman's remark that "men are likely to forget in the heat of action where their best interest lies and let their emotions carry them away" is surely apt. We should ask ourselves whether other rational causes might lead to violation of the principle. This is a matter for the sequel.

We say that KP is a *Nash equilibrium* (or NE) outcome because no player could change its strategy *unilaterally* and do better.[1] KP is an NE because for each player, given the play by the other player(s), its strategy is best. Specifically, KP is an NE because *given that Gutman chooses Press*, Spade can do no better with Blab, and *given that Spade chooses Keep Silent*, Gutman can do no better with Extreme Press. A Nash equilibrium outcome is a strategic standoff: no player can do better, given what the other players have played. One might, and classical

[1] Named in honor of John Nash, 1928–, who invented and developed the concept. His home page is http://www.math.princeton.edu/jfnj/. See http://www.nobel.se/economics/laureates/1994/nash-autobio.html for his biography for his The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel, 1994, prize.

game theory does, predict that game outcomes among rational players will be Nash equilibria. Again, we should ask ourselves whether rational causes might lead to violation of this *Nash equilibrium principle.* Again, this is a matter for the sequel.

We say that an outcome is *Pareto optimal* if there is no other outcome for which every player does better.[2] If an outcome is not Pareto optimal, we say it is dominated. (Notice the difference between a dominated strategy and a dominated outcome.) In our Interrogation game, Figure 2.2, BE is a dominated outcome; both Spade and Gutman do better with BP. Similarly, KE is dominated, this time by KP. Both KP and BP are Pareto optimal. We say that the set of Pareto optimal outcomes constitutes the *Pareto frontier*. The *Pareto frontier principle* has it that among rational players game outcomes will always be in (or on) the Pareto frontier. Once again, we should ask ourselves whether rational causes might lead to violation of this, the Pareto frontier principle. And what if the principles fail to apply or even even conflict?

Summing up, we can label the outcomes in the game matrix, [N] for Nash equilibrium, [P] for Pareto optimal, [NP] for Nash and Pareto, and unlabeled for none of the above. Here is Interrogation with labeling.

[2]Named in honor of Vilfredo Pareto, 1848-1923, who invented and developed the concept. See http://cepa.newschool.edu/het/profiles/pareto.htm.

	Press (P)	Extreme Press (E)
Blab (B)	4 [P] 3	3 1
Keep Silent (K)	2 [NP] 4	1 2

Figure 2.3: Interrogation with Labeling

A last word on the Interrogation game. Dashiell Hammett, the author of *The Maltese Falcon,* actually worked for a time as a private detective. He knew whereof he wrote. *The Maltese Falcon* was published on February 14, 1930, well before the flowering of game theory.

2.2 Prisoner's Dilemma

The Prisoner's Dilemma is the most famous and the most studied of games in the abstract. Invented about 1950 (and attributed to A.W. Tucker), experimentation began on it immediately [32, 33] and continues to this day. Entire books have been devoted to it, including [2, 73, 75]. Luce and Raiffa give the standard interpretation [61, page 95]. There is no need to revise it:

> Two suspects are taken into custody and separated. The district attorney is certain that they are guilty of a specific crime, but he does not have adequate evidence to convict them at trial. He points out to each prisoner that each has two alternatives: to confess to the crime the police are sure they have done, or not to confess. If they both do not confess, then the district attorney states he will book them on some very minor trumped-up charge such as petty larceny and illegal possession of a weapon, and they will both receive minor punishment; if they both confess they will be prosecuted, but he will recommend less than the most severe sentence; but if one confesses and the other does not, then the confessor will receive lenient treatment for turning state's evidence whereas the latter will get "the book" slapped on him.

The game is standardly taken to hinge on cooperation. The prisoners (Row and Column) may behave cooperatively (C) by *refusing to confess* or they may behave uncooperatively and defect (D) on each other. If both prisoners cooperate (refuse to confess) each receives a reward, R, for cooperation. R=3 in much of the literature and in what I will call the *Default* Prisoner's Dilemma (PD) game. Think of 3 as the number of years the prisoner gets to live out of jail during the next 5 years. More is better. If both prisoners defect (confess), each receives a reward of P, the penalty for mutual defection. P=1 in the Default PD. If one player cooperates (C) and the other player defects (D), the cooperator gets S, the sucker's payoff and the defector gets T, the temptation to defect. In the Default PD, S=0 and T=5. These assumptions are recorded in the left-hand game matrix of Figure 2.4. Typically,

	D	C
D	1 [N] 1	0 [P] 5
C	5 [P] 0	3 [P] 3

	D	C
D	P [N] P	S [P] T
C	T [P] S	R [P] R

Figure 2.4: Default and Canonical Prisoner's Dilemma: $T > R > P > S$ and $2R > (T + S)$

and in what I call the *Canonical* Prisoner's Dilemma (right-hand game matrix in Figure 2.4), the payoffs to each player are symmetric in the sense that T for row chooser equals T for column chooser, and so on. This is not strictly required for PD and wasn't true in the original PD experiments [32, 33].[3] Prisoner's Dilemma strictly requires that $T_i > R_i > P_i > S_i$ for each player i. Further, it is usually, although not always, required that $2R_i > (T_i + S_i)$ for each player i. Both the Default and the Canonical PDs meet both of these conditions.

The PD game is a dilemma because our rationality principles are in conflict. The principle of dominance advises both players to defect, D, since it is a dominant strategy for each of them. The Nash equilibrium principle concurs. DD is the only Nash equilibrium outcome of this game. DD, however, is not Pareto optimal: *both* players would do better if CC were the outcome. The Pareto frontier in this game is: $\{CD, CC, DC\}$. On one side we have the dominance principle and the

[3]The game matrix used was:

(-1, 2)	($\frac{1}{2}$, 1)
(0, $\frac{1}{2}$)	(1, -1)

Nash equilibrium principle and on the other we have the Pareto frontier principle. The two sides are, in this game, directly in conflict.

2.3 Hawk-Dove

Two players confront each other over a resource whose full value is V to either of them. Each player may play one of two strategies: H (Hawk) or D (Dove). Doves signal that they wish to share the resource equally. Hawks signal they are willing to fight to get the resource. When two Doves meet, each gives the characteristic sharing signal and the resource is divided equally, or, perhaps, a fair coin is tossed and the winner gets all. In any case, the expected return to each of the two Doves is $V/2$. When a Hawk meets a Dove, the Hawk (as it always does) signals fight, the Dove (as it always does) signals share, then the Dove retreats and the Hawk takes the entire resource. Finally, when two Hawks meet, each signal fight, neither retreats, both fight at a cost of C. In the end, the resource is shared equally, minus the cost, or, perhaps, half the time one Hawk gets the entire resource and half the time the other Hawk gets it. In any case, the expected return to each of the Hawks is $\frac{1}{2}(V - C)$. We can summarize the Hawk-Dove game in the following strategic form representation. Let us assume, sensibly and without loss of generality, that

	H	D
H	$\frac{1}{2}(V-C)$ $\frac{1}{2}(V-C)$	0 [P] V
D	V [P] 0	$V/2$ [P] $V/2$

Figure 2.5: Hawk-Dove Game: $C > V > 0$

$V > 0$ and $C > 0$. Notice that the outcomes HD, DD, and DH are Pareto optimal: in each case it is impossible to find another outcome that will not make at least one of the players worse off. Note especially what happens when $C > V$ and when $V > C$. DD is not a Nash equilibrium in either case, since Row would prefer HD and Column would prefer DH. If $V > C > 0$, then HD is not Nash because Column would prefer H; similarly for DH, and HH is a Nash equilibrium. Conversely, if $C > H > 0$ fighting is very damaging. Row would refer DH to HH and Column would prefer HD to HH, so HH is not a Nash equilibrium, while HD and DH are.

	H y	D $(1-y)$
H x	$\frac{1}{2}(V-C)$ $\frac{1}{2}(V-C)$	0 V
D $(1-x)$	V 0	$V/2$ $V/2$

Figure 2.6: Hawk-Dove Game with Mixed Strategies

What about *mixed equilibria*, in which strategies are played according to a probability distribution? Let Row play H with probably x and D with probability $(1-x)$. Similarly, let Column play H with probability y and D with probability $(1-y)$. See Figure 2.6, in which these probabilities are recorded in the margins. The expected return to Row, or the game value for Row, G_R in this regime is

$$G_R = xy(\frac{1}{2}(V-C)) + x(1-y)V + 0 + (1-x)(1-y)(\frac{V}{2}) \quad (2.1)$$

$$= x[y(\frac{1}{2}(V-C)) + (1-y)V - (1-y)(\frac{V}{2})] + (1-y)(\frac{V}{2}) \quad (2.2)$$

Taking a derivative with respect to x (see the Addendum to this chapter, §2.11.1):

$$\frac{dG_R}{dx} = [y(\frac{1}{2}(V-C)) + (1-y)V - (1-y)(\frac{V}{2})] \quad (2.3)$$

Setting this to 0, solving for y, and simplifying we get:

$$y = \frac{V}{C} \quad (2.4)$$

Making the same calculation for G_C, the value of the game to Column, we also get:

$$x = \frac{V}{C} \quad (2.5)$$

It is an equilibrium (indeed a stable one) for Row to play H with probability $x = \frac{V}{C}$ and for Column to play H with probability $y = \frac{V}{C}$. Note that this assumes $V < C$, which is what we'll assume in our subsequent studies of the Hawk-Dove game. Note further that when $V - C$ is positive, Hawk-Dove is a degenerate Prisoner's Dilemma: $T = V, R = V/2, P = (V - C)/2, S = 0$ violates the $2R > T + S$ condition.

The method we used immediately above to find the mixed equilibria of a game in strategic form is entirely general, and especially tractable in a 2×2 game. We will see it again. There is another, very fruitful perspective we can take on "solving" the game. Suppose now that the regime of play is repeated or iterated. We have an infinite population of players, some of whom play H and some of whom play D. We draw them at random from the population and have them play each other. We record the returns the two players get and we adjust their frequencies accordingly in the next generation. See Figure 2.7 for a representation relevant to our current regime of play, which is called a *replicator dynamic.* Figure 2.7 is a special version of Table 2.6, with $x = y$ and the perspective of the row player.

	H x	D $(1-x)$
H	$\frac{1}{2}(V-C)$ $\frac{1}{2}(V-C)$	0 V
D	V 0	$V/2$ $V/2$

Figure 2.7: Hawk-Dove Game with a Mixed Population

What about equilibrium? Think of it this way. Suppose you could pick which strategy to play, H or D, knowing the current value of x. At what value of x would you be indifferent between playing H and playing D? You would be indifferent when the expected return from playing H, $E(H)$, equaled the expected return from playing D, $E(D)$.

$$E(H) = \frac{1}{2}(V-C)x + V(1-x) \tag{2.6}$$

$$E(D) = 0x + \frac{V}{2}(1-x) \tag{2.7}$$

Setting them equal

$$\frac{1}{2}(V-C)x + V(1-x) = \frac{V}{2}(1-x) \tag{2.8}$$

and solving for x we again get

$$x = \frac{V}{C} \tag{2.9}$$

Thus, the equilibrium reached by the replicator dynamic is a Nash equilibrium. This *replicator dynamic equilibrium* is also stable.[4] A replicator dynamic equilibrium is a special case of a Nash equilibrium; it is an equilibrium reached by an infinite population of strategies under the regime of the replicator dynamic. Finite populations over finite times, driven by evolution and natural selection, may approximate it. The replicator dynamic equilibrium, like the Nash equilibrium, will be a useful benchmark and point of comparison for us.

2.4 Stag Hunt or Assurance

Two agents go hunting and take up their places in a blind, which hides them both from each other and from any stags that happen by. Together they can expect to bag a stag, which will feed them each for 3 days. If, however, one of the players reneges and goes hunting for hare, that player can expect to bag two hares, enough to feed him for two days. The other player will receive nothing, neither stag nor hare. If both players renege, each can expect to bag one hare, a day's worth of food. The game, presented in strategic form in Figure 2.8 is so named in honor of a passage in Rousseau's *A Discourse on Inequality*:

> If it was a matter of hunting a deer, everyone well realized that he must remain faithful to his post; but if a hare happened to pass within reach of one of them, we cannot doubt that he would have gone off in pursuit of it without scruple...

The Stag Hunt is also called the Assurance game. What assurance does a player have that the other player won't renege? The game has been used to model arms races. To see why, relabel. For the row player, change Hunt stag to Refrain from deploying missile defense and Hunt hare to Fully deploy missile defense. For the column player change Hunt stag to Refrain from deploying missile defense penetration system and Hunt hare to Fully deploy missile defense penetration system. Hunting stag (or its strategic equivalent) is a cooperative play, as chasing hare is uncooperative. The Stag Hunt game is thus another kind of strategic context in which issues of cooperation arise.

[4] And it qualifies as an ESS (Evolutionary Stable Strategy). See [65] for the seminal introduction of the Hawk-Dove game and the concept of an ESS.

	Hunt stag (S)	Chase hare (H)
Hunt stag (S)	3 [NP] 3	2 0
Chase hare (H)	0 2	1 [N] 1

Figure 2.8: Stag Hunt (aka: Assurance game)

Note that there are two equilibria: SS (both Hunt stag) and HH (both Chase hare), only one of which is Pareto optimal, SS. What about a mixed equilibrium? Symbolizing for the general case, with allusion to Prisoner's Dilemma we have the game matrix of Figure 2.9:

	Hunt stag (S)	Chase hare (H)
Hunt stag (S)	R [NP] R	T S
Chase hare (H)	S T	P [N] P

Figure 2.9: Generic Stag Hunt (aka: Assurance game): $R > T > P > S$

Calculating the replicator dynamic equilibrium we have

$$E(S) = Rx + (1-x)S \tag{2.10}$$
$$E(H) = Tx + (1-x)P \tag{2.11}$$

Equating and solving for x gives us

$$x = \frac{P+S}{R+P-T-S} \tag{2.12}$$

For the specific game in Figure 2.8

$$x = \frac{1+0}{3+1-2-0} = \frac{1}{2} \tag{2.13}$$

The expected return received by a player of the game at this equilibrium is

$$G_{x=0.5} = \frac{1}{4}(3 + 0 + 2 + 1) = 1.5$$

The single Pareto optimal outcome continues to dominate.

See “The Stag Hunt” by Brian Skyrms [88] for a thoughtful discussion of the significance of this game.

2.5 Chicken

Each player drives a car, racing it directly on a collision course with the other player’s car. If both players Drive Straight they will crash into one another with dire consequences for each. If one player swerves, leaving the road clear for the other, that player is “chicken” and the non-swerving player gets accolades for bravery. If both players swerve, both are “chicken” and both dishonored, but less so than being the sole “chicken.” Figure 2.10 abstracts this strategic context as a strategic form game. In our game, the Pareto frontier consists of SS (both swerve),

	Swerve	Drive Straight
Swerve	2 [P] 2	3 [NP] 1
Drive Straight	1 [NP] 3	0 0

Figure 2.10: Chicken

SD (row swerves, column drives straight), and DS. SD and DS are both Nash equilibria. Symbolizing for the general case, with allusion to Prisoner’s Dilemma gives us Figure 2.11. Calculating the replicator dynamic equilibrium we have:

$$E(S) = Rx + S(1 - x) \quad (2.14)$$

$$E(D) = Tx + P(1 - x) \quad (2.15)$$

Equating and solving for x gives us

$$x = \frac{P - S}{R - S - T + P} \quad (2.16)$$

	Swerve	Drive Straight
Swerve	R [P] R	T [NP] S
Drive Straight	S [NP] T	P P

Figure 2.11: Generic Chicken: $T > R > S > P$

In our specific case

$$x = \frac{0-1}{2-1-3+0} = \frac{1}{2} \tag{2.17}$$

This game apparently was originated in the movie, “Rebel without a Cause,” starring James Dean and Natalie Wood. In the movie’s “Chickie Run” scene, the two players race stolen cars towards a cliff overlooking the ocean. Whoever bails out first is “chicken.” One of the players is unsuccessful in abandoning his car before it lurches over the cliff. James Dean survives and gets the girl.

2.6 Battle of the Sexes

We meet Della, loving wife of loving Jim in O. Henry’s short story “The Gift of the Magi”[5] Della has a problem. They are a young couple, times are tough, and Jim’s salary has been cut. They are poor.

> Della finished her cry and attended to her cheeks with the powder rag. She stood by the window and looked out dully at a gray cat walking a gray fence in a gray backyard. Tomorrow would be Christmas Day, and she had only $1.87 with which to buy Jim a present. She had been saving every penny she could for months, with this result. Twenty dollars a week doesn’t go far. Expenses had been greater than she had calculated. They always are. Only $1.87 to buy a present for Jim. Her Jim. Many a happy hour she had spent planning for something nice

[5] Freely available at http://www.auburn.edu/˜vestmon/Gift_of_the_Magi.html thanks to Project Gutenburg.

> for him. Something fine and rare and sterling—something just a little bit near to being worthy of the honor of being owned by Jim.

How is she to get money for his Christmas present?

> Now, there were two possessions of the James Dillingham Youngs in which they both took a mighty pride. One was Jim's gold watch that had been his father's and his grandfather's. The other was Della's hair. Had the queen of Sheba lived in the flat across the airshaft, Della would have let her hair hang out the window some day to dry just to depreciate Her Majesty's jewels and gifts. Had King Solomon been the janitor, with all his treasures piled up in the basement, Jim would have pulled out his watch every time he passed, just to see him pluck at his beard from envy.

So Della sells her hair and buys a gold chain for Jim's watch. Jim, of course, comes home with an expensive set of combs for Della's hair. He has sold his watch to buy them.

Abstracting this story we get the Battle of the Sexes as it is called in the game theory literature. Its story is slightly different. He (say Row) would like to go to the fight (F, boxing match). She (Column) would like to go to the opera (O). They both would prefer to attend the same event and they each have to commit to an event without communicating with the other (cell phones have been lost, time is short, etc.). Figure 2.12 gives a reasonable abstraction. (B=best return; S=second best return; 0=worst.) In pure strategies, there are two Nash equilibria and two

	F	O
F	1 [NP] 3	0 0
O	0 0	3 [NP] 1

	F y	O $\overline{x}$
F x	S [NP] B	0 0
O $\overline{x}$	0 0	B [NP] S

Figure 2.12: Battle of the sexes: Specific and Symbolic ($\overline{x} = (1 - x)$)

Pareto optimal outcomes: FF and OO. There is a mixed Nash equilibrium at

$$x = \frac{B}{B+S} \tag{2.18}$$

$$y = \frac{S}{B+S} \tag{2.19}$$

Because this is not a symmetric game (in a sense to be made clear in the sequel), the replicator dynamic equilibrium is not (yet) well defined.

2.7 Inspector versus Evader

A two-period game is played between the Inspector (Column) and the Evader (Row). Evader might be a drug smuggler, Inspector the Coast Guard, or Evader might be a country bent on developing nuclear weapons, Inspector the United Nations, and so on. Inspector can only inspect during one of the two periods. The evader has two strategies:

- E: Evade the rules during the first period; evade during the second period if and only if Inspector inspects during the first period.

- $\neg E$: Do not evade the rules during the first period; evade during the second period if and only if Inspector inspects during the first period.

Inspector has two strategies:

- I: Inspect during the first period (and not during the second).

- $\neg I$: Inspect during the second period (and not during the first).

In terms of outcomes, let us assume Evader's preferences are $E\neg I > \neg EI > \neg E\neg I > EI$ and Inspector's preferences are $\neg E\neg I > \neg EI > EI > E\neg I$. Let $0 < b, c < 1$, $0 < a, d$, and assign the outcome values as in Figure 2.13.

	I y	$\neg I$ $(1-y)$
E x	$-a$, 0	1, $-d$ [P]
$\neg E$ $(1-x)$	c, b	0, 1 [P]

Figure 2.13: Inspector versus evader game

The game has no Nash equilibrium in pure strategies. Using the technique of §2.11.1 we find that there is an equilibrium at

$$x = \frac{1-b}{1-b+d} \tag{2.20}$$

$$y = \frac{1}{1+a+c} \tag{2.21}$$

Checking further shows that this is a stable equilibrium. Notice that the larger a and c are the smaller y is, and the larger d is the smaller y is. Is the rationality of this plain?

See [74, chapter 11] for a thorough discussion of this game. I have honored the specifics of his example.

2.8 A Zero-Sum Game

The game in Figure 2.14 has the special property that for any outcome, Row's gain is Column's loss, and *vice versa*. Such *zero-sum* games are contexts of pure

	C_1 y	C_2 $(1-y)$
R_1 x	2 [P] -2	-4 [P] 4
R_2 $(1-x)$	-2 [P] 2	1 [P] -1

Figure 2.14: A zero-sum game

opposition. Every outcome is on the Pareto frontier simply because if one player is relatively better off, the other player is relatively worse off. It happens in this particular game that there is no Nash equilibrium outcome, i.e., no Nash equilibrium in pure strategies. There is a Nash equilibrium in mixed strategies. Using the general results from the Addendum to this chapter, §2.11.1:

$$x = \frac{(1+2)}{(2+1)-(-4-2)} = \frac{1}{3} \tag{2.22}$$

$$y = \frac{-1-4}{(-2-1)-(4+2)} = \frac{5}{9} \tag{2.23}$$

2.9 PD Property Games: ##12, 47, 48 & 57

Recall the canonical game matrix for the 2×2 game (Figure 2.1, page 28), reprinted below:

	C_1	C_2
R_1	r_1, c_1	r_2, c_2
R_2	r_3, c_3	r_4, c_4

Figure 2.15: Canonical game matrix for the 2×2 game in strategic form

If we accept limitations on the numerical values assigned to the rewards, the r_is and the c_js, then the number of 2×2 games can be restricted to a tractable size and the class studied systematically. If each reward value for an agent is unique and drawn from $\{1, 2, 3, 4\}$, then there are only 576 = $4! \times 4!$ distinct 2×2 games. Many of these are really equivalent, e.g., one can be transformed to another simply by exchanging rows, or one can be transformed to another by switching the rôles of the players: Row becomes Column, Column becomes Row. It turns out that there are exactly 78 unique 2×2 games under these assumptions. Anatol Rapoport and his co-workers have studied them all [76].[6]

Among these 78 unique $2{\times}2$ games, it turns out that exactly 4 have the *PD property*—first seen in the Prisoner's Dilemma—of having Pareto frontier outcomes distinct from Nash equilibrium outcomes. Figure 2.16 presents Rapoport's game #12.

[6] See [76, pages 14–7] for details on counting the number of 2×2 games.

	C_1	C_2
R_1	2 [N] 2	1 [P] 4
R_2	4 [P] 1	3 [P] 3

Figure 2.16: Game #12: Prisoner's Dilemma

Game #12 is a Prisoner's Dilemma: $T = 4, R = 3, P = 2, S = 1$. Figures 2.17, 2.18, and 2.19 present games #47, #48, and #57 respectively, with their [N] and [P] outcomes labeled.

	C_1	C_2
R_1	3 [N] 2	1 [P] 4
R_2	2 1	4 [P] 3

Figure 2.17: Game #47

	C_1	C_2
R_1	2 [N] 2	1 [P] 4
R_2	3 1	4 [P] 3

Figure 2.18: Game #48

	C_1	C_2
R_1	3 [N] 2	2 [P] 4
R_2	1 1	4 [P] 3

Figure 2.19: Game #57

Points arising:

1. Prisoner's Dilemma, as in game #12, is typically specified in a *symmetric* fashion. In the special case in which we have a *game in symmetric form* it happens that $V_R(i, j) = V_C(j, i)$: the value that the Row player receives if Row plays strategy i and Column plays strategy j is equal to the value to the Column player of Row playing j and Column playing i. Here, e.g., $V_R(R_2, C_1) = 1 = V_C(R_1, C_2)$. Symmetric form games are more easily—hence more often—studied. In Part II we begin with symmetric form games, then move on to asymmetric games (aka: bimatrix games).

2. Games ##47, 48, and 57 are asymmetric games. They are further distinguished from game #12 and indeed all Prisoner's Dilemma games by having a Pareto dominated outcome that is *not* a Nash equilibrium. Thus is a nice setup arrived at to test our three principles (so far) of rationality. The principle of (strategy) dominance and the Nash equilibrium principle predict the

outcome R_1C_1 for all three games. The Pareto optimality principle predicts $\{R_1C_2, R_2C_2\}$. No principle predicts R_2C_1. Who is right and under what conditions?

3. None of these four games has a Nash equilibrium in mixed strategies.

4. It is possible to have a Pareto outcome in mixed strategies, for example a mixture of the elements in $\{R_1C_2, R_2C_2\}$. Can it actually happen? If so, under what conditions?

2.10 Other Games, Other Forms

Games in the wild are natural phenomena and as such need to be represented or modeled for purposes of investigation. To do so, we abstract the phenomena into simpler, more tractable representations, which afford our inquiries. This is a general point, and it applies straightforwardly in contexts of strategic interaction.

My purpose in this chapter has been to provide an inventory of abstract games, linked insofar as possible to naturally-occurring games. We draw upon this stock in the sequel. The discussion has proceeded and will proceed by example, raising concepts and terminology as they are needed.

Each of the games discussed in this chapter have been a 2×2 game in strategic form. There are other important 2×2 games, there are plenty of important games that are not 2×2 games, and there are other game forms than the strategic. We shall see examples of each of these in the course of our discussion. We begin with the 2×2 game because it is an excellent place to begin our algorithmic, constructivist—"from the ground up"—study of contexts of strategic interaction.

> [I]f there is any hope of eventually constructing scientific theories of human behavior, we must first learn to perform controlled experiments with a view of drawing inferences from them that at least have *apparent* relevance to human motivations, learning, decisions—above all, to interactions. Gaming experiments include all these features, and experiments on 2×2 games are the simplest and most tractable that include the most important of them. The value of such experiments is that they can teach us not necessarily how people behave in real life but how we can study certain aspects of characteristically human behavior systematically, from the ground up, as it were.
>
> –*The* 2×2 *Game*, Rapoport & Guyer [76, page 13, underline added]

2.11 Addendum

2.11.1 Solving for Mixed Equilibria in 2×2 Games

Recall Figure 2.1, our canonical game matrix for the 2×2 game in strategic form, which is reprinted below with a bit of relabeling and additional information: Row

	C_1 y	C_2 $(1-y)$
R_1 x	a_r, a_c	b_r, b_c
R_2 $(1-x)$	c_r, c_c	d_r, d_c

Figure 2.20: Canonical game matrix for the 2×2 game in strategic form

plays R_1 with probability x and R_2 with probability $(1-x)$. Similarly, Column plays C_1 with probability y and C_2 with probability $(1-y)$. The expected return for Row, G_R, is

$$G_R = xya_r + x(1-y)b_r + (1-x)yc_r + (1-x)(1-y)d_r \tag{2.24}$$

The equilibrium values of x, y, $0 < x, y < 1$ if they exists, are found by taking the partial derivatives $\partial G_R/\partial x$ and $\partial G_C/\partial y$, setting the results to 0, and solving. We have

$$\frac{\partial G_R}{\partial x} = ya_r + (1-y)b_r - yc_r - d_r + yd_r = 0 \tag{2.25}$$

Solving for y gives us:

$$y = \frac{d_r - b_r}{(a_r + d_r) - (b_r + c_r)} \tag{2.26}$$

The analogous calculation for G_C yields

$$x = \frac{d_c - c_c}{(a_c + d_c) - (b_c + c_c)} \tag{2.27}$$

If (and only if) the resulting values for x and y are legitimate probabilities, we have found a mixed equilibrium for the 2×2 game.

2.11.2 Replicator Dynamic Equilibrium for 2×2 Games

Given our standard 2×2 game matrix: the replicator dynamic equilibrium may be

	S_1 x	S_2 $\overline{x}$
S_1 x	A A	C B
S_2 $\overline{x}$	B C	D D

Figure 2.21: Canonical game matrix for the symmetric 2×2 game in strategic form

found by solving

$$Ax + B(1 - x) = Cx + D(1 - x) \tag{2.28}$$

which gives

$$x = \frac{(D - B)}{(D - B) + (A - C)} \tag{2.29}$$

2.11.3 Pareto Optimal Mixed Equilibrium

Consider a canonical 2×2 *symmetric* game under the replicator dynamic, with x the proportion of C players and $\overline{x} = (1 - x)$ the proportion of D players:

	S_1 x	S_2 $\overline{x}$
S_1 x	A A	C B
S_2 $\overline{x}$	B C	D D

Figure 2.22: Canonical game matrix for the symmetric 2×2 game in strategic form

Although the labeling here resembles that often used for the general Prisoner's Dilemma, the case at hand should be taken generally. The expected value of the game, G, for a player is

$$\begin{aligned} G &= Ax^2 + Bx(1-x) + C(1-x)x + D(1-x)(1-x) \qquad (2.30) \\ &= Ax^2 + Bx - Bx^2 + Cx - Cx^2 + D - 2Dx + Dx^2 \qquad (2.31) \end{aligned}$$

Differentiating

$$\begin{aligned} \frac{dG}{dx} &= 2Ax + B - 2Bx + C - 2Cx - 2D + 2Dx \qquad (2.32) \\ &= 2(A - B - C + D)x + (B + C - 2D) \qquad (2.33) \end{aligned}$$

Setting to 0 and rearranging:

$$x = \frac{(B + C - 2D)}{2((B + C) - (A + D))} \qquad (2.34)$$

which will be maximal when

$$\frac{d^2G}{x^2} = (A + D - C - B) < 0 \qquad (2.35)$$

Chapter 3

Computational Explanation

This chapter is an essentially unedited copy of my paper [50].

> *. . . what counts as an explanation has become more and more difficult to distinguish from what counts as a recipe for construction. [47, page 203]*

3.1 Kinds of Explanation

Scientific research reliably and routinely produces substantial progress within its several disciplines. Physicists increase our knowledge of physics, biologists advance biology, social scientists advance social science, and management scientists advance management science. Much more unusually, scientific research produces—invents or discovers—new ways of doing science. My topic in this paper is one such development. I aim to describe and characterize it in an introductory fashion and then to comment on what it means—on the opportunities it presents—for the management sciences.

The innovation I have in mind has been noticed by others, and indeed as we shall see has developed at least since the 19th century. It has not, as far as I know, been recognized with a commonly-accepted name, even though the idea has been accepted in at least parts of several disciplines. So to begin, I shall give it a name: computational explanation.

The core thought is that a computational explanation is an explanation that appeals to a computation to do its explaining. Phenomena are explained as resulting from a computational process. If the phenomena in question are produced by a computer program, then a computational explanation is surely one we would think appropriate, even mandatory.[1] What is new, and perhaps surprising, is the applicability of computational explanations in the biological, social, and management sciences. That is what I want to discuss.

There are other kinds of explanation and they have predominated in scientific study. Without pretending to cover the topic with any justice, it will suffice for present purposes to focus on two contrasting types of explanation: covering law, and axiomatic. Covering law explanations are most familiar from physics. A general rule or law is presented, typically in the form of a differential or other sort of equation, and given initial conditions the state of a system can be predicted and explained subsequently. Think of Newtonian physics. Such explanations are something of a "gold standard" in science; the problem outside of physics has been the scope of their application. If more could be found, that would be welcome; in the meantime progress accumulates under other standards.

In what I'm calling an axiomatic explanation, a system of axioms replaces the covering law. Often, the axioms are motivated as much or more by normative considerations as by empirical findings. Think of utility theory. An ideal kind of preference is axiomatized. With initial conditions we may then deduce—and explain and predict—behavior of an ideally rational agent. Much of economics and game theory follows this basic pattern of reasoning. The challenge is to abstract a reasonable representation, formalize a description of it axiomatically, derive properties of the formal system, and test the conformance of the formal system with particular concrete instances.

All of this is well and good. Why should anyone be interested in any other form of explanation? The general complaint has been that, at least outside of physics, the techniques of covering law and axiomatic explanation have been insufficiently productive. There is, many have thought, much more to be achieved with scientific thought than can be achieved with these two methods. In particular, the successes of computational explanation in its various forms are, on the positive side, what has so encouraged its pursuit.

How does computational explanation differ from covering law and axiomatic explanations? To a first approximation, a computational explanation is:

- Procedural

  In contract to declarative (both covering law and axiomatic explanations tend to be declarative).

[1] Recognizing, of course, the legitimacy of asking for explanations of non-computational features, e.g., the purpose of the program.

- Constructive.

  In some sense it answers to "If you can't make one, you don't know how it works." Each step in the procedure is doable by the entity in question; a mechanism is available for performing each of the steps in the procedure.

- Representational.

  Semantic, or meaningful. Basically, the procedure has an interpretation; its states carry information or meaning about the world outside of the system. Purely random arrangements do not, but once we have selection, there is (some) such information and meaning present.

  Computational explanations add to the more basic procedural or constructive explanations that they are about something. Computations aren't just procedures. They are (at least) procedures that are the way they are because of what they are.

These requirements are in order of increasing stringency. I'll try to make these points clearer in the sequel. We'll proceed by example.

3.2 Modeling the Firm: Cyert & March

Perhaps the earliest monograph—still read today—emphasizing a procedural view of management studies is Cyert and March's 1963 *A Behavioral Theory of the Firm* [16]. They open by announcing a perspective that has come to be known as, or associated with, the Carnegie School of thought on economics, organizations, and decision making.

> This book is about the business firm and the way it makes economic decisions. *We propose to make detailed observations of the procedures by which firms make decisions and to use these observations as a basis for a theory of decision making within business organizations.* Our articles of faith are simple. We believe that, in order to understand contemporary economic decision making, we need to supplement the study of market factors with an examination of the internal operation of the firm – to study the effects of organizational structure and conventional practice on the development of goals, the formation of expectations, and the execution of choices. [16, page 1, emphasis added]

The views they oppose and wish to offer alternative to are, or are associated with, neo-classical economics and its assumptions of ubiquitous knowledge and heroic rationality, and hence of the irrelevance to economics of decision processes by finite beings.

> The assumptions of rationality in the [neo-classical] theory of the firm can be reduced to two propositions: (1) firms seek to maximize profits; (2) firms operate with perfect knowledge. [16, page 8]

Cyert and March politely suggest that such views are unrealistic. Even more politely, they assert their interest in questions not addressed in—and they argue not addressable by—the neo-classical theory of the firm.

> Our conception of the task we face is that of constructing a theory that takes (1) the firm as its basic unit, (2) the prediction of firm behavior with respect to such decisions as price, output, and resource allocation as its objective, and (3) an explicit emphasis on the actual process of organizational decision making as its basic research commitment. [16, page 19]

Here are examples of questions Cyert and March propose to investigate with a behavioral theory, in contrast to the neo-classical theory.

> What happens to information as it is processed through the organization? What predictable screening Biases are there in an organization? What is the effect of conflict of interest on communication? What difference does time pressure make?
>
> . . .
>
> In general, there are many questions about the behavior of business firms but only a few answers. Existing theory is not equipped to answer most of the questions we have raised. Where an answer can be derived by brute force, it tends to be ambiguous or conspicuously inadequate. [16, pages 21–2]

In consequence

> . . . we need to consider in somewhat more detail the actual procedures used by business firms to make economic decisions. . . . [B]usiness firms adapt over time by learning a number of simple decision rules and procedures and . . . a behavioral theory of the firm should deal

> both with that adaptive process and with the procedural implications of long-run adaptation. [16, page 117]

Without saying these views of the Carnegie School have triumphed since 1963, we can say they are now widely accepted if not prevalent. The debate continues and is surely relevant to the subject of this paper, but it is not central for present purposes. What is central is their views on methodology and the structure of their results. First, a word on their methodology.

A Behavioral Theory of the Firm is quite self-conscious about its methods and approaches. Nearly as much space is given over to explaining what the results mean and why they are legitimate as is given to presenting the results themselves. A summary will have to suffice. Happily, Cyert and March are obligingly articulate.

> [This study] involved four major research commitments. They are commitments that evolved during the course of the research, but they constitute a general retrospective characterization of our research strategy:
>
> 1. *Focus on a small number of key economic decisions make by the firm.* In the first instance, there were price and output decisions; subsequently they included internal allocation and market strategy decisions.
> 2. *Develop process-oriented models of the firm.* That is, we viewed decisions of the firm as the result of a well-defined sequence of behaviors in that firm; we wished to study the decisions by studying the process.
> 3. *Link models of the firm as closely as possible to empirical observations* of both the decision output and the process structure of actual business organizations. The models were to be both explicitly based on observations of firms and subject to empirical test against the actual behavior of identifiable firms.
> 4. *Develop a theory with generality beyond the specific firms studied.* We wanted a set of summary concepts and relations that could be used to understand the behavior of a variety of organizations in a variety of decision situations.
>
> [16, pages 1–2; emphasis in the original]

Point 2 is central to their approach: actual processes matter. Point 1, that there should be "Focus on a small number of key economic decisions make by the firm," is a modeler's point. Simplifications must, can and will be made. Points 3, that the models should be more or less directly testable and tested empirically, and 4, that the behavioral models will generalize, can be seen as a "reputation bet" against the neo-classical tradition. That tradition is noted, at least in some quarters, for models of great generality and attenuated empirical validity. In any event, parsimony of focus, empirical validation, and generalization have to be seen on their own as desirable goals for this kind of modeling. Cyert and March are assiduous in making their case in this regard throughout the book.

Now to the structure of their results. *A Behavioral Theory of the Firm* reports on a series of detailed field studies undertaken with American firms in the 1950s. Cyert and March describe unusually intimate access to their subjects. Interviews were done, first-hand participant observations were conducted, and corporate archives were thoroughly read. As a result, a number of decisions were closely observed.

The resulting models typically have the following form:

1. A flowchart describing a decision process at a high level

2. Detailed, behavioral models of the (key) processes within the flowchart

Figure 3.1, above, restates one of their flowchart models, in this case a model of how a department store responds to sales information.[2] For key steps in the process, e.g., revising and using the reorder rules, Cyert and March offer rather simple formulas or if-then rules (depending on the case), which they arrive at by observation. They then apply the resulting detailed model to an independent data set and test the predictions. Here is a typical summary of results obtained:

> In order to test the ability of the model to predict the price decisions that will be made by the buyer on new merchandise, an unrestricted random sample of 197 invoices was drawn. The cost data and classification of the item were given as inputs to the computer model. The output was in the form of a predicted price. Since the sample consisted of items that had already been priced, it was possible to make a comparison of the predicted price with the actual price.

[2]This example was chosen for its simplicity. Most of the models are considerably more complex.

1. Form sales estimates
2. Advance orders
3. Observe sales feedback
4. Is sales goal being achieved?
    (a) If yes, go to 7
    (b) Else, continue
5. Search
    (a) Renegotiate constraints
    (b) Mark-down
    (c) New items
6. Revise reorder rules
7. Use rules to reorder
8. Re-evaluate sales goals and return to 1

Figure 3.1: General form of reaction to sales goal indicators. After Figure 6.1, [16, page 138]

> The definition of a correct prediction was made as stringent as possible. Unless the predicted priced matched the actual price to the exact penny, the prediction was classified as incorrect. The results of the test were encouraging; of the 197 predicted prices, 188 were correct and 9 were incorrect. Thus 95 percent of the predictions were correct. An investigation of the incorrect predictions showed that with minor modifications themodel could be made to handle the deviant cases. However, at this point it was felt that the predictive power was good enough so that further expenditure of resources in this direction was not justified. [16, pages 158–9]

In retrospect, *A Behavioral Theory of the Firm* offered not only a *procedural* theory, but a *computational* one as well. The behavioral models for the elements of the flowcharts appeal only to simple rules and formulas, e.g.,

> RULE 1 The estimate for the next six months is equal to the total of the corresponding six months of the previous year minus one-half of the sales achieved during the last month of the previous six-month period. [16, page 143]

The upshot is that Cyert and March were able to offer a theory of the firm, at least a sketch of one, that relied on sometimes complex arrangements (flowcharts) of typically rather simple computations (viz., RULE 1 above), and that came with impressive predictive power and generalization. An impressive achievement of computational modeling.

3.3 Molecular Genetics

Historians and philosophers of science have lately noted and much commented upon how biologists, especially in developmental biology and in molecular genetics, have increasingly resorted to computational models of biological systems.[3] In 1998, Yuh, Bolouri, and Davidson published an account of "an experimental analysis of the multiple functions of a well defined cis-regulatory element [the promoter region] that controls the expression of a gene [*Endo16*] during the development of the sea urchin embryo" [100]. Remarkably, "The outcome is a computational model of the element, in which the logical functions mediated through its DNA target site sequences are explicitly represented. The regulatory DNA sequences of the genome may specify thousands of such information-processing devices" [100]. The authors focus on module A of the promoter (controlling) region for Endo16. Here is where they find a molecular computer. A sense of how they think it works can be had by examining Figure 3.2 (published later).

[3] I am much indebted to Robin Fox Keller [47, cf. pages 239ff] for her discussion in this regard of several especially striking examples. The Yuh, Bolouri, and Davidson, and the Wray papers constitute one of her examples. The interested reader should consult her excellent monograph. My treatment of the examples here relies on the original sources and uses more recent informtion. For an early analytical treatment of genetic regulation (the lactose operon) see [53].

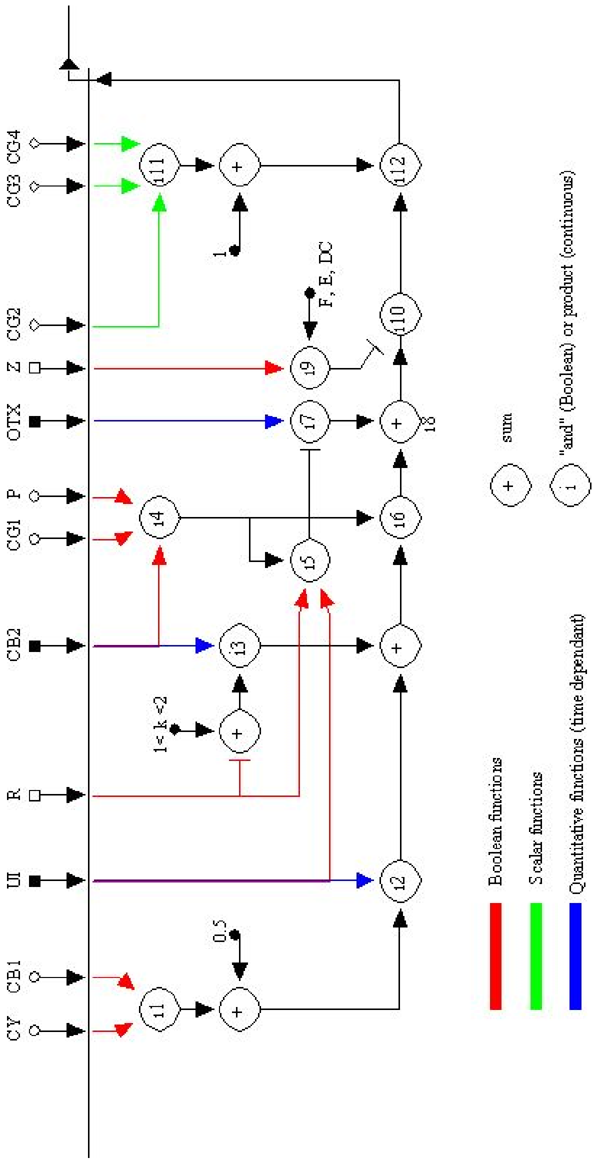

Figure 3.2: The Endo16 model of cis-regulatory control; from [101, 27 July 2002].

The paper finds that

> [T]he DNA sequence of module A specifies what is essentially a hard-wired, analog computational device. The requirement for this logic device is that there are many different inputs to the regulatory system that must be sorted appropriately. It is to us a remarkable thought that every developmentally active gene in the organism may be equipped with devices of this nature. ... The various functions mediated by module A are precisely encoded in the DNA sequence. Each target site sequence has a specific, dedicated function: Take the site away and the function is abolished; put it back in the form of a synthetic oligonucleotide and it reappears. [100]

Summing up, the authors write that

> The properties of the module A regulatory apparatus enable it to process complex informational inputs and to support the modular,polyfunctional organization of the *Endo16* cis-regulatory system. Perhaps the main insight from this experimental exploration is that these system properties are all explicitly specified in the genomic DNA sequence. [100]

That these are remarkable findings is confirmed by the presence of an editorial review of the piece appearing in the same issue in which it was published. The editorial, by Gregory A. Wray [96], serves to make the results of the research article more broadly accessible and to emphasize their significance. The following passage from Wray's editorial is methodologically significant for our purposes.

> To test their understanding of the *Endo16* promoter, Yuh and colleagues wrote a computer model that simulates these regulatory interactions. With the model, they made predictions about the consequences of specific promoter manipulations on transcription levels that were then tested experimentally. That these predictions were largely confirmed demonstrates not only an unusally complete understanding of how a particular promoter functions, but also the degree to which the *Endo16* promoter operates as an analog device. The "program that runs this tiny computer is directly in DNA as regulatory elements; its inputs are single molecules whose composition varies in time and among various cells of the embryo, and its output is a precise level of transcription.

The methodological commonality with Cyert and March is truly striking. Nor is it unusual in the biological literature. Many examples could be given. I'll close this section with an illustrative passage from a recent review article on morphogenesis. Notice how naturally the language of computation slips in, not as a metaphor, but as a description of how things are working.

> A major challenge now is to explore the possibility that there is also a conserved "morphogenetic code—a set of rules common to processes that are used repeatedly in different combinations to make functional organs. These instructions fall into two categories. First, there are basic subroutines that define essentially mechanical operations such as the packaging of cells into segments, the folding of epithelial sheets into tubes or cups, and the outgrowth of buds. Each of these modules utilizes sets of genes controlling properties such as differential cell adhesion, cell motility, cell-matrix interactions, and cytoskeletal organization.
>
> The second category determines how these subroutines are coordinated with cell proliferation and cell fate determination. This "project management depends on signaling centers that arise in the organ primordia or progenitor fields at positions initially determined by the primary embryonic axes. Each center is a group of cells that regulates the behavior of surrounding cells by producing positive and negative intercellular signaling molecules. [43]

3.4 Beginning with Darwin

Charles Darwin's scientific and conceptual achievements continue to be celebrated and probed in search of deeper understanding. It is by now standard to note that evolutionary theories of adaptation (e.g., Lamarck's and that of Charles's grandfather Erasmus Darwin) predated *The Origin of Species* (1859). Darwin's achievement was not to originate a theory of evolution. Rather it was (with Alfred Russell Wallace) to propose a workable (and largely correct) mechanism or *procedure* by which evolution comes about. In fact, Darwin did not use the term evolution in the first edition of the *Origin.* Instead, he consistently wrote of his theory of "descent with modification by natural selection." Most fundamentally (and sufficient for present purposes) Darwin (and Wallace) put together three ideas or observations:

1. *Profusion* of individuals with *variance* of traits

   Every species has the capability to, and tends to, produce more offspring each generation than can possibly survive and reproduce. The individuals so produced are not all identical.

2. *Selection* among the variants

   Natural selection operates on populations of varying individuals, selecting for properties favorable to survival and reproduction.

   The individuals in a species vary in many different ways, including their capacity, at least in expectation, for survival and reproduction.

3. *Reproduction* of variants favored by selection, with *inheritance* of favorable traits.

   Inheritance may be approximate and far from perfect. What is required is a heritable association between the traits favorable to the parents and the traits of their offspring.

Given such a regime—of profusion with variance of traits, selection by traits, and reproduction with inheritance—it is nearly inevitable that evolution or "descent with modification by natural selection" will occur. Darwin and Wallace claimed that in fact it did routinely and that this process has in the main been responsible for adaptation and speciation. Such, boiled down for present purposes, is the (biological) theory of evolution by natural selection.[4] So successful has this theory become that now when we say something is an evolutionary theory or account, we *mean* it appeals to, or posits, a profusion-selection-reproduction process.[5] More accurately, we describe such accounts as *selectionist.* I shall employ the two terms as equivalents.

Once you have an evolutionary or selectionist perspective (theory), scientific progress largely consists in applying the theory in order to create particular explanations and predictions. In this way the theory both becomes useful and receives support. Darwin excelled in using his theory to explain and account for biological phenomena. Among other things, he provided evolutionary explanations for the structure of the phylogenetic tree, for sexual selection, for the fossil record, for the expression of emotions, and much else. And generally his accounts continue to be seen as correct in the main.

[4]For a more detailed analysis see [49].

[5]With this qualification: reproduction may be weakened to mere survival, and profusion may be weakened to growth. These should be seen as limiting cases.

For present purposes, however, we are less interested in the biology and more interested in scientific methodology. From this perspective, Darwin's great achievement was to find a simple process (evolution by natural selection) that could be harnessed to model (for purposes, e.g., of explanation, prediction, and intervention) extremely complex and otherwise refractory phenomena (biology). Darwin offered a procedural model of adaptation and speciation. He did not offer a equational model or an axiomatic model (or indeed any other kind of model). Although the details of Darwin's procedure have been challenged and refined, serious models of adaptation and speciation since Darwin have not abandoned the core procedurality of the theory. (Gould's recent book [39] is an example of a serious effort to revise and refine evolutionary theory in biology.) Further, Darwin provided excellent examples of how to exploit a procedural theory. One observes the "pattern in evolution" (Eldredge's felicitous phrase, [24]), finds or posits special conditions and uses them in conjunction with the theory to account for the patterns.

Darwin (and Wallace) offered a procedural theory or model of biological adaptation and speciation. Lacking at the time a workable theory of how variation is produced and maintained, it might be questioned whether Darwin's was a constructive theory. (In his defense, variation, reproduction, etc. were clearly observable, even if their underlying mechanisms and causes were not well understood.) Nor could have Darwin conceived of the evolutionary process as computational, since that latter concept was unavailable in his time. The concept of computation is, however, available to us and we now turn to it.

3.5 Church's Thesis and Computation, Broad and Narrow

The fundamental concepts and properties of computation were developed and discovered before the Second World War and before anything like a modern computing machine was built. It is easy to agree that determining the output of a function on the natural numbers should count as a computation. Given a well-defined function, f, taking one or more natural numbers as arguments, the process of applying the function to a particular configuration of argument values and determining the resulting value of the function is by all accounts a computation. It is sufficient to be a computation. And it is also assumed to be necessary, since no one has come up with a broader notion of computation. I shall call this the *broad concept of computation.*

Given this notion of computation, we can ask after the design of machines or collections of primitive functions that will perform computations. Alan Turing famously described an abstract machine, now called the Turing maching, which is able to perform some computations. That is, it can be set up so as to determine the outputs for some functions on the natural numbers. Turing's simple machine is surprisingly general and powerful (neglecting speed and other details of implementation). Yet it is known, originally by what is called the *Church-Turing theorem*, that there are some functions (on the natural numbers) the Turing machine cannot compute. Moreover, we have *Church's thesis* which says, essentially, that the Turing machine is maximally powerful in the sense that no other machine (or collection of primitive functions) can compute a function that the Turing machine cannot.

The Church-Turing theorem has been proved and is not in doubt. Church's thesis is not specific enough to be amenable to proof, but it can be disproved by a single counterexample. Because no one has come up with one since it was proposed in the 1930s, the thesis is generally judged to be true, or at least to be a warranted working assumption.

It is important to appreciate how truly broad the broad notion of computation is. Take any discrete procedure at all. We may think of it as transforming the (discrete) states of a given system from one set of values to another. We can always encode the values of the relevant states as natural numbers. Given this, the procedure can be viewed as a function on the natural numbers. Thus, in a certain sense, everything that happens (discretely) is a computation. Conversely, we can see that there is nothing special about the natural numbers other than that they are convenient entities for being the subject of functions. We could have used jelly beans instead, except that there aren't enough of them.

3.6 Evolutionary Mechanisms

Evolution in Darwin's sense of variation and selective retention has come to loom larger in importance than even Darwin might have imagined. There are broadly two ways in which evolution or selectionism has expanded its scope beyond (merely!) explaining adaptation and speciation in biology.

First, evolutionary mechanisms have been found to underlie other biological processes, notably the immune response, and learning and cognition. A long-standing puzzle in immunology has been to explain how an immune system can react differentially to about 4×10^9 antigens (foreign bodies provoking a reaction), when there are only on the order of 10^5 different genes in the human genome. The answer, it was discovered in the 1970s, is (roughly) that immune cells individually reshuffle their DNA, which allows production of new antibodies. Selection occurs by reinforcement of those cells producing successful antibody. Thus, the Darwinian process operates within the body of an organism, over a period of days.

Can Darwinian processes underlie behavior and cognition? As Richards describes in his masterful study, *Darwin and the Emergence of Evolutionary Theories of Mind and Behavior* [77], the positing and search for evolutionary mechanisms underlying behavior and cognition began with Darwin himself and were expanded upon dramatically by, among others, William James. Behaviorism in psychology succeeded in discouraging such investigations from the beginning of the twentieth century until the rise of cognitive psychology, roughly in the 1960s. Since then, evolutionary or Darwinian theories of mind have been espoused with increasing enthusiasm by scientists from a range of fields, including pyscholology (e.g., Donald Campbell) and neurophysiology (e.g., William Calvin). Gerald Edelman, who won a Nobel prize for his contributions to immunology, has also been active.

The second way in which a Darwinian, selectionist perspective has proved valuable is in computational learning. As Holland (e.g., [44]) and others have shown, the simple genetic algorithm (GA) and its evolution programming variants are general-purpose computational algorithms. Evolution is a form of computation. Not only can we understand natural evolutionary processes as computations, but we can abstract and emulate them for practical purposes. A small industry—academic and commercial—has sprung up to do just that (see GECCO and other such conferences).

I assume that a manage science audience is familiar with GAs and evolution programs, and so I shall continue without discussing them in any detail. I do want to note, however, that not all *computational* theories of behavior and cognition are Darwinian. The following passage provides a convenient example.

> The idea that the representation of objects is essentially hierarchical and combinatorial was developed by the late David Marr in his influential 1982 book, *Vision.* Marr gave an account of how our brains process visual information, from the point at which light enters the eye to that at which we recognize the array of objects in the world around us; ... This account was *computational* in the sense that the processes of vision were conceived as a series of computations of the sort that might be carried out on a digital computer. It was also based

> on what was known of the neurophysiology of vision. The way in which we perceive and represent the visual world is far from fully understood, however, and Marr's theory is in many respects incomplete. Nevertheless it is a useful starting point for a discussion of how objects are represented in the brain. [15, page 219]

The goal of completing a computational understanding of behavior and cognition remains one of the grand challenges of science, and one of the great opportunities of applications of computing.

3.7 Evolutionary Economics and Game Theory

We have touched on computational modeling in biology and in psychology. A computational, and especially evolutionary, perspective has also been taken and proved useful in the social sciences. I will focus on economics and game theory, where most of the work has been done and is being undertaken.[6]

Nelson and Winter's important work, *An Evolutionary Theory of Economic Change* [70] is an appropriate focus for our discussion, in addition to their recent review article [71]. Along with many others, Nelson and Winter have argued that the preoccupation in neoclassical economics with equilibria gives short shrift to explanation of many economic phenomena. General equilibrium theory and neoclassical economics, it is claimed, simply are not up to accounting for dynamics and bounded rationality, and do not actually give accurate descriptions of economic systems. Nelson and Winter describe their alternative, evolutionary approach as follows.

> [T]he modeling approach that we employ does not use the familiar maximization calculus to derive equations characterizing the behavior of firms. Rather, our firms are modeled as simply having, at any given time, certain capabilities and decision rules. Over time these capabilities and rules are modified as a result of both deliberate problem-solving efforts and random events. And over time, the economic analogue of natural selection operates as the market determines which firms are profitable and which are unprofitable, and tends to winnow out the latter. [70, page 4]

[6] I note only in passing the popular concept of social evolution, based on what Dawkins called *memes*. This suggestive concept remains under development. I note, however, the related notion of "routines as genes" in Nelson and Winter [71].

I refer the reader to their review article [71] for an overview of the extensive literature on evolutionary economics appearing since their book in 1982.

Epstein and Axtell's 1996 book *Growing Artificial Societies* [28] is an apt exemplar for a now active body of work that nicely complements evolutionary economics, although its origins and motivations are somewhat different. Epstein and Axtell (and the related work) take bounded rationality for granted. Methodologically, they are committed constructivists and avidly subscribe to the constructivist's motto: "If you can't build one, then you don't understand it." Or, from the title of their book: "Social Science from the Bottom Up." (See also [27] for further reflections on constructivism.) This outlook coheres with, and lends support to, the general theme of emergence of surprising properties from (computationally) simple substrata that has captured the attention of so many on the leading edges of computational modeling and explanation.[7] Like Nelson and Winter (and the evolutionary economists generally), Epstein and Axtell (and that related literature) dispute many of the underlying assumptions of classical economics and present results (contrary or extending) that rely only on more realistic assumptions of what can be attributed to a finite agent. This is a rich, fruitful, and rapidly growing mode of research.

Finally, for our all-too-brief survey, evolutionary game theory has become a robust part of the study of games, along with classical game theory and behavioral game theory. (Samuelson provides a recent review from a rather classical or mainstream perspective see [80].) For our purposes, evolutionary game theory is concerned with

- The *behaviors* and especially *dynamics* generated by realistic strategies, and
- *Design principles* to support the fielding of artificial agents in strategic contexts

and assumes a sceptical attitude towards the adequacy of the Nash equilibrium as a solution concept for games played by actual agents.

Axelrod's work on iterated prisoner's dilemma (e.g., [2, 5]) is a paradigmatic example, although the relevant active literature is very large. It is known that there is exactly one Nash equilibrium in single-shot prisoner's dilemma and in prisoner's dilemma repeated a fixed (and known) number of times: mutual defection, to the detriment (hence 'dilemma') of both players. Also, if the game is to

[7] To mention just a few: Resnick and the StarLogo folks, SWARM, swarm intelligence, ant colony optimization, Wolfram and the cellular automata crowd, artificial life, natural computation [7].

be repeated an indefinite number of times, there are a large number of Nash equilibria. In the definite and indefinite cases, human subjects tend to obtain rewards much higher than mutual defection. Thus, classical game theory either predicts poorly (the definite case) or not at all (the indefinite case). And classical game theory has little to offer on how to play the game, in either case. Axelrod held a series of contests in which entrant's computer programs played iterated prisoner's dilemma against each other. The data for these tournaments, quite surprisingly, revealed properties of generally successful strategies and of the remarkable (but situationally conditioned) power of TIT FOR TAT. This is but one example of surprising and useful results obtained from evolutionary game theory.

As noted, classical game theory has little to offer regarding how to design and field artificial agents in strategic contexts (see [22] for elaboration of the point). In response, most researchers with an interest in the question have looked to learning computations—usually either evolutionary computation or some form of reinforcement learning, including learning classifier systems—to instruct their agents. If we don't know what to tell the agents, at least we can help them learn how to behave in strategic situations. Results to date are promising and manifold, but confined to the laboratory. Many of the studies have looked at standard game theory examples, such as prisoner's dilemma, stag hunt, or the ultimatum game (e.g., [102]). Results are beginning to appear for more realistic applications, such as supply chain management (e.g., [52]), and bidding at auction. The prospects for effective applications are genuinely encouraging.

3.8 Principles and Prospects

What does the foregoing intellectual history indicate for INFORMS researchers, researchers in Information Systems and/or Management Science? Where are relevant and promising applications likely to be found? Are there opportunities to contribute to science? In short, what work is there for us to do?

So far, I have introduced the concept of computational explanation, sought to clarify it enough for present purposes, and have proceeded by recounting examples. I now want to offer some organizing remarks in preparation for addressing the questions of the previous paragraph.

It is worth noting, in summary form, the academic disciplines that have been involved with computational modeling. What follows is merely a convenience sample, and an idiosyncratic one at that, meant only to indicate something of the breadth of the substantive contributions of computational modeling. (I apologize to all of those who should have been mentioned and aren't. Convenience is arbitrary.) Note that disciplinary distinctions here are especially problematic. To cite but one example, norms and conventions, are discussed by economists (e.g., [27, 98, 99, 97]), political scientists (e.g., [2, 3, 4, 5]), and philosophers (e.g., [8, 17, 87]). Also, connections among the disciplines are rampant, e.g., "The Biological Basis of Economic Behavior" [79].

1. Biology

   In addition to the papers cited above, Evelyn Fox Keller [47] has written an insightful and accessible monograph on computational models in biology, emphasizing embryology. Neurophysiologists, such as [23] and [12] have offered selectionist theories brain development and cognition. And most fundamentally, biology is the origin and home of selectionist theories of evolution.

2. Psychology and cognitive science

   Already mentioned is that selectionist accounts of cognition arose beginning in the nineteenth century and that detailed computational versions are in play today in biology (e.g., [12, 23]). Indeed, most of the field of cognitive science, and very much of artificial intelligence, can be understood as seeking computational models of cognition.

3. Economics (and commerce)

   The work of Nelson and Winter was discussed at length above [70, 71]. It has produced, or is associated with, a now large related literature. More recently, but also very productively, economists associated with the Brookings Institute and with the Santa Fe Institute have been active in exploring computational models of economic phenomena. See, e.g., [6, 28, 27], and [98, 99, 97]. Others have been active as well, e.g., [36].

4. Political science, organization science

   Robert Axelrod, political scientist at the University of Michican, has been a seminal contributor to computional modeling in the social sciences, cf., [2, 3, 4, 5]. Ian Lustick's work, e.g., [62], is an example of subsequent computational modeling in political science. In organizational science, I discussed Cyert and March's work at length, cf., [16]. A recent example in this tradition, drawing explicitly on developments in evolutionary explanation, is [57].

5. Philosophy

   A number of philosophers have been active in reflecting upon, and indeed producing, computational explanations. Much of the work is concerned with providing a naturalistic account of norms and conventions, e.g., Skyrms [87], Danielson [17, 18], and [8]. Paul Thagard, who holds a faculty position in both philosophy and computer science, has been active as a philosophically-oriented computational cognitive scientist, cf., Thagard [92, 93].

6. Computer and computational science

   Just as economists and game theorists have turned their attention to computational issues, so computer scientists and others with a computational science bent have begun to focus on game theory and economics, e.g., [81]. The recent DIMACS Workshop on Computational Issues in Game Theory and Mechanism Design,

   http://dimacs.rutgers.edu/Workshops/gametheory/

   featured a representative selection of this work.

   Complementing the applications and explanations in the various fields, there has been a flowering of invention and investigation of algorithms for computational modeling. Genetic algorithms are perhaps the best known, but they are only a special case of a larger class that includes evolution programming, genetic programming, learning classifier systems, and evolutionary computation generally. See the GECCO conference for recent work, e.g.,

   http://gal4.ge.uiuc.edu:8080/GECCO-2002/.

   Yet more generally, ant colony optimization, particle swarm optimization (PSO) [48], and various forms of "natural computation," algorithms inspired by naturally-encoun-tered (normally, biological) processes are very much in play [7].

Focusing now on the concerns of an INFORMS audience, what, we must ask, are we to make of all of this in the context of the applications-oriented world of management science and information systems? A small point is that this world has always recognized the distinction between, and the validity of both, theory and applications. Theory in this context has to include substantive scientific theory in biology, economics, political science, and so on. Our community should treat this as an opportunity to make contributions.

The larger point is that application opportunities are manifold. The basic applications methodology largely consists of exploring and modifying algorithms developed originally as computational explanations or abstractions of phenomena in the various sciences. Genetic algorithms are a clear example. I hope it is clear by now that they are just one of a rich family of algorithms originally developed as computational explanations or inspired by natural phenomena. There is much work to do to investigate (and even define) this larger class and to discover how it can best be used for practical purposes. Here, then, is a convenient classification of application areas pertaining to the management sciences. The comments are brief and meant to be only indicative of the attendant richness. Each taxon in the classification scheme—prediction, optimization, etc.—merits and receives the attention of entire careers. My aim here is to suggest and stimulate, not to pronounce and entomb.

1. Prediction

   It is surprising but true that despite many excellent techniques (e.g., statistical time series models), the prediction problem has not be definitively solved nor does progress on it appear to be exhausted. Koza [54] produced an early demonstration that evolutionary computation could be effective in finding good prediction models. Among others, Allen and Karjalainen [1] have applied evolutionary computation with success to stock market prices.

2. Optimization

   Genetic algorithms and evolutionary computing have long been applied to practical optimization problems. Goldberg's Ph.D. thesis under Holland [37] recorded an early and conspicuous success. Good reviews can be found in [13, 38, 68, 69]. Innovation and progress continues, cf., [51] and much else at the various GECCO conferences.

   Remarkably, during the last ten years several intriguing computational techniques, modeled on natural phenomena, have appeared, or been revived, and have shown much promise. These include ant colony optimization (e.g., [19, 21, 20]), particle swarm optimization (e.g., [48]), memetic algorithms (see [48, pages 245–255] for a not entirely favorable review; and [78] for a commerce-oriented application), reinforcement learning (often applied to game-playing agents; [91]), and various kinds of "natural computation" (e.g., [7]). These *families* of algorithms are only beginning to be explored.

3. Encoding

Rule-based expert systems were an early and moderately useful way to encode knowledge for subsequent use. Their range of application has been limited by the cost and trouble of "knowledge acquisition," the process of obtaining information in a form that fits the encoding scheme. Holland's "classifier systems" [11, 38, 44], a software architecture that facilitates automated knowledge acquisition, were conceived, in part, as a response to this problem. There is now an active community working on generalizations of classifier systems, the class usually referred to as Learning Classifier Systems (LCS), [94, 95, 56]. See the annual International Workshop on Learning Classifier Systems.

4. Design

Computational modeling is likely, I believe, to prove useful in design of supply chains, logistics systems, auctions, and generally in systems in which agents must confront strategic behavior. See [52] for a preliminary study on artificial agents and supply chains. The governing thought is that, in designing automated systems with strategic interactions, adaptive, learning artificial agents can be used to discover effective strategies and to test proposed system designs. See also [78] for an application in strategy formation. And there is much literature on design of game-playing agents, e.g., [29, 81, 30, 102]

5. Extraction

Just as expert systems have long been used to encode information, neural nets have long been used to extract information from, and to obtain classifications of complex phenomena. Such applications continue, along with underlying innovations that make the use of neural nets more effective (e.g., by using evolutionary computation to adjust connection weights). In addition, the broader classes of algorithms, already discussed, show much promise in this task. John Holmes, for example, reports good success with using Learning Classifier Systems to perform data mining [45]. Indeed, by whatever algorithmic method, data mining and text data mining are very promising application areas for the methods here discussed.

3.9 Summary and Conclusion

The computational modeling enterprise seeks to model naturally-occur-ring processes as computations, either broadly or narrowly defined. Model and, of course, support explanations, predictions, interventions, and all the purposes of science generally. It is not (primarily) the investigation of algorithms as abstract entities (as in computer science); rather it is the attempt to discover and understand computational, or algorithmic, processes in nature and practice. Computational modeling may be contrasted with equational and axiomatic modeling, and as such it presents a genuinely distinct mode of doing science and engineering, i.e., of seeking explanations and predictions, and of supporting interventions and innovations.

The successes of computational modeling reach back at least to Charles Darwin, who (along with Alfred Russell Wallace) originated a theory of biological adaptation and speciation as due to “descent with modification by natural selection,” of evolution as fundamentally proceeding by a process of variation and selective retention. Evolutionary theories—involving essentially processes of variation and selective retention—are characteristically procedural and (usually) constructive. They come with explanatory mechanisms that instantiate the variants on which selection acts. They are a special category of computational explanation: evolutionary computation. They are not, however, the whole story. Remarkably examples of a more general class have been discovered or invented of late and are showing much promise both for support basic science and for applications. Indeed, the two purposes are complementary.

It is an exciting time to be a computational modeler.

Acknowledgement

This material is based upon work supported by, or in part by, NSF grant number SES-9709548.

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Part II

Introducing Societies

Chapter 4

Players without Memory

Play Prisoner's Dilemma once, one on one, and the defector will always beat the cooperator. Suppose that a strategy converts to (imitates, is taken over by) its alternative if the alternative scores more points. Then a single play of cooperation against defection will produce two defectors. Does anything change if, instead of one pair of players, we have an entire society? Will the defectors conquer the (mirco)world? What happens in other games?

In this chapter we work with a simple and remarkably powerful style of modeling agents in *social* contexts of strategic interaction, and in doing so we begin to explore answers to these questions. We shall model a society as a collection of individuals arrayed on a two-dimensional ("territorial" [2] or "spatial" [41]) grid that I call the *gridscape.* Think of a checker board, with one agent on each square. Agents interact with—play a game with—each of their neighbors, count of up their points, and convert to (imitate, are conquered by) the strategy of the neighbor with the most points. Agents keep their existing strategies if none of the neighbors do better. (Ties are resolved randomly.)

To illustrate, this is a randomly-populated 12$\times$12 gridscape, which we'll call generation 0:

101001000011
110010000111
100001010001
011011000100
100000110110
011110111110
101011000010

101111011101
101001010010
111010110011
110100100010
011111110010

The two strategies in play—the two types of players—are represented as 1 and 0. Letting the agents play Prisoner's Dilemma with its default reward structure (see Figure 2.4, page 33)—

	Defect (0)	Cooperate (1)
Defect (0)	1, 1	5, 0
Cooperate (1)	0, 5	3, 3

—this is how strategies are distributed in generation 1 after one round of play and updating:

000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000

Cooperation has been extinguished in one generation!

Before continuing, three words on the details of the update process. First, in this example "neighborhood" was defined as the so-called *Moore* neighborhood: the eight immediately adjacent cells to the cell whose points are being tabulated. Below, that is the cell labeled C; the eight neighbors are N, S, E, W, NW, NE, SE, and SW.

NW	N	NE
W	C	E
SW	S	SE

Also popular in this sort of model is the *von Neumann* neighborhood of four cells:

	N	
W	C	E
	S	

We shall mostly, and unless otherwise noted, use the Moore neighborhood of eight. Second of our two words of detail, is to note that we assume (mainly for convenience) that every cell has 4 or 8 neighbors as the case may be. This is achieved by "wrapping" the grid: the column on the far right has as its right-hand neighbors the column on the far left. Similarly, the row on top has the row on the bottom as its northerly neighbor. Think of the gridscape as really having the shape of a doughnut. The third word is to note that each player plays each of its neighbors twice: once as the center cell (C) and once as a neighbor.[1] So, if player i is C and plays j to i's north, then when j has the role of C, i is the player to j's south. In the special case in which we have a *game in symmetric form* it happens that $V_R(i,j) = V_C(j,i)$: the value that the row player receives if row plays strategy i and column plays strategy j is equal to the value to the column player of row playing j and column playing i. Our Prisoner's Dilemma game—above and as played here—is a game in symmetric form. Figure 4.1 shows the general or canonical form for symmetric 2×2 games.

	S_1 x	S_2 $\overline{x}$
S_1 x	A (row), A (column)	B (row), C (column)
S_2 $\overline{x}$	C (row), B (column)	D (row), D (column)

Figure 4.1: Canonical game matrix for the symmetric 2×2 game in strategic form

[1]It is not necessary to play the games twice, simply convenient to do so from the point of view of implementation. Conceptually, each agent/cell plays one game with each of its neighbors in succession.

With these remarks to hand, it should be clear how to calculate the points for a cell. For example,

1	0	1
1	C=1	0
1	0	0

gives a value for C (starting with N and going clockwise) of $(0+3+0+0+0+3+3+3)\cdot 2 = 24$. (The presence of symmetry allows use to multiply by 2 and dispense with the second batch of 8 neighborly calculations.) It should be apparent that the point totals for generation 0 are

30.0	64.0	24.0	56.0	56.0	24.0	48.0	32.0	32.0	48.0	30.0	36.0
36.0	24.0	32.0	32.0	12.0	40.0	40.0	24.0	32.0	12.0	30.0	42.0
30.0	56.0	40.0	40.0	48.0	18.0	40.0	0.0	40.0	40.0	56.0	24.0
48.0	18.0	6.0	32.0	12.0	18.0	56.0	40.0	48.0	12.0	48.0	48.0
12.0	56.0	56.0	56.0	48.0	56.0	24.0	24.0	64.0	30.0	24.0	40.0
40.0	24.0	18.0	24.0	18.0	56.0	24.0	24.0	24.0	30.0	24.0	56.0
18.0	64.0	30.0	80.0	36.0	30.0	56.0	56.0	64.0	64.0	24.0	56.0
18.0	64.0	18.0	30.0	30.0	24.0	56.0	12.0	18.0	18.0	48.0	30.0
30.0	72.0	24.0	64.0	56.0	24.0	64.0	24.0	56.0	48.0	24.0	72.0
30.0	36.0	24.0	48.0	12.0	48.0	24.0	18.0	32.0	40.0	18.0	36.0
30.0	36.0	72.0	30.0	56.0	64.0	30.0	56.0	32.0	40.0	18.0	64.0
56.0	30.0	30.0	24.0	24.0	24.0	24.0	12.0	24.0	40.0	18.0	64.0

and that these produce the generation 1 gridscape of all 0s.

4.1 Inevitable Conquest?

After initialization, our gridscapes operate deterministically (except for ties). Is it certain that cooperators will be eliminated in Prisoner's Dilemma? Here are summary statistics from a typical run on a 200×200 gridscape, with 99% 1s at initialization.

Generation	0s	1s	Total Points
0	418	39582	1913228.0
1	3608	36392	1828420.0
2	8983	31017	1667816.0
3	14390	25610	1504112.0
4	20501	19499	1307108.0
5	25794	14206	1133364.0
6	30150	9850	986952.0
7	33556	6444	869640.0
8	35954	4046	785384.0
9	37434	2566	733044.0
10	38409	1591	698488.0
11	39057	943	675044.0
12	39378	622	663176.0
13	39519	481	658040.0
14	39609	391	654840.0
15	39669	331	652624.0
16	39700	300	651436.0
17	39711	289	650980.0
18	39711	289	650980.0

After generation 17 the system is static; no further changes occur. A few cooperators have survived. Why?

If we look at the gridscape for generation 18 we see mostly 0s, of course. The 1s appear in distinct patterns. Here is an example with labeling:

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	1	1	1	1	0	0
3	0	0	1	1	1	1	0	0
4	0	0	1	1	1	1	0	0
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0

A little calculation should convince you that this is a stable pattern: a rectangle of 1s (at least 3 deep) cannot be invaded by 0s. Here is the table segment with the 1s replaced by their point totals.

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	18	30	30	18	0	0
3	0	0	30	48	48	30	0	0
4	0	0	18	30	30	18	0	0
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0

In this (kind of) configuration, the largest number of 1s that a 0 can have as neighbors is 3, so the highest number of points a 0 can get is $(3 \cdot 5 + 5 \cdot 1) \cdot 2 = 40 = 2 \cdot (3T + 5P)$.[2] Each 1 is adjacent to a 1 surrounded by 1s, yielding $2 \cdot 8 \cdot 3 = 48 = 2 \cdot 8R$. So long as $8R > 3T + 5P$, cooperators can survive (assuming without loss of generality that $S = 0$). Satisfying this condition is not required for a Prisoner's Dilemma. Here are summary data from a typical run with $S = 0, P = 100, R = 101$, and $T = 103$ ($8R = 808 < 809 = 309 + 500$).

Generation	0s	1s	Total Points
0	394	39606	6.4020608E7
1	3430	36570	6.2332296E7
2	8836	31164	6.1245808E7
3	15374	24626	6.0852688E7
4	21922	18078	6.0959208E7
5	27755	12245	6.1415856E7
6	32406	7594	6.2057232E7
7	35670	4330	6.2734128E7
8	37732	2268	6.3273848E7
9	38895	1105	6.3601376E7
10	39522	478	6.3763824E7
11	39874	126	6.3904016E7
12	39981	19	6.398384E7
13	39998	2	6.3997288E7
14	40000	0	6.4E7
15	40000	0	6.4E7

[2]Recall that in Prisoner's Dilemma, P is the penalty for mutual defection, R is the reward for mutual cooperation, T is the temptation to defect, and S is the sucker's payoff.

4.2 Special Forms

Rectangular blocks of 1s, at least 3 wide and deep are indeed special forms. They are, in our present regime, safe from invasion by 0s. We can see this by experimenting with some non-random initializations.

Starting with this nonrandom generation 0 and the default Prisoner's Dilemma game values ($T = 5, R = 3, P = 1, S = 0$)

```
000000000000
000000000000
000000000000
000111100000
000111100000
000111100000
000111100000
000000000000
000000000000
000000000000
000000000000
000000000000
```

the system never changes. If we add a pair of 1s on the right the rectangle is filled out on the next round:

```
000000000000    000000000000
000000000000    000000000000
000000000000    000000000000
000111100000    000111110000
000111110000    000111110000
000111110000    000111110000
000111100000    000111110000
000000000000    000000000000
000000000000    000000000000
000000000000    000000000000
000000000000    000000000000
000000000000    000000000000
```

If we let a single 0 invade, disaster happens.

Generation 0:

```
000000000000
000000000000
000000000000
000111100000
000111110000
000011110000
000111100000
000000000000
000000000000
000000000000
000000000000
000000000000
```

Generation 1:

```
000000000000
000000000000
000000000000
000111110000
000001110000
000001110000
000001110000
000000000000
000000000000
000000000000
000000000000
000000000000
```

Generation 2:

```
000000000000
000000000000
000000000000
000000110000
000000110000
000000110000
000001110000
000000000000
000000000000
000000000000
000000000000
000000000000
```

And Generation 3:

```
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
```

(But not all invasions work. Which do not?)

Finally, here's what happens to a 4×4 initial block when $S = 0, P = 100, R = 101$, and $T = 103$.

Generation 0:	Generation 1:
000000000000	000000000000
000000000000	000000000000
000000000000	000000000000
000111100000	000000000000
000111100000	000011000000
000111100000	000011000000
000111100000	000000000000
000000000000	000000000000
000000000000	000000000000
000000000000	000000000000
000000000000	000000000000
000000000000	000000000000

And after that it's oblivion for the 1s.

One upshot of these remarks is that numerical values matter on the gridscape. Recalling §2.11.1, there are no mixed equilibria for the Prisoner's Dilemma, given the defining requirement that $T > R > P > S$. We have just seen, however, that whether cooperators survive or not on the gridscape depends on the particular values assigned to T, R, P & S. Gridscape, and more generally, structured interactions depart from equilibrium—notably replicator dynamic—interactions.

When default Prisoner's Dilemma is played with random initialization on a sufficiently large gridspace, indefinite survival of rectangular blocks of cooperators (1s) regularly occurs. In these blocks, each 1 has a 1 as a neighbor who is surrounded by 1s and gets $2 \cdot R \cdot 8$ points. When a line of 0s faces a line of 1s, the 0s (assuming they have 0s behind them) get $2 \cdot (3 \cdot T + 5 \cdot P)$ points each. The 1s will be protected from invasion so long as

$$2 \cdot R \cdot 8 > 2 \cdot (3 \cdot T + 5 \cdot P)$$

Without loss of generality, let $R = T - \epsilon$. Then the 1s are safe from invasion so long as

$$8(T - \epsilon) > 3T + 5P)$$

or

$$5T > 5P + \epsilon$$

At $T = 5, P = 1, \epsilon = 2 = T - R$ safety is assured. We saw, however, what happens when $S = 0, P = 100, R = 101$, and $T = 103$: the cooperators are wiped out.

Can it go the other way? For the 1s to expand at the expense of the 0s, it must happen that a 1 adjacent to a 0 gets itself more points in a round of play than are gotten by the 0 or any of its neighbors. If two homogeneous blocks of 1s and 0s meet on a line, the 1 will get $2(4 \cdot R + 4 \cdot S)$ and the 0 will get $2(4 \cdot T + 4 \cdot P)$. Since Prisoner's Dilemma requires that $T > R$ and $P > S$, the 1s cannot conquer territory in this case. Consider now the case in which a 0 with three 1s as neighbors abuts a 1 with five 1s as neighbors. The 0 gets $2(3T + 5P)$ points and the 1 gets $2(3S + 5R)$ points. Let $S = 0$, $P = 1$, and $T = R + 1$. The 1s can invade if

$$3T + 5 < 5R$$

or

$$R > 4$$

A case in point: $T = 7, R = 6, P = 1$, and $S = 0$. Here are the summary statistics from a typical run.

Generation	0s	1s	Total Points
0	5020	4980	558400.0
1	9535	465	197200.0
2	9487	513	201040.0
3	9108	892	231360.0
4	8563	1437	274960.0
5	7985	2015	321200.0
6	7412	2588	367040.0
7	6751	3249	419920.0
8	6041	3959	476720.0
9	5380	4620	529600.0
⋮	⋮	⋮	⋮

Generation	0s	1s	Total Points
⋮	⋮	⋮	⋮
27	963	9037	882960.0
28	965	9035	882800.0
29	1002	8998	879840.0
30	965	9035	882800.0
31	952	9048	883840.0
32	999	9001	880080.0
33	957	9043	883440.0
34	951	9049	883920.0
35	997	9003	880240.0
36	959	9041	883280.0
37	949	9051	884080.0
38	999	9001	880080.0
39	957	9043	883440.0
40	951	9049	883920.0
41	997	9003	880240.0
42	959	9041	883280.0
⋮	⋮	⋮	⋮

Generation	0s	1s	Total Points
⋮	⋮	⋮	⋮
86	999	9001	880080.0
87	957	9043	883440.0
88	951	9049	883920.0
89	997	9003	880240.0
90	959	9041	883280.0
91	949	9051	884080.0
92	999	9001	880080.0
93	957	9043	883440.0
94	951	9049	883920.0
95	997	9003	880240.0
96	959	9041	883280.0
97	949	9051	884080.0
98	999	9001	880080.0
99	957	9043	883440.0

Here is the gridscape for generation 99:

1111111110111111111111111111111111111111111110011111111111111111111111111110111111111111111111111111
1111111110011111111111111111111111111111111101111111111111111111111111111110111111111111111111111111
1111111110011111111111111111111111111111111101111111111111111111111111111110111111111111111111111111
1111111110011111111111111111111111111111111101111111111111111111111111111110000111111111111111111111
1111111110111111111111111111111111111111111101111111111111111111111111101111111000001111111111111111
1111111111101111111111111111111111111111111101111111111111111111111111101111111111110111110011111110
0111111111110000111111111111111111111111111101111111111111111111111111101111111111110111110000000000
1111111111110111011111111111111111111111111101111111111111111111111111101111111111110111111111111111
1100111111110111100000000000000000000000000000111111111111111111111111101111111111110111111111111111
1110111111110111111111111111111111111111111101111111111111111111111111101111111111110111111111111111
1111111111110111111111111111111111111111111101111111111111111111111111001111111111110111111111111111
1111111111110111111111111111111111111111111101111111111111111111111111101111111111110111111111111111
1111111111110111111111111111111111111111111101111111111111111111111111101111111111110111111111111111
1111111111110111111111111111111111111111111101111111111111111111111111101111111111110111111111111111
1111111110000111111111111111111111111111111101111111111111111111111111101111111111110111111111111111
1111111101111111111111111111111111111111111100001111111111111111111111101111111111110111111111111111
1111110111100111111111111111111111111111110111001111111111111111111111101111101111110111111111111111
1111101111110111111111111111111111111111101111000111111111111111111111100000000011110111111111111111
1111100111111111111111111111111111111111011111101111111111111111111111111111111111110111111111111111
1111111111111111011111111111111111111110111111111111111111111111111111011111111111111111001111111111
1110111111111111100111111111111111111110111111111111111111111111111111001111111111110111111111111111
1100111111111111111111111111111111111011111111111110011111111111111111110011111111111111111111111111
1111111111111111110111111111111111111101111111111111110111111111111111100000000111111111111111111111
0111111111111111111011111111111111110111111111111111101111111101111111101111111111111111110111111111111
0111111111111111111111111111111111101111111111111111100000000011111111110111111111111110011111111111110
1111111111111111111101111111111110111111111111111111101111111101111111110111111111111111110111111111111
1111111111111111111111111111111110111111111111111111011111111101111111111100111111111111111111111111001
1111111111111111111101111111111101111111111111111111011111111101111111111100111111111111011111111111011
1111111111111111111101111111110011111111111111111111011111111101111111111111111111111111001111111111011
1111111111111111111101111111011111111111111111111111111111011101111111111111111111111111110000000000011
1111111111111111111101111110111111111111111111111111111111101101111111111111111111011111111001111110011
1111111111111111111100000000111111111111111111011111111111110111111111111111111111001111111101111111011
1111111111111111111101111110111111111111111111011111111111110111111111111111111111111111111111111111111
0111111111111111111100111100011111111111111111011111111111110111111111111111111111001111011111111111110
0111111111111111111111111110001111111111111111011111111111110111111111111111111111011100111111111111111
1111111111111111111111111110000111111111111111011111111111110111111111111111111111100000011111111111111
1100111111111111111111111110001111111111111111011111111111110111111111111111111111110111011111111111111
1110111111111111111111111110001111111111111111011111111111110111111111111111111111011111011111111111111
1111111111111111111111111110001111111111111111011111111111110111111111111111111111011111101111111111111
1111101111111011111111111110000111111111111111011111111111100011111111111111111111011111101111111111111
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1111111111110111111111111110001111111111111111011111111111100000111111111111111111011111111111111011111
1111111100000111111111111111111000101111000000000000000000011111111111111111111111011111111111111101111
1111111110001111111111111111111000000110011111111111111110001111111111111111111111011111111111111110001
1111111110111111111111111111111000100011111111111111111111111111111111111111111111011111111111111110001
1111111110111111111111111111111111111111011111111111111111111111011111111111111111011111111111111110001
1111111110111111111111111111111111111111011111111111111111111111001111111111111111011111111111111111100
0111111110011111111111111111111111111111011111111111111111111111011111111111111111000011111111111111100
0111111101111111111111111011111111111111011111111111111111111111110111111111111111001111011111111111111110
1011111101111111111111111011111111111111011111111111111111111111110111111111111111011111101111111111111111
1111111110011111111111111011111111111111011111111111111111111111110111111111111111001111111110011111111111
1111111110000000000000000011111111111111011111111111111111111111110111111111111111111111111111110011111111
1111111111111111111111101111111111111011111111111111111111111111110111111111111111011111111111111001111111111
1111110111111111111111101111111111111011111111111111111111111111110000000001111111111111111111111000111111111
1111100111111111111111101111111111111111111111000000000000111011111111111111011111111111111111111111111111101111111
1111000011111111111111101111111111111111111111111111111000000111111110111111111111111111111111111111111111110011
0000111111111111111111101111111111111111101111111111111111111111100000000011111111111111111111111111111111111000
1011111100111111111111101111111111111111111111111111111111100111111111111111111111111111111111111111111111111111
1011111101111111111111101111111111111110011111111111110111111111111111111111111111111111111111111111111111
1011111111111111111111101111111111111111111011111111111111111111111111111111111111101111111111111111111111111111
1011111111111111011111110111111111111111111111111111111111111111111111111110011111111111111111111101111111111111111111111111111
0001111111111111000011111101111111111111111111111111111101111111111111111111011111111111111111111111111100
1111111111111111000011111101111111111111111111111111111100111111111111111111011111111111111111111111111001
1110111111111111111111111110111111111111111111111111111111100111111111111111111101111111111111111111111111111
1111001111111111111111110000011111111000111111111111111111111110111111111111111111101111111111111111111111001111
1111101111111111111111111010011111111100111111111111111111111111111111111111111111101111111111111111111111011111
1111101111111111111111111111111111111111101111111111111111111111111111111111111111101111111111111111111000111111111
1111101111111111111111111111111110011100111111111111111111011111111111111111111111101111111111111111000111111111
1111101111111111111111111111111110111111111111111111111111001111111111111111111111100000000000000000000000111111111111
1111101111111111111111111111111111111111011111111111111111111111011111111111111111111111111011111111111111111111111
1111101111111111111111111111111111111111101111111111111111111111011111111111111111111111111101111101111111111111111111111111
1111101111111111111111111111111111111111101111111111111111111111011111111111111111111110011111011111111111111111111111
1111101111111111111111111111111111111111101111111111111111111111011111111111111111111111111011111111111111111111111111
1111000111111111111111111111111111111111101111111111111111111111011111111111111111111111011111111111111111111111111111
1100111001111111111111111111111111100000011111111111111111111011111111111111111111111111111111110011111111011111111111111111111011
0011111101111111111111111111111111000111110111111111111111111111111111111111111111110111111111111111111111111111110000
1111111110000000000000000011111111111111011111111111111111111111111111111111111111101111111111111111111111111111000111

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1111111111001111111111111111111011111111111111101111111111111111100111111111111111101111111111111111011111
1111111110111111111111111111111011111111111110111111111111111101111111111111111111111111111011111
1111111110111111111111111111111011111111111111111111111111111111111111111111111111111111111111101111
```

Here is the crucial generation 1 gridscape, one round removed from a random 50:50 start, and the low point of the 1s:

```
0000000000000000000000000000000000000000000000000001111100000000000000000000000000000000000000110000
0000000000000000000000000000000000000000000000000000011000000000000000000000000000000000000000110000
0000010000000000000000000000000000000000000000000000000000000000000000000000000000000000000000010000
0000000000000000000000000110000000000000000000000100000001000000000000000000000001000000000000010000
0000000000000000000000001110000000000000000000000110000000000000000000000000000001000000000000000000
0000000000000000000000001110000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000001110000100000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000111000000000000000000000000000000000000000000000001100000000000000000000000000000
0000000000000000001111100000000000000000000000000000000000000000001000000000000000000000000000000100
0001000000000000000011000001000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000000000010000000001100000000000000000000111000000000000000000011100000000
0000000000000000000000000000000000000110000000110000000000000000000000000000100000000000000011100000000
0000000000000000000000000000000000000000000000001111000000000000000000001110000000000000000000000000
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0000000000000000000000000000000000000000000000000001100000110000000000000000000000000000000000000001111
0000000000010000000000000000000000000000000000000001000000000001000000000000010000000000000000001111
0000000000110000000001000000000000000000000000000001000000000001000000000000000000000000000000001111
0000000000000000000001000000000000000000000000000000000000000000011000000000000000000000000000000000000
0000000011000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000010000000000000000000000000000000000010000000000000000000000000000000000000000000111000000000
1110000000000000000000000000000000000000000010000000000000000000000000000000000000000000111000000001
1110000000000000000000000000000000000000000010100000000000000000000000000000000000000000111000000001
1110010000000000000000000000000000000000000010100000000000000000000000000000000000000000011000000001
1000000000010000000000000000000000000000000000000000000000000000000000110000000000000000000000000000
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000001000000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000000001000000000000000000000000000000000000000000110000000000000000000000
0000000000000000010000000000000000000000000000000000000000000000000011000000110000000000000000000000
0000000000000001110000000000000000000000000000000000000000000000000000000000110000000000000000000000
0000000010000001110000000000000000000000000000000000000000000000000000000000111000000000000000000000
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0000000000000000000000000000000000000000000010000011000000000000000000000000000000000000000000000000
0000000000000000000000000000000000001000000010000000000000000000000000000000000000000000000011110000
0000000000000000000000000000000000000000000000000000000000000000000000000000000000000110000001110000
0000000000000000000000000000000000000000000000000000000000000000000000000000000000001110000001110000
0000000000000000000000000000000010000000000000000000000000000000000000000000000000001110000001110000
0000000000000000000000000000000011100000000000000000000000000000000000000000000000000001000000000000
0000000000000000000000000000000011100000000000000000000000000000000000000000000000000000000000000000
0000000000000000000000000000010000000000000000000000000000000000000000000000000000000000000000000000
```

Remarkably, the 1s immediately crash to a low point of 465 out of 10,000 cells, then recover to consistently just over 9,000. In fact, at generation 32 the system enters a stable cycle, lasting 6 rounds.

4.3 Quasi-Battle of the Sexes

The Battle of the Sexes game (§2.6 page 40) is not symmetric, but can be transformed into a symmetric version I'll call the Quasi-Battle of the Sexes Game.[3] The version we'll investigate is the following.

	C (0) x	D (1) $\overline{x}$
C (0) x	2 2	3 [NP] 4
D (1) $\overline{x}$	4 [NP] 3	1 1

	C (0) x	D (1) $\overline{x}$
C (0) x	R R	S [NP] T
D (1) $\overline{x}$	T [NP] S	P P

Figure 4.2: Quasi-Battle of the Sexes ($\overline{x} = (1 - x)$)

In addition to the two noted Nash equilibria in pure strategies, there is a third Nash equilibrium (also a replicator dynamic equilibrium) in mixed strategies, with C (coded as 0 in the simulations of this section, since it is the more cooperative strategy) played with probability $x = \frac{3}{4}$. In general this equilibrium will be at

$$x = \frac{T - P}{(T + S) - (R + P)} \tag{4.1}$$

Notice that the two Nash equilbria—and Pareto optimal outcomes—in pure strategies cannot be achieved in the sense that it is impossible for everyone to play them. If everyone plays C (0) then C (00) is achieved and if everyone plays D (1) then DD (11) results. Only a mixture of D and C will produce equilibrium or Pareto outcomes. Further, note that if both players play C at the equilibrium probability of $\frac{3}{4}$ the expected return to each player is 2.5. That is not Pareto optimal, however. If both players played C with probability $\frac{5}{8}$ their expected returns would be 2.5625 each. In general the mixed Pareto optimum is at

$$x = \frac{T + S - 2P}{2(T + S - R - P)} \tag{4.2}$$

(Recall §2.11.3 page 49.) Thus, the replicator dynamic finds an equilibrium which is not on the Pareto frontier. How will the gridscape perform?

[3] Colman [14, page 110] calls this specific game the Battle of the Sexes.

Here are summary data from a typical run:

Generation	0s	1s	Total Points
0	7161	7239	575856.0
1	14227	173	467056.0
2	13706	694	470864.0
3	14063	337	468592.0
4	13636	764	471440.0
5	14072	328	468832.0
6	13604	796	472096.0
7	14002	398	469856.0
8	13586	814	472448.0
9	13988	412	469328.0
⋮	⋮	⋮	⋮
295	13955	445	469472.0
296	13536	864	472032.0
297	13955	445	469472.0
298	13536	864	472032.0
299	13955	445	469472.0

Note that the point total declines from the original 1:1 distribution of 0s and 1s. This run uses as 120×120 grid, so there are 14,400 cells in all. For convenience, each agent/cell plays 16 games (8 as initiator with its 8 neighbors and 8 as responder to each of its 8 neighbors). In generation 0 the grid produces 575856.0 points for an average of $2.499375 = \frac{575856.0}{14400\times16}$ points per game. In generation 299 we have entered an oscillating stable condition and in that generation the cells receive an average of $2.037638889 = \frac{469472.0}{14400\times16}$ points per game. Further, in generation 299 the percentage of 0s (Cs) is $0.969097222 = 13955/14400$. Under random play, as in the replicator dynamic, if both players played 0 (C) with this probability, their expected return would be 2.088888407 each. The corresponding return for generation 298 is 2.1656, but the players on the gridscape realized on avaerage only 2.04875. Things have gotten worse since initialization and generation 0.

4.4 Leader

Leader is very similar to Quasi-Battle of the Sexes:[4]

	C (0) x	D (1) $\overline{x}$
C (0) x	2 2	4 [NP] 3
D (1) $\overline{x}$	3 [NP] 4	1 1

	C (0) x	D (1) $\overline{x}$
C (0) x	R R	T [NP] S
D (1) $\overline{x}$	S [NP] T	P P

Figure 4.3: Leader ($\overline{x} = (1 - x)$)

This version has a mixed Nash equilibrium playing C (0) with probability = $\frac{1}{2}$. The general formula is

$$x = \frac{S - P}{(T + S) - (R + P)} \tag{4.3}$$

It has a Pareto optimal mixture playing C with probability = $\frac{5}{8}$, the general formula being the same as that for Quasi-Battle of the Sexes, expression (4.2).

[4]The specific game and its name are Colman's in [14, page 109].

The following summary statistics are from a typical run. The results are robust for random initialization with the probability of 0 ranging between 0.25 and 0.75.

Generation	0s	1s	Total Points
0	7226	7174	575616.0
1	3051	11349	365296.0
2	5782	8618	458256.0
3	6352	8048	482352.0
4	5566	8834	470672.0
5	5936	8464	483344.0
6	5356	9044	471296.0
7	5834	8566	478464.0
8	5360	9040	470256.0
9	5894	8506	480256.0
⋮	⋮	⋮	⋮
490	5086	9314	455024.0
491	6972	7428	493232.0
492	5053	9347	455232.0
493	6944	7456	490512.0
494	5149	9251	457728.0
495	6814	7586	489232.0
496	5113	9287	457072.0
497	6881	7519	488560.0
498	5239	9161	458096.0
499	6898	7502	490656.0

Note that in generation 499 the average cell earns a return of $2.129583333 = \frac{490656.0}{14400 \times 16}$. Again, the expected return at equilibrium under the replicator dynamic is 2.5. Note particularly, the sustained bias towards the 1s, defying both the predictions of the replicator dynamic (Nash) equilibrium and the Pareto optimal value.

Interestingly, if we set T to 400, R to 200, S to 300 and leave P at 1, the results we get are not much different. Summary data from a typical run:

Generation	0s	1s	Total Points
0	7245	7155	5.2168624E7
1	7142	7258	3.6738936E7
2	8419	5981	4.5269956E7
3	7590	6810	4.3500676E7
4	7735	6665	4.470886E7
5	7050	7350	4.2635604E7
6	7612	6788	4.4425012E7
7	6839	7561	4.1875956E7
8	7733	6667	4.4443012E7
9	6712	7688	4.163526E7
⋮	⋮	⋮	⋮
490	8176	6224	4.523676E7
491	6306	8094	4.0025192E7
492	8190	6210	4.520948E7
493	6330	8070	4.011558E7
494	8177	6223	4.5229964E7
495	6301	8099	4.0013264E7
496	8187	6213	4.5217892E7
497	6336	8064	4.011672E7
498	8179	6221	4.5228348E7
499	6300	8100	4.0024052E7

Note throughout the rather violent shifts between generations, averaging near the 1:1 ratio predicted by the replicator dynamic equilibrium. When, however, the 0s are in the majority, their number approaches the $\frac{5}{8}$ value of the mixed Pareto optimum. Using generation 498 as an example, $8179/14400 = 0.567986111$ while $5/8 = 0.625$.

4.5 Stag Hunt

Recall the Stag Hunt game (Figure 2.8, page 38), reproduced below:

	Hunt stag (0)	Chase hare (1)
Hunt stag (0)	3 [NP] 3	2 0
Chase hare (1)	0 2	1 [N] 1

Figure 4.4: Stag Hunt (aka: Assurance game)

This game, and Stag Hunt generally, does have a mixed equilibrium, which is also a replicator dynamic equilibrium. For our particular game, at equilibrium Hunt Stag occurs with probability $\frac{1}{2}$ and players get a return of $1\frac{1}{2}$ on average. This is only half of what they would get at the Pareto outcome, further reinforcing Stag Hunt's connection with dilemma. There is, however, no Pareto optimum in mixed strategies. Generically we have for Stag Hunt:

	Hunt stag (0)	Chase hare (1)
Hunt stag (0)	A [NP] A	C B
Chase hare (1)	B C	D [N] D

Figure 4.5: Generic Stag Hunt (aka: Assurance): $A > C > D > B$

If a mixed Pareto optimum exists it is at

$$x = \frac{B + C - 2D}{2((B + C) - (A + D))} \tag{4.4}$$

requiring

$$B + C - 2D \leq 2((B + C) - (A + D)) \tag{4.5}$$

or

$$B + C - 2D \quad \leq \quad 2((B + C) - (A + D)) \tag{4.6}$$

or

$$2A \quad \leq \quad (B + C) \tag{4.7}$$

which is impossible for Stag Hunt, given our assumption that $A > C > D > B$.

If you simulate the replicator dynamics for this game, the results are highly sensitive to the initial distribution of strategies. Starting with a 1:1 ratio of 0s and 1s, the system is driven in about half the cases to all 0s. In the other half of the cases the 1s conquer the population. If the initial ratio differs from 1:1, the favored strategy reliably goes to fixation. The equilibrium is not stable under the replicator dynamic.

Running this particular Stag Hunt on the gridscape, starting with a uniform (1:1) random distribution of 0s and 1s, we get in a typical run:

Generation	0s	1s	Total Points
0	7167	7233	344960.0
1	8715	5685	451224.0
2	12288	2112	599424.0
3	13891	509	668312.0
4	14306	94	686720.0
5	14392	8	690712.0
6	14400	0	691200.0

Conquest by the Stag Hunters! Even when the initial frequency of 0s (Stag Hunters) is very low, conquest will often occur, since small patterns such as

```
11111111111111111111
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11100011111111111111
11100111111111111111
11111111111111111111
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```

will expand to conquest.

What is an agonizing dilemma for an individual appears to be a no-brainer for this society. Stag hunting wins out decisively.

4.6 Chicken

Recall Chicken in its canonical form (page 39):

	Swerve (0)	Drive Straight (1)
Swerve (0)	2 [P] 2	1 [NP] 3
Drive Straight (1)	3 [NP] 1	0 0

Figure 4.6: Chicken

This instance of Chicken has a mixed Nash (and replicator dynamic) equilibrium at Swerving with probability $\frac{1}{2}$, giving each player in expectation $1\frac{1}{2}$. By alternating play with 01 and 10, the players could expect a return of 2. When simulated the replicator dynamic equilibrium is reliably approached and proves to be very robust to the initial distribution of strategies.

Here are summary statistics for a run with uniform (1:1) initialization on a 12×12 gridscape. What we see is a steady fluctuation about the 1:1 ratio of 0s and 1s (Swerve and Drive Straight).

Generation	0s	1s	Total Points
0	67	77	3288.0
1	11	133	624.0
2	29	115	1384.0
3	72	72	2968.0
4	91	53	3544.0
5	95	49	3680.0
6	94	50	3720.0
7	92	52	3568.0
8	87	57	3520.0
⋮	⋮	⋮	⋮

Generation	0s	1s	Total Points
⋮	⋮	⋮	⋮
189	80	64	3144.0
190	103	41	4008.0
191	73	71	2904.0
192	101	43	3976.0
193	74	70	2976.0
194	99	45	3840.0
195	79	65	3176.0
196	97	47	3816.0
197	70	74	2888.0
198	94	50	3864.0
199	61	83	2504.0

As we might expect from these numbers, the gridscape undergoes a great deal of change as its history unfolds.

Generation 3:	Generation 193:
111111000000	000000011000
111111000000	000000011100
100111000000	001110000000
110101000000	111110001111
110100000001	101110001111
011100000111	100011111111
011100011111	000011111110
000000011100	111111111111
000001111100	111111011111
000001111111	111100000011
111111111111	000000000000
111111000001	000000000000

Scaling up, we get these summary statistics for a run with uniform initialization on a 100×100 gridscape.

Generation	0s	1s	Total Points
0	4993	5007	239904.0
1	1024	8976	54320.0
2	3175	6825	141360.0
3	4532	5468	190752.0
4	5748	4252	230656.0
5	5690	4310	226128.0
6	6323	3677	247912.0
7	5854	4146	231000.0
8	6298	3702	245880.0
9	6089	3911	237648.0
⋮	⋮	⋮	⋮
990	6611	3389	253200.0
991	6601	3399	252600.0
992	6443	3557	248304.0
993	6385	3615	246792.0
994	6420	3580	247952.0
995	6402	3598	247200.0
996	6447	3553	248448.0
997	6469	3531	249288.0
998	6466	3534	248464.0
999	6458	3542	248112.0

Here is the gridscape for generation 999:

```
1111110000001100000001100000001111111001000001110000001111000001000000001000000111111111111111111
1111000000001110000001110000001101111100100000111110000000000100110000000000000000111111111111111111
0111000111011110000111110000000101111001100000111110000000000100001011111000000111000000100000000000
0111000111110000000111000000000000000000110000111000000000000000000111111110000011100000010000000000
0111000111000000000111000110000000000000110000111000000001101110001111110110000111000000100000000000
0011111111000000000000000110000000000000111011111000000001111110000000000010000111000000100000000000
0111111111000000000000000000000001110111111111111000000000000111000000000001110011100000100000000000
0111111110000000000000000110000101111111110100000110111100000000000000000000111001111111111110000000
0111000010000000000000001110000000000001110000000000100000000000000000000000000000111111111111110000
0111000011000010000111010000000100000000110000001110000000000000000000011101010000011111101111111100000
0111000000100000001111100000000100000000100000001110000000011110000000111111000001110000000011110000
0111000011100000000111100000000100000000000000001110000000111100000000011111100001110000000000000000
0000000011100000000000000001110000111001000000011100011100001000000000101111000011100000000011100000
0000000011100000000000000001100000111011000000011110011000001000000000000011100001111000000011111000
0000000111000000000000111000000000001110000000000111111000000000100000000000000000000000111000000011111000
0000000111000000100000111110000000111000001111011111111000000000000000000000000000000011100000001111110000
0011000010000011100111111000000001100000000000000111111000000011110000000000000000000011100000000000010000
0011000000111011100011111000000000010000000111000111111000000011110000000111111000000011100000011000111000
0000000111111001100011100000000000010000000110000000111000111011110000001111111110000111000011100011100
0000000111000000111111000000000110111111111100000000111000111100000000011111111100000011100011110000000000
0000000111000000111111000000111110000000100110000000111000010000000000111000000000000111000111101100000
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11111111000000001111011100111100000000100100000011110000111110000011100000000000011101111111111000000
11010111100000001000011100111100000000111110000011110000101111000011110000000111111111111111011100000
01010011100000001110000000111000000000111111000111111111100011000011110000000111111110100011001110000
00110011111001000010000000000000000001111111001100011000100001000011110001101111110000000011011000000
00110000011111111111000000000000011111111000000100000000100001000001110001111011000000000001111111000
11110011111111110001000000000000011111011100000011000000010000110000000001110000000000000001111111110
00100011111001011111000000111111111110110000000110000000100001000000000011100001100000001110000111110
000000111110110000000001111111100000000000000000000000000100011100001000011110001100000001110000011010
0000000111101100000000111101111110000000010111100001111111000111110110000011100011100000001000000011011
0000000111101000000000100100000000000000001011111000111111000011111010000001111001110000000110000011010
00001110001110000001110000000000000000001110001100011111000000001111000000111100111100000100000011010
101111100001100000111000010100000000000111100000000011100001000011111100000111110011100000000000000011
101111100000000000011100011111000000000011110000000000000000010000001111000011111100111000000000000000011
1111111000010000000000000100000000000000001000000000000000010000011101100011111100011100000000011111111
1100000000011000000000000000000000000000110000000000000001000000000000110001100001110000111111111110
00000000111111111111000000000011110000000001000000111000000100000000000001100011100011100001111111111110
0111000011111111101100000000000011000000001000000111000000100000010000000000001000011111111111111000000
01110000011111111000000001001100000000000001000000000000000001000000100000000110010000111111111111111000000
001000000111111110000000110010000100000000100011100000000111111111100000011011111111111111111111000000
0001111110000001000000011100000000110000001111111000011110000000111000000000111111000001111111110000
0011100010000000000011101110000000000011000111111110000011111100001111000000000101111100000111011111000
000111110000000010001111111100000000001111100001100000000111100000000111000000001000111000001110001111000
00001110000000001001111111000000000011111100000000000001110000000000010000000110000110000000000001111000
01000000001100110111011100000000111111100000000000000011100000001100000000011110011000111100000000000
1100000000111111011101111110000011110110000000000000011100000000111111101011110000011111100000000000
0100000000011000000000001111000000111100100000000000000111000000000110000011001111100011111100000000000
0100000000010000000000000111100001100000100000000000000111100000000110000010000001000000111000000000000
010000000001000000000000010000000000000010000000001100111111100011100000010000110000000000000000010111000
0100000000110000011000000010000000000111010000000011001111111000111100000100001100000000000000001111111
000000000001110001100000001000000000011101100000000000000001100001110110001000000000000000000000010111111
0010000000001100010000000010000000000111010001000000000000001000000000111111000000000000000011110000111
00100000000000000000111000010000000011100111011100000000000011110000000011000110000000001111000111110000111
11100000000000011101110000100000001011110111111000000000001101000000000000001000000111111000111110000111
111000000100110000111000010000000000000011000100000000001100001110110000011111110011111100111110000111
111000000110010000000110000000000100000000000000100000000000000110100110000011111110011111100111000000001
1110101111000000000110111000000000000000000000001000000001100001111111100000111001100111111001110000000000
1110111111001000000001111111111111000000000000000101111111100010011111100000000001111111100001100000000100
000011111100000000000100110111111000011011111100001000011100000011111100000000011010000000011011100000100
0000010001000000000000001000111000011000010000010000111110000001111010100000000000000000000001110000100
0000001100000000000000000111001110000000000001000001000000011000000111000111100000000000000110001110000100
0000001100000000001110001110001110000001000010000010000000000000001110000001100000000011000110000100000100
0000001100000000001110000100001111100100000100000100000111000000111000111100000000111111011101000000100
000000110000000000000000100000000110000000001000001000000110000000000000011011111111001011101111100000100
00000001100000000011001100000000110000000010000010000011011111111000000000000111110010111011111100000100
00000000100000000011011011000000000000000001000001000001111111111000000000000001111000001100000111100100
0001110000000000001100011110000000000000000110000100000000111111100000000000000000111111111000000011100000
0011000000100000000111100111110000000000000001111100100000000000000000000000000000000001000001000000010111000
000100000001011000111100111000000000000111111111001000000010000000000110011000000000000000100000000000011000
00000000000011000111100111000000000001111111111001000000010000000000111111000000000000010011111111101111000
0010000000000000011100011100000000000011111111001100000001000000000000011100000000000000110010000001110110000
00100000000000001100000011100000001000000000111001000000000100000000000011000000000001110000000001000111000
110000000000000010000111011111111110000000000100011000000000011111111000111000011111111111111000010000000011
010000011010001110001110111111110000000000010111100000000000000100110001110000110000000011000000000000000
00000000111110011100011111111111111001100000001000110000000000000010010000111000000000000000011000000000000000
011000000011001110000011110000000001111111110001100000000000000100000000111000000000000000100000000000000000
111000001111111110000011110000000000000000100011011100100000100000111110000000011000011000111110000000
11111111111111111110000001000000000000000000011110111001100000100001111100000000111000010001111100000000
111110000111111111110000000000000000000000000010111000100000111110011100000000111000001111110000000010
1111100001110111111110000001000000000000000000000000001000000100000111111111111110000000110000000111111
111110000111011011111000000100000001000000000000000000000001000000100000111111111111110000000000000000111110
1111100000111100011110000001000001100000000000000000000000001000000111111111111111110100000000000000000010000
1110000000001110001111000001110000001000000000000000000111000010000001111111111100000000110011111000000000111000
0000000000001100011000011111000000100000000000000111100001000011011111100000000000100001111000000000111000
1100000000001110011111111111100000100000000110001111000010000111100111000000000100000111000000000111001
110000000000111110000011110000000010000000011100111100001000000111111110000000000000000111000000111111101
1100000000111111100000001000000000001001000000101000000110000000001001111111000000000110000011100000111111111
110000000001111100000000100000000001111100000011100011000000101000111110100000000100000111000000000000111
100000000000011111000000010000000001111000000000111000100000001110000000101110000001000000111000000010000111
10000110000000011100000110011111111000000000000111000000000001000000001110100000000100000111000000010000111
100001000000000111000001100111111111110100110000000000000000001000011111100111000011000001110000000010000001
11000000000111111000001100111111111111011111000000000000111111100111111001111100100000001100000010000001
1000000001111111100000000000000000000111010011000000000000111111100111111001111100100000000110000000000001
11100000001111111100000000000000000011111100000001110000011100000001011101111110101110000000000111110000011
0110000001110100111000000000000000001111110000000111000001110000001111000111111111110000000001111110000011
00000000111111011100000000000000000111000000111011100000111000000111000011111111111100100001111111100000000
00000010111011111000000000000100001110000001111100000001110000001000000000000111000000110000111000000000000
0000011100000000111000111111111001110000000111110000011111000000000000000000111000000111000111000000000000
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0111001000000000000000111000001100111000001000000000011111111010000000000001110000011100011100000000000
11111100000000000000011100000000011100000100000111001011111111100000000001110000001111111111111111
```

Here is generation 999 from a different run of a 100×100 gridscape:

```
0000001110011100000000000000000110000000000000000011111000000000000000000000000000001000001100000000011111
11111111100111000000100000001110001111011111001100110111100000000000000000000010000001000001100011111
11111111000111000000100000001110001111101101110000000100000000000000000000001000001100000110000011 1
0011111100011100000010000000000000101100000001000000011000000000000111110110000100000000000001110
00000001110111001110000001110000000001100000000000000001100000001111001111011100000000011000000001110
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01000011100111110000111111000000111000000000100000011111100000000000000100000110011000000111111110000011111
00000011100111110000001110000000111000000000100000011001000000011111000000000000000110000111111110000011111
```

And here are the summary statistics from this run:

Generation	0s	1s	Total Points
0	1001	8999	60896.0
1	5228	4772	220096.0
2	4783	5217	202640.0
3	5403	4597	225216.0
4	5273	4727	216456.0
5	5562	4438	225632.0
6	5713	4287	229312.0
7	6014	3986	238584.0
8	5942	4058	234768.0
9	6074	3926	238792.0
⋮	⋮	⋮	⋮
990	6353	3647	243680.0
991	6678	3322	255784.0
992	6385	3615	245800.0
993	6673	3327	255712.0
994	6187	3813	240272.0
995	6480	3520	250464.0
996	6204	3796	240736.0
997	6699	3301	256384.0
998	6298	3702	243312.0
999	6663	3337	255960.0

The runs give indistinguishable results, yet the first began with a 1:1 ratio of 1s and 0s, while the second began with on 10% 0s (see table above, generation 0). Suppose instead we start with 10% 1s and 90% 0s uniformly (randomly) distributed? Here are the summary statistics from a typical run.

Generation	0s	1s	Total Points
0	8995	1005	316768.0
1	3866	6134	165128.0
2	5773	4227	238064.0
3	5119	4881	207528.0
4	6032	3968	239632.0
5	5780	4220	229704.0
6	6243	3757	244272.0
7	6051	3949	236880.0
8	6310	3690	246984.0
9	6069	3931	238232.0
⋮	⋮	⋮	⋮

Generation	0s	1s	Total Points
⋮	⋮	⋮	⋮
990	6456	3544	248864.0
991	6499	3501	249888.0
992	6449	3551	248016.0
993	6501	3499	249456.0
994	6741	3259	257048.0
995	6377	3623	245624.0
996	6724	3276	256344.0
997	6464	3536	248072.0
998	6583	3417	253272.0
999	6422	3578	247664.0

And here is the gridscape for generation 999:

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0001100000000000000000000000000000000000000000011000111111110100111000000001100001000011100011111110
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0000001100000011100000000000000000100000011111110000000000000001011111111100001100001111000011000011110
```

Again, the results are indistinguishable from the earlier runs.

Consider now another version of Chicken: As always, there is a Pareto optimal

	x Swerve (0)	$\overline{x}$ Drive Straight (1)
x Swerve (0)	A=2 [P] A=2	C=4 [NP] B=1
$\overline{x}$ Drive Straight (1)	B=1 [NP] C=4	D=0 D=0

Figure 4.7: Chicken Version 2: $C > A > B > D$

mixed strategy if $2A < B + C$. Version 2 of Chicken, above, is like version 1, except that C has been increased from 3 to 4. At these values there is a Pareto mixed optimum at

$$x = \frac{B + C - 2D}{2((B + C) - (A + D))} = \frac{5}{6} \tag{4.8}$$

The replicator dynamic (and mixed Nash) equilibrium is at

$$x = \frac{D - B}{A - B - C + D} = \frac{-1}{2 - 1 - 4} = \frac{1}{3} \tag{4.9}$$

Here are summary statistics from a typical run:

Generation	0s	1s	Total Points
0	7212	7188	403680.0
1	79	14321	5828.0
2	452	13948	24628.0
3	1098	13302	51312.0
4	2008	12392	89144.0
5	2959	11441	126104.0
6	3958	10442	165500.0
7	4800	9600	198528.0
8	5407	8993	224600.0
9	5720	8680	242596.0
⋮	⋮	⋮	⋮
490	3255	11145	192348.0
491	3281	11119	193408.0
492	3271	11129	191252.0
493	3345	11055	194664.0
494	3156	11244	188940.0
495	3322	11078	195704.0
496	3453	10947	202020.0
497	3050	11350	183100.0
498	3438	10962	201504.0
499	3236	11164	192088.0

Note the decline in total points after the initial random start. The system reliably settles down, as we see. The percentage of 0s in generation 499, $3236/14400 = 0.224722222$ is not near Nash and very far from Pareto.

4.7 Hawk-Dove

The generic Hawk-Dove game (Figure 2.5, page 34), is copied below. There is

	H (0)	D (1)
H (0)	$\frac{1}{2}(V-C)$ $\frac{1}{2}(V-C)$	0 [P] V
D (1)	V [P] 0	$V/2$ [P] $V/2$

Figure 4.8: Hawk-Dove Game: $C > V > 0$

a replicator dynamic mixed equilibrium at playing H with probability $\frac{V}{C}$. This is not changed if we add a constant, K, to each return.

	H (0)	D (1)
H (0)	$\frac{1}{2}(V-C)+K$ $\frac{1}{2}(V-C)+K$	K [P] $V+K$
D (1)	$V+K$ [P] K	$(V/2)+K$ [P] $(V/2)+K$

Figure 4.9: Hawk-Dove Game with arbitrary constant K: $C > V > 0$

Letting: $V = 10, C = 12, K = 1$ we get the concrete payoff matrix:

	Hawk (0)	Dove (1)
Hawk (0)	0, 0	11, 1
Dove (1)	1, 11	6, 6

At the replicator dynamic equilibrium the Hawk strategy is played with probability $\frac{V}{C} = \frac{5}{6}$. Simulation of the replicator dynamic robustly approximates this probability. Similarly, letting $V = 100, C = 200, K = 50$ reliably and robustly results in Hawk being played with approximately probability of $\frac{V}{C} = \frac{1}{2}$.

Matters are different on the gridscape. At $V = 100, C = 200, K = 50$, here are typical summary data from a run. The results are robust and the initial random proportion of 0s and 1s does not, within a broad range, much influence the outcome.

Generation	0s	1s	Total Points
0	7559	2441	6848400.0
1	5512	4488	1.00748E7
2	6838	3162	7569200.0
3	5451	4549	1.00144E7
4	5599	4401	9590400.0
5	4996	5004	1.0508E7
6	4751	5249	1.08724E7
7	4531	5469	1.10944E7
8	4516	5484	1.10756E7
9	4106	5894	1.1706E7
10	4199	5801	1.15308E7
11	3738	6262	1.22172E7
⋮	⋮	⋮	⋮
990	3689	6311	1.21924E7
991	3403	6597	1.26508E7
992	3611	6389	1.23052E7
993	3601	6399	1.23488E7
994	3416	6584	1.25616E7
995	3501	6499	1.24712E7
996	3518	6482	1.24476E7
997	3371	6629	1.26896E7
998	3579	6421	1.2368E7
999	3456	6544	1.2614E7

At the replicator dynamic equilibrium the probability of playing Hawk is $\frac{1}{2}$; on the gridscape it stabilizes near $\frac{1}{3}$. A player in a replicator dynamic regime can expect a return per play of $\frac{1}{2}(\frac{1}{2}(0) + \frac{1}{2}150) + \frac{1}{2}(\frac{1}{2}50 + \frac{1}{2}100) = \frac{1}{4}(300) = 75$. A player in this gridscape society can expect at stability roughly $\frac{1}{3}(\frac{2}{3}(150)) + \frac{2}{3}(\frac{1}{3}(50) + \frac{2}{3}(100)) = \frac{2}{9}200 + \frac{4}{9}100 = \frac{800}{9} \approx 88.9$. The imposition of the gridscape regime has resulted in higher average returns to the players.

Here is generation 999 from this run. Notice, once again, the emergence of organization highly distinct from a random distribution of the strategies.

1111110011111111111111111111000111110000110001110000111111000000000000001011111110011111111011111011

0000110000111111111111111111110111111000011000111000011011100000100001011100111111001000000000111111011
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Things go differently with $V = 10, C = 12, K = 1$ (and $\frac{V}{C} = \frac{5}{6}$). Here are the summary data from a typical run.

Generation	0s	1s	Total Points
0	2472	7528	901296.0
1	9236	764	121488.0
2	9575	425	63768.0
3	9754	246	37752.0
4	9840	160	24624.0
5	9861	139	20520.0
6	9858	142	20736.0
7	9877	123	18288.0
8	9860	140	20208.0
9	9890	110	16464.0
10	9894	106	16032.0
11	9906	94	14352.0
12	9917	83	12768.0
13	9920	80	12480.0
14	9938	62	9792.0
15	9946	54	8424.0
16	9946	54	8424.0

The Doves are all but wiped out. Only 54 Doves out of 10,000 players survive indefinitely. A look at generation 15, the first "stable" generation, is revealing. Notice the 9 cases of 6 Doves (1s) in various rotations.

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These *6-pointer* emergent patterns of Doves have their own dynamic, which can be followed over generations.

Generation 0	Generation 1	Generation 2
0000000000	0000000000	0000000000
0000000000	0000000000	0000000000
0000000000	0000000000	0000000000
0000000000	0000000100	0000000000
0001111000	0000000100	0000000000
0000011000	0000001100	0000000000
0000000000	0000001100	0000011000
0000000000	0000000000	0000011110
0000000000	0000000000	0000000000
0000000000	0000000000	0000000000

Generation 3	Generation 4
0000000000	0000000000
0000000000	0000000000
0000000000	0000000000
0000000000	0000000000
0000000000	0001111000
0000110000	0000011000
0000110000	0000000000
0000100000	0000000000
0000100000	0000000000
0000000000	0000000000

The surviving 6-pointer patterns of Doves (1s) rotate clockwise 90° each generation. Notice that in generation 2 *all* of the locations occupied by Doves in generation 0 have been taken over by Hawks, albeit temporarily. Once established in a field of Hawks, this emergent pattern of Doves cycles forever.

The interest of these observations is not diminished by adding labels to the Hawk-Dove game in strategic form (C_l is a *label*, not to be confused with the outcome parameter C):

	Hawk (0)	Dove (1)
Hawk (0)	$\frac{1}{2}(V-C)=D$ $\frac{1}{2}(V-C)=D$	$0=B$ [P] $V=C_l$
Dove (1)	$V=C_l$ [P] $0=B$	$\frac{V}{2}=A$ [P] $\frac{V}{2}=A$

Figure 4.10: Hawk-Dove Game: $C > V > 0$

and noting that $C_l > A > B > D$. Hawk-Dove is a game of Chicken.

4.8 Observations

These simple models demonstrate and suggest much. It is time now to pause for summary and interpretation.[5]

4.8.1 Brief Points

1. We have examined symmetric 2×2 games in some detail, comparing outcomes of play under the replicator dynamic with outcomes of play on the gridscape.

   We used a very simple regime on the gridscape—we might call it the *basic gridscape regime*. Under this regime every cell on the gridscape plays each of its (4 or, normally, 8) neighbors twice, once as initiator and once as responder, without memory of previous play. Points are totaled up and each cell keeps its strategy if it equals or exceeds all of its neighbors, and failing that converts to the strategy of its most successful neighbor, ties being resolved by chance. This completes one generation of play. Subsequent generations ensue until a stopping condition is met, usually a specified maximum number of generations.

   Under the replicator dynamic regime each strategy is represented as a portion of the population. Strategies are drawn at random according to their proportion in the population during a given generation. Pairs of strategies play against each other and the return each gets is accumulated in association with the strategy producing it. Each generation consists of a number of such plays, 5000 in the studies for this chapter. At the end of a generation, a new population is constituted with the proportions of strategies in accordance with the total returns obtained in the previous generation. The replicator dynamic thus models in a basic way an evolutionary dynamic: strategies obtaining comparatively higher returns from their play increase in frequency.

2. The strategies we investigated are maximally simple. They have no memory of previous play, they are not reactive or conditional in any way, they do

[5] All of the gridscape results reported in this chapter were generated by the program Symmetric2x2M0GridScape.java, v 1.2. Replicator dynamics simulation results were generated by ReplicatorDynamic2x2.java, v 1.2. Both programs available from the author at http://opim-sun.wharton.upenn.edu/~sok/agebookcode/.

not benefit from learning. Yet complex patterns are produced deterministically from random starts. Will this change, and if so how, as we investigate more realistic models, with more intelligent agents, with post-initialization elements of chance, and so on?

3. The results are often surprising. Even granting—rather a stretch—that careful reflection on these games would yield the results seen here, I think it has to be said that reflection would have produced surprising results. This should–and will—occasion some reflection on the concept and nature of rationality. If Prisoner's Dilemma is to be played once and outside of a society, I suppose we would all choose to play 0, Defect. What if you had to bet on the 1s or the 0s on balance in an evolving society on the gridscape?

4. The gridscape systems often display optimum-seeking behavior. In the Prisoner's Dilemma run described beginning on page 86 at initialization the grid produces 558,400 points for the players. The Cooperators immediately undergo a near-catastrophic population decline. At their low point the gridscape yields only 197,200 points. As the Cooperators recover productivity of the gridscape increases. By generation 99, 90% of the players are Cooperators and the gridscape yields 883,440 points. In Stag Hunt, we saw for the standard game that the gridscape was entirely conquered by those who would Hunt Stag. This extracts the maximum possible value from the system. The gridscape regime did not do especially well in maximizing value in the game of Chicken we investigated. It was able to recover from an initial excess of those who would Drive Straight, but it failed to exploit an initial excess of those who would Swerve. Finally, in one version of Hawk-Dove the Hawks nearly conquer the gridscape, yielding a net value well *below* that obtainable at the replicator dynamics equilibrium (RDE). Yet in another version, the gridscape extracted significantly more "wealth" than is available at the RDE.

5. Reward quantities matter. Two specific games may satisfy the requirements for Prisoner's Dilemma, Hawk-Dove, and so on. Their Nash equilibrium, Pareto, and replicator dynamics properties will be identical or very similar. Yet they may produce very different results on the gridscape.

6. Our method of simulating—of executing play on the gridscape—is useful for making discoveries. For example, it might be difficult to predict that

rectangular blocks of 1s—and only rectangular blocks of 1s—would survive in the canonical Prisoner's Dilemma. Similarly, we found dynamically stable 6-pointers in the Hawk-Dove game. Once we see something empirically, analysis can more easily be undertaken to explain it.

7. Patterns in the gridscape often arise from random starts. We have here a clear and unmysterious case of emergence: Play of the game converts a random, high entropy grid to a highly non-random, low entropy grid. Structure has been created from an underlying process. Further, the process is distributed: there is no central control; what happens happens as a result of local actions taken in parallel.

8. Geometry matters. Suppose we run the Default Prisoner's Dilemma ($T = 5, R = 3, P = 1, S = 0$) on our 4×4 block of 1s, but use the von Neumann neighborhood instead of the Moore neighborhood.

Generation 0:	Generation 1:	Generation 2:
000000000000	000000000000	000000000000
000000000000	000000000000	000000000000
000000000000	000011000000	000010000000
000111100000	000111100000	000011000000
000111100000	001111110000	000111100000
000111100000	001111110000	000111110000
000111100000	000111100000	000011100000
000000000000	000011000000	000001000000
000000000000	000000000000	000000000000
000000000000	000000000000	000000000000
000000000000	000000000000	000000000000
000000000000	000000000000	000000000000

Generation 3:	Generation 4:
000000000000	000000000000
000000000000	000000000000
000000000000	000000000000
000010000000	000000000000
000011100000	000000100000
000111100000	000001000000
000001000000	000001000000
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9. The search spaces presented by the gridscapes are more than astronomical. 10^{80} is an astronomical number; it is the estimated total number of atomic particles in the universe. Our 200×200 gridscape has $2^{200\times200} = 2^{40000} = 2^{20^{2000}} > 10^{6^{2000}} = 10^{12000}$ possible states.

10. More intelligent play may well yield very different dynamics. What would happen, for example, if agents were introduced that cooperated on the first game played with a neighbor but defected on the second, if the neighbor defected on the first? In the next chapter we take up *reactive strategies.* Agents that meaningfully learn are not far behind.

In conclusion, a few slightly longer comments.

4.8.2 Computational Explanation

Gridscape models, such as we've seen in this chapter, afford us means to provide *explanations,* indeed computational explanations (see Chapter 3). The essential pattern is:

- Observe phenomena to be explained; then
- Ask how the phenomena could have (or did) come about; then

- Construct a gridscape model whose behavior mimics (perhaps very roughly) the observed phenomena; and
- Appeal to the model in explanation of the phenomena.

There is, of course, a big difference between *an* explanation (which the above pattern offers) and *the* (true, correct, best) explanation and even *a good* explanation. These are interesting issues, and need to be kept in mind, but they need not divert us. Instead, we can judge each case on its merits. Sometimes calls for explanation can be resolved rather straightforwardly with these gridscape (and similar) models. We observe games in the wild and find cooperation occurring. We ask Hobbes's question: Can—and if so, how can—cooperation emerge and be sustained in a society of individuals not controlled by a central authority? The experiments on Prisoner's Dilemma and on Stag Hunt described in this chapter provide existence proofs that there are indeed circumstances in which cooperation not only can be sustained but can triumph. The models have also provided some insight into how this can come about. In virtue of being *computational* models we have full access to their machinations and, in principle at least, to understanding of why they behave as they do. None of this guarantees that the computational mechanisms mirror in any accurate way the ”real world” mechanisms that produced the original phenomena. But it's a start.

4.8.3 Societies

How is it that gridscapes (and related structures, such as networks) can be said to model societies? A *game*, recall, is a context of strategic interaction in which 2 or more players engage in interdependent decision making. The return that a given player receives from a play of the game depends on the reward structure of the game, what strategy the player plays, and what strategies the other players play. A *society*, at least as I shall use the term technically, is also a context of strategic interaction: agents (players) interact with one another and receive returns as a function of their decisions as well as the decisions (the strategies played) by their counter-players. Societies, however, have an additional, special feature: the games played by an agent in a society are affected—via their payoff structures or via the strategies employed by the agent's counter-players—by games the agent does not play, i.e., by games played by other agents in the society. Games require at least 2 players, societies at least 3.[6] The pattern of who plays whom—the

[6] I am reminded of a recorded interview I once heard with Bertrand Russell. The interviewer began by discussing Russell's family background (privileged). “It would not be accurate to describe

social network—constitutes (or at least is a major feature of) the *social structure* of a society.

Our gridscape models fit this characterization of society exactly. The rewards received by an agent, as well as the strategies of the agent's counter-players, are dependent in part upon the strategies and rewards received of the agent's counter-players' counter-players. The agent itself never plays against these important counter-players' counter-players. Put differently, if I am a player on the gridscape, it matters much to me what strategy will be played by the guy to the east. And it matters to that guy what the guy to his east plays. Hence, over the generations, it matters (at least potentially) much to me what the entire distribution of strategies on the gridscape is. In time, social effects can reach everywhere.

The *social principle* holds that social structure matters. *Social explanation* appeals to social situations, situations in which what agents see (directly interact with) depends on things (games played by other agents) the agents do not see. The extent to which social explanation matters is a matter of investigation for us.[7]

4.8.4 Negotiation and Contracting

If players/agents are able to negotiate enforceable contracts, many if not all of the games of this chapter would have a very different flavor. The dilemma in Prisoner's Dilemma would disappear if the players could negotiate an iron-clad contract; presumably or at least often both would agree to cooperate. In Stag Hunt, both players would better off if they could bind themselves to Hunt Stag. Players of Chicken could arrange never to collide. And so on. The point is not that negotiation is trivial or without its puzzles, for it is not. Rather, the point is that negotiation brings new factors into play.

Negotiation, at least the non-trivial sort, serves the purpose of restricting the players in their actions. Such restrictions may be in the interest of the players, or not. Similarly, the gridscape is an imposed structure that restricts actions in the society it defines. This happens without courts, lawyers, or even agents with

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the Russells as well-connected. Instead, the Russells were the people to whom the well-connected were connected."

[7] The term *ecology* is perhaps more apt than *society* because the latter suggests a grouping homogenous in some sense, while the former does not. We speak of human society and the ecological balance among humans and the bacteria living within them. Which term we use does not much matter. I have chosen *society* mainly because it is the term used by John Dewey (see *Human Nature and Conduct*) and other pragmatist philosophers when discussing phenomena of the ilk addressed here and because I am broadly sympathetic with those discussions.

intentions or purposes. The gridscape, notably with its capturing of an important aspect of a society, similarly gives us conceptual tools for understanding—for explaining and intervening in—broadly social phenomena in plants, animals, and humans. That, at least, is the promise.

4.8.5 Social Structure

We have explored a two-dimensional gridscape regime for symmetric 2×2 games. Other games will follow. Why just a two-dimensional gridscape? Why indeed.

Consider the Prisoner's Dilemma played on a one-dimensional (1-d) gridscape. As usual, 0s are Defectors and 1s are Cooperators. Suppose a field of 0s abuts a triple of cooperators:

$$\ldots 0000011100000\ldots$$

This triple is safe from invasion if

$$P + T < S + R \tag{4.10}$$

which is never true for the Prisoner's Dilemma. The triple is also safe if

$$P + T < 2R \tag{4.11}$$

which can happen, e.g., $T = 101, R = 100, P = 1, S = 0$. Notice, however, that two triples of 1s separated by a single 0 and surrounded by 0s are not safe from invasion. What about invasion of the 0s by the 1s? This will happen if

$$R + S > \max\{2P, T + P\} \tag{4.12}$$

In Prisoner's Dilemma, it is certainly possible that $R + S > 2P$, but definition of the game disallows $R + S > T + P$. In 1-d, a clustered triple of 1s may or may not be safe from an invasion by a clustered triple of 0s, and the latter are always safe from invasion by the former.

In general, we are interested in what happens at the interface between 3-cubes of strategies. In one dimension the 3-cubes are three of one strategy in a row. In two dimensions, as we have seen, they are a 3×3 square of one strategy. In three dimensions we have a 3×3×3 = 3^3 cube, and in general we have a 3^d hypercube.

Recall now our canonical symmetric 2×2 game in strategic form:

	S_1 x	S_2 $\overline{x}$
S_1 x	A, A	B, C
S_2 $\overline{x}$	C, B	D, D

Figure 4.11: Canonical game matrix for the symmetric 2×2 game in strategic form

In d dimensions, a hypercube of S_1s can invade an abutting hypercube of S_2s if

$$A[2 \cdot 3^{d-1} - 1] + B3^{d-1} > \max\{D[2 \cdot 3^{d-1} - 1] + C3^{d-1}, D[3^d - 1]\} \quad (4.13)$$

Similarly, a hypercube of S_2s can invade an abutting hypercube of S_1s if

$$D[2 \cdot 3^{d-1} - 1] + C3^{d-1} > \max\{A[2 \cdot 3^{d-1} - 1] + B3^{d-1}, A[3^d - 1]\} \quad (4.14)$$

Mapping these to Prisoner's Dilemma (0s for Defectors, 1s for Cooperators): $A \Rightarrow P, B \Rightarrow T, C \Rightarrow S$, and $D \Rightarrow R$. Then a cube of the Cooperators will invade an abutting cube of the Defectors if

$$R[2 \cdot 3^{d-1} - 1] + S3^{d-1} > \max\{P[2 \cdot 3^{d-1} - 1] + T3^{d-1}, P[3^d - 1]\} \quad (4.15)$$

Setting $S = 0$ and noting that $P[2 \cdot 3^{d-1} - 1] + T3^{d-1} > P[3^d - 1]$, this simplifies to

$$R[2 \cdot 3^{d-1} - 1] > P[2 \cdot 3^{d-1} - 1] + T3^{d-1} \quad (4.16)$$

or

$$R > P + \frac{T3^{d-1}}{[2 \cdot 3^{d-1} - 1]} \quad (4.17)$$

In 2-d:

$$5R > 5P + 3T \quad (4.18)$$

In 3-d:

$$17R > 17P + 9T \quad (4.19)$$

In the limit as $d \to \infty$:

$$2R > 2P + T \tag{4.20}$$

And unneglecting S gives us

$$2R > 2P + T - S \tag{4.21}$$

in the limit. So, the Cooperators *can* expand in 2-d and higher (depending on the actual values of T, R, P and S), and things don't change much after 3-d.

Here are the summary statistics from a run with $T = 150, R = 100, P = 5$ and $S = 0$. We begin with a single 3-cube (3×3 square) of Cooperators (1s) in a 20×20 gridscape of Defectors (0s).

Generation	0s	1s	Total Points
0	391	9	48560.0
1	379	21	68240.0
2	363	37	94000.0
3	339	61	132640.0
4	327	73	151600.0
5	319	81	164480.0
6	283	117	220640.0
7	243	157	282880.0
8	195	205	358000.0
9	159	241	413440.0
10	119	281	476400.0
11	115	285	485360.0
12	123	277	472480.0
13	107	293	495360.0
14	75	325	546160.0
15	95	305	517200.0
16	79	321	540800.0
17	79	321	541520.0
18	95	305	516480.0
19	63	337	565840.0
20	107	293	497520.0
21	43	357	594080.0
22	107	293	497520.0
23	43	357	594080.0
⋮	⋮	⋮	⋮

At generation 20 the system settles into a two-state oscillation, dominated by Cooperators.

Generation 0:	Generation 1:	Generation 2:
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00011100000000000000
00000000000000000000	00011100000000000000	00111110000000000000
00011100000000000000	00111110000000000000	01111111000000000000
00011100000000000000	00111110000000000000	01111111000000000000
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00000000000000000000	00011100000000000000	00111110000000000000
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00000000000000000000	00000000000000000000	00000000000000000000
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00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000

Generation 19:	Generation 20:	Generation 21:
11111111111111011111	11111111111110001111	11111111111111011111
11111111111110001111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111110001111	11111111111110001111	11111111111111011111
11111111111110001111	11111111111110001111	11111111111111011111
11111111111110001111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
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11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
10100010111110001111	00000000000001010000	11111111111110001111
00000000000000000000	00000000000000000000	00000000000000000000
10100010111110001111	00000000000001010000	11111111111110001111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111

The behavior is simplified on an infinite gradiscape. Assume:

1. An infinite gridscape in d dimensions.

2. Prisoner's Dilemma is being played.

3. Expansion of a Cooperating 3-cube obtains, as in Expression 4.16, page 125.

4. The gridscape is filled with Defectors, except for a single 3-cube, V, (in d dimensions) composed entirely of Cooperators.

   Remark: Results will hold if the 3-cube is extended to a larger hyper-rectangle (in d dimensions)

Then:

Proposition 1 *The 3-cube, V, cannot be invaded at any of its cells.*

Remark: Expression 4.16 governs. Again:

$$R[2 \cdot 3^{d-1} - 1] > P[2 \cdot 3^{d-1} - 1] + T3^{d-1} \tag{4.22}$$

Invasion occurs on the surface of V. Every surface cell is adjacent to an interior cell. Every interior cell receives a return of $R[3^d - 1]$ each generation and $R[3^d - 1] > R[2 \cdot 3^{d-1} - 1]$. No cell on the surface of V is exposed to more than 3^{d-1} Defectors. Since $T > P$, V cannot be invaded.

Proposition 2 *Every expansion is permanent; conversion of a cell from Defect to Cooperate cannot be reversed.*

Remark: This is not true, as we have seen, in a finite gridscape, since a defector may abut more than one side of V. In support of the proposition, note that every converted cell belongs to some 3-cube in d dimensions and no 3-cube can be invaded.

Proposition 3 *Every cell adjacent to V will be converted to Cooperate within two generations.*

Remark: Corners and edges are even more favorable for expansion than other surface points, since they abut nearly internal cells; they have more than $[2 \cdot 3^{d-1} - 1]$ Cooperating neighbors.

Proposition 4 *V—a cluster of Cooperators playing Prisoner's Dilemma in a field of Defectors—will expand forever on the gridscape; every Defecting cell within a finite distance from V will be converted to Cooperate in a finite number of generations.*

There are other ways to generalize our gridscape models. So far, we have focused on the Moore neighborhood of depth 1. In two dimensions, this is the 8 immediately adjacent cells to a given cell. The depth 2 Moore neighborhood (in two dimensions) adds to this the 16 immediately surrounding cells. Whereas in the depth 1 Moore neighborhood we focused on 3-cubes in d dimensions, we attend to 5-cubes for the depth 2 Moore neighborhood. For the sake of simplicity (and without substantive consequence), let us assume that in each generation each cell plays itself. Thus, for the depth 1 neighborhood in d dimensions, each cell has 3^d neighbors (including itself). The revision of Expression 4.13 (for depth 1 Moore models) is:

In d dimensions, a 3-hypercube of S_1s can invade an abutting 3-hypercube of S_2s if

$$A[2 \cdot 3^{d-1}] + B3^{d-1} > \max\{D[2 \cdot 3^{d-1}] + C3^{d-1}, D[3^d]\} \tag{4.23}$$

Setting $A \Rightarrow R, B \Rightarrow S, C \Rightarrow T$, and $D \Rightarrow P$ for Prisoner's Dilemma and rearranging gives us what was earlier the limiting formula:

$$2R + S > T + 2P \tag{4.24}$$

Under this regime even 3-cubes in 1 dimension may expand in Prisoner's Dilemma, depending on actual values of the rewards.

Generalizing to a wider Moore neighborhood, a 5-hypercube of S_1s can invade an abutting 5-hypercube of S_2s if

$$A[3 \cdot 5^{d-1}] + B[2 \cdot 5^{d-1}] > \max\{D[3 \cdot 5^{d-1}] + C[2 \cdot 5^{d-1}], D[5^d]\} \tag{4.25}$$

With the usual translation to Prisoner's Dilemma this simplifies to

$$3R + 2S > 2T + 3P \tag{4.26}$$

which is *less* favorable to expansion by the Cooperators. Generalizing further, let δ be the depth of the Moore neighborhood in use, then Expression 4.25 goes to

$$A[(\delta+1)\cdot(2\delta+1)^{d-1}]+B[\delta\cdot(2\delta+1)^{d-1}] > \max \left\{ \begin{array}{l} D[(\delta+1)\cdot(2\delta+1)^{d-1}]+ \\ \quad C[\delta\cdot(2\delta+1)^{d-1}] \\ D[(2\delta+1)^d] \end{array} \right. \tag{4.27}$$

With the usual translation to Prisoner's Dilemma this simplifies to

$$(\delta + 1)R + \delta S > \delta T + (\delta + 1)P \tag{4.28}$$

and in the limit as $\delta \to \infty$

$$R + S > T + P \tag{4.29}$$

As neighborhoods expand Cooperators have an increasingly difficult time expanding at the expense of Defectors.

There is more to say about the mathematics of play on the gridscape and we shall return to this topic in due course. Larger issues lurk, however. Although the two-dimensional gridscape is a good place to start, we should think more generally, in terms of social networks. The gridscape is such a network, with certain properties, such as regularity (e.g., everyone has the same number of neighbors). But societies of agents will often be able to find, design, or impose social networks with other forms, and these forms will matter. The creation, exercising, and destruction of social networks pervades social systems. It can be seen as among the primary drivers of human history. (The delightful and provocative essay by J.R. and William H. McNeill, *The Human Web: A Bird's-Eye View of World History*, argues just this [67].) It has also become part of common folklore, via the notion of "six degrees of separation" between typical individuals in our society. (Note that on the gridscape there are very many pairs of cells more than 6 cells apart.)

The gridscape is among the simplest of social networks. One should expect *greater* social effects to attend other less simple social network structures. That we have found as much of this as we have strikes me as remarkable.

4.8.6 The Shadow of Society

When agents play games repeatedly it is well known that the "shadow of the future" may greatly affect play. Similarly, the gridscape society has greatly affected the play we have examined in this chapter. It is important to see that these are two distinct factors. The agents in this chapter have no memory and no capacity to respond to experience. They do or die and that's all. The fact that neighboring agents play each other twice each generation is merely a computational convenience; the agents are unable to exploit any information so gained.

The system behavior we have seen is due to the structure of the games played, the strategies played by the agents, *and* the imposed gridscape structure with its attendant rules. The latter, as I have argued, brings to the table inherently social factors. What we might call the *shadow of society* affects outcomes just as does the shadow of the future. But the two are distinct factors. We turn next to agents with the barest modicum of intelligence: they can respond to experience and alter their behavior. The shadow of the future may interact with the shadow of society.

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