Draft:

Agents, Games, and Evolution

An Essay on Constructive Rationality

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Part I, (chapters 1–4) Part II, (chapters 5–7)

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Part I

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Chapter 1

Contexts of Strategic Interaction

Ideas have lives of their own. They arise at surprising times and come from surprising places. Ideas interact with other ideas. They become associated and these associations—or societies—lead to new social structures and to new ideas. These in turn may calve off in groups and form new societies. Of course, ideas live and have their being in human minds and cultural artifacts. Without us they wouldn't exist. Nor would most of us exist without the benefit of ideas that have created and sustained our civilization. Nor would there be animals without photosynthesizing plants, or photosynthesizing plants without photosynthesizing bacteria. Interdependence pervades.

The subject of this book is a certain society of ideas. I shall call it the AGE society (Agents, Games, and Evolution), without making any claim that the name ideally describes its subject. What name does? As with any interesting and reasonably complex society, AGE is a product, and continues to be a producer, of history. Its structure, its dynamics, even its constituents are often opaque and puzzling, and everything is in flux. Hence the need for study.

I shall not attempt to define the subject at hand. It likely can't be done and wouldn't be of much value even if it could. Hyperprecision would only be a distraction at this point. Rather, I will plunge in, immerse us at the center of AGE society, and explore from where we find ourselves. Better to start somewhere reasonable, then ask questions, attend to relevant problems and data, refine and recombine concepts and hypotheses, and build models and conduct experiments. All this is to be undertaken in an iterative, exploring, probing, nondeterministic search for sharper clarity, deeper understanding, and useful results. This shall be our mode throughout. If the process seems to be a sort of groping, adaptive muddling, so be it. The means are informed by the main results.

Let us begin, then, by discussing *contexts of strategic interaction* (CSIs), also known as *games.* Here is a—perhaps *the*—main theme in our AGE society of ideas.

1.1 Games in the Wild

Games, or more descriptively *contexts of strategic interaction* (CSIs), are everywhere.[1] They pervade social situations and occur quite naturally (or appear "in the wild" as geneticists say of certain alleles). Two people play backgammon. They are in a game, or context of strategic interaction (CSI), because the reward (winning or losing) for each player depends at least in part on decisions made by the other player. One player cannot make a series of decisions that results in winning or losing, *independently of what the other player does.* The other player has to make decisions, too, and they matter. The context is interactive—two or more players are involved—and it is strategic because both players have interests, which they take into account in making their decisions.

Backgammon is representative of many games in that it is purely competitive.[2] One player's win is the other player's loss. The interests of the players are, we may assume, entirely opposed. In other CSIs (or games) the players' interests are entirely coincident. These are what we call *games of pure coordination.* Two people are conversing by telephone when the connection is suddenly dropped. How should they attempt to resume the conversation? If both call back simultaneously both will get a busy signal or perhaps voice mail. They share a joint interest in mutually divining a decision that results in prompt and unfrustrated resumption of their conversation. Here we may assume the interests of the two agents are identical. Neither really cares who makes the new call, so long as it results in immediate resumption.

Lying between games of pure competition (e.g., backgammon) and games of pure coordination (e.g., resuming a broken phone call) are *mixed motive* games (or

[1] The term game is perhaps an unfortunate one for a number of reasons. It suggests a certain frivolity, also that only contexts of pure competition are of interest. More importantly, we need a distinction between a situation involving strategic interaction and a model of such a situation. *Game* gets used for both. When necessary to differentiate, I'll use game$_S$ for the situation, not always well defined with all vagueness left out, and game$_M$ for a model, presumably specified with great precision, of a game$_S$. Or, CSI for game$_S$ and game for game$_M$.

[2] At least approximately or often. Consider playing with a tyro and playing to lose for purposes of instruction.

CSIs). A small group negotiates where to have dinner. No two people have identical preferences, but everyone agrees that failing to come to a congenial decision quickly is the worst outcome. Remarkably subtle moves will typically attend this familiar situation. Bluff, bluster, threat, compromise, accommodation, probing, retreating, appeal to norms, humor, and much else are routinely employed with facile skill by everyone who participates in such groups.

How are we to understand games? In particular, how are we to predict and explain both behavior and outcomes in games? This is a large and important question. I remind the reader that our mode here is to make some progress through an "iterative, probing, nondeterministic search for sharper clarity and deeper understanding." To this end, a rough characterization of our topic will be helpful:

> Games, or CSIs, essentially involve at least two *agents* (or players) who make *choices* and receive *rewards* (or payoffs).[3] The reward to an individual agent is based in part on its choices *and the choices made by the other agent(s)*, as well as the underlying structure of the situation.

Now consider a few representative, idiosyncratically-chosen examples of contexts of strategic interaction.

1.1.1 War

Much more than a pure, brutal contest of strength, war has been recognized from the earliest writings as a field of interactive decision making. Deception especially has been and remains a primary theme; it is inherently a strategic concept. Think of the Trojan horse incident told in the *Iliad* and the story of the Cyclops in the *Odyssey.* Think of the elaborate obfuscations undertaken by the Allies in World War II concerning the time and place of D-Day. Sun Tzu, in *The Art of War* (http://www.chinapage.com/sunzi-e.html), the oldest known military treatise, wrote this:

[3]Later, it will be useful to distinguish *rewards,* which are received after each move in a strategic situation, and *returns,* which are the net of the rewards obtained in a multi-move strategic context. Unless otherwise noted, what I say about rewards applies to returns and *vice versa.*

18. All warfare is based on deception.
19. Hence, when able to attack, we must seem unable; when using our forces, we must seem inactive; when we are near, we must make the enemy believe we are far away; when far away, we must make him believe we are near.
20. Hold out baits to entice the enemy. Feign disorder, and crush him.
21. If he is secure at all points, be prepared for him. If he is in superior strength, evade him.
22. If your opponent is of choleric temper, seek to irritate him. Pretend to be weak, that he may grow arrogant.
23. If he is taking his ease, give him no rest. If his forces are united, separate them.
24. Attack him where he is unprepared, appear where you are not expected.
25. These military devices, leading to victory, must not be divulged beforehand.

Other themes abound, but deception and surprise remain keystones to military strategy. Other works on the short list of classics in military strategy include: *On War,* by Karl von Clausewitz, *The Prince,* by Niccola Machiavelli, and *A Book of Five Rings,* Miyamoto Musashi.[4] Liddell Hart, e.g., [48] is an especially persuasive spokesman for the importance of military deception and surprise. Thomas Schelling is uniformly insightful and a joy to read, e.g., [68, 66, 67, 69]. The *Memoirs* of Ulysses S. Grant are chock full of material to stimulate reflection on war and on interactive decision making in general. Here is my favorite passage. Grant is describing his first field command in the American Civil War.

> My sensations as we approached what I supposed might be a 'field of battle' were anything but agreeable. I had been in all the engagements in Mexico that it was possible for one person to be in; but not in command. If someone else had been colonel and I had been lieutenant-colonel I do not think I would have felt any trepidation. ...As we approached the brow of the hill from which it was expected we would

[4] You may find these on-web at http://www.gametheory.net/html/books.html#classics.

> see the enemy... my heart kept getting higher and higher until it felt as though it was in my throat. I would have given anything to have been back in Illinois, but I had not the moral courage to halt and consider what to do; I kept right on. When we reached a point from which the valley below was in full view I halted. The place where the Confederates had been encamped was still there but the troops were gone. My heart resumed its place. It occurred to me at once that [Colonel Thomas] Harris had been as much afraid of me as I had been of him. This was a view of the question I had never taken before; but it was one I never forgot afterwards. From that event to the close of the war, I never experienced trepidation upon confronting an enemy, though I always felt more or less anxiety. I never forgot that he had as much reason to fear my forces as I had his. The lesson was valuable.
>
> —Ulysses S. Grant, *Memoirs*

1.1.2 Trading and Investing

"Buy low, sell high" is great advice if (and only if) you know what to do. As the song says about that special form of trade and investment called love, "Nice work if you can get it, And you can get it if you try."

Examples of buying low or selling high? This is from *The Reader's Digest,* June 2003, pages 76–7:

> Customers at The Home Depot who overestimate how much paint they need return the unopened cans, which are stocked in the "Oops Paint" section. The "remnant" paint—perfect for bathrooms and other small projects—sells for $5 a gallon and $1 a quart (regular gallon prices are $21 to $25). "And it's not all chartreuse," says The Home Depot spokesperson Mandy Holton. "There are usually a lot of great neutrals." Best time to buy: Sundays and Mondays, because folks return unwanted paint over the weekend.

More generally, traders and investors are in the business of finding assets that are either under-valued or over-valued in the market. In other words, they seek opportunities for *risky arbitrage.* Risky because—unlike the paint at The Home Depot—the values of the assets in question are typically not known with much certainty. Arbitrage because the traders are looking to buy assets that are underpriced (and then resell them at their proper prices) or looking to sell ("unload") assets that are over-priced. In any event, the trick is to have and exploit knowledge that is superior to what is represented in the market. The nature of this knowledge and the means of getting it vary greatly. An investor in equities may look deeply and carefully at the fundamentals of the companies. Which are and which are not well managed, well positioned, in possession of new products and alliances? An investor may look at the "technical" data, the trends and other movements in prices. In the extreme, so-called day traders do this in real time, attempting to out-guess the market, that is out-do the other traders in discerning what is over-valued or under-valued. Note that investing on analysis of fundamentals would seem to have less strategic content than investing on technical grounds.

On-line, Internet-based examples are readily available for those who wish to trade or just to study and learn. Tradesports (http://www.tradesports.com) is a real-time, on-line trading market that affords an excellent case for study. Academic analogs—but with real money if you want—are available for elections at Iowa Electronic Markets (IEM; http://www.biz.uiowa.edu/iem/; http://www.biz.uiowa.edu/iem/markets/) and the University of British Columbia Election Stock Market (http://esm.ubc.ca/). A *BusinessWeek* article, "The 'Election Futures' Market: More Accurate than Polls?"[5] presents the case in a popular format that these markets predict election outcomes better than opinion polls. The Bush administration even toyed with creating a similar market for the purpose gauging intelligence in the Middle East (see "Betting on Terror: What Markets Can Reveal" by Floyd Norris in *The New York Times,* August 3, 2003). The idea was dropped after being exposed to public ridicule. Is it ridiculous? Consider: What would it take to "game" (distort for ulterior purposes) these markets? When would anyone want to do this? What might be done to prevent manipulation? Does it make sense to have an SEC for markets in international affairs?

There are always the public equity markets. Consider this comment on the bond market by a Salomon trader in the 1980s.

> The men on the trading floor [Salomon's bond trading area] may not have been to school, but they have Ph.D.'s in man's ignorance. In any market, as in any poker game, there is a fool. The astute investor Warren Buffett is fond of saying that any player unaware of the fool in the market probably *is* the fool in the market. In 1980, when the bond market emerged from a long dormancy, many investors and even Wall Street banks did not have a clue who was the fool in the new game. Salomon bond traders knew about fools because that was their job.

[5] 1996; http://www.businessweek.com/1996/46/b3501116.htm.

> Knowing about markets is knowing about other people's weaknesses. And a fool, they would say, was a person who was willing to sell a bond for less or buy a bond for more than it was worth. A bond was worth only as much as the person who valued it properly was willing to pay. And Salomon, to complete the circle, was the form that valued the bonds properly.
>
> —*Liar's Poker,* Michael Lewis [46, page 35]

1.1.3 Athletic Contests

There are sports, called games in common parlance, that have little or no strategic content. They amount more or less to contests of skill. Among them are golf, bowling, darts, skiing, track and field events, and bob sledding. Still other competitive games, such as billiards, have strategic content only with fairly advanced play. These are not, for the most part, of interest as CSIs, contexts of strategic interaction, and will not concern us further.

Many other athletic contests most unambiguously count as CSIs. Baseball has given us a wonderful strategic slogan, entirely appropriate for war and other games: "Hit 'em where they ain't." Wee Willie Keeler hit .432 in 1897. Asked how a man of his diminutive size could put together such an average, Keeler responded: "Simple. I keep my eyes clear and I hit 'em where they ain't."[6] Deception—or the fake-out—plays as prominent a role in these athletic contests as it does in warfare. Think of the pitcher-batter duel in baseball, the fake-out moves in basketball, or the mixing of plays in American football.

Management of sports teams is as much a matter of strategic interaction as the play itself. In *Moneyball: The Art of Winning an Unfair Game,* Michael Lewis describes how the Oakland A's baseball team, with consistently small amounts spent on player salaries, is consistently able to contend in major league baseball and reach the playoffs [47]. In two words: risky arbitrage. The A's, and in particular their general manager Billy Bean, have identified predictive measures superior to those used by other teams, for example using on-base percentage instead of batting average to evaluate the worth of a batter. Better measures of value allow the A's to 'buy' (hire) under-priced players. They may not have the best team in baseball, but they have one of the best. Their efficiency in the sense of what it costs them to win a game is tops and they operate at a profit in a media market dominated by the San Francisco Giants baseball team.

[6] From http://www.baseballtips.com/slang.html.

1.1.4 Gambling

Many forms of gambling do not involve strategy or even much skill. Examples include playing slot machines and playing roulette. Not so with poker at the professional level. Poker is prototypical. It is to competitive games of strategy what robins are to birds: a standard, familiar, readily available example, displaying in typical form many of the characterizing features of the subject. Everyone over time gets roughly the same quality of hands, yet there is an enormous difference among players in their success rates. Everyone can count cards and figure the odds. What actually matters is bluffing, reading your opponents (discerning their "tells," behavior such as slamming down chips that indicates what is in their hands), and preventing your opponents from reading you. The following fine passage from a master poker player is well worth quoting at length:

> Let's take a quick glimpse at the high-stakes poker world, an enterprise that yields several of my friends over a million dollars a year! At this level, too, luck is a factor on any given day, week, or month, but what's different is that if you play better poker than your opponents do, pretty consistently, you'll find that over almost any *two*-month period your winnings have exceeded your losses. Furthermore, if you play better poker than your opponents over a *six*-month period, your results will have moved very solidly in the winning direction. Making a few well-timed bluffs each day will add up to a lot of money each year!
>
> In fact, if an inexperienced poker player were to sit down for a few hours with a group of world-class poker players, he would have virtually no chance to win over even an eight-hour period. This very fact is why five or six top pros might be willing to sit down in the same game with this fellow and each other: the money that even one amateur is likely to contribute makes it work their while to do battle with so many respected opponents.
>
> This is why so many of the top poker players today drive fine cars and live in palatial homes [the author of this passage lives with his family in Palo Alto]. Right now, as you're reading this book, there is a $600–$1,200-limit poker game at the Bellagio Casino in Las Vegas and a $400-$800-limit poker game at the Commerce Casino in Los Angeles. There is ...
>
> If that's not enough action for you, four nights a week in Los Angeles,

> there is a $2,000-$4,000-limit Seven-Card Stud game a Larry Flynt's Hustler Club Casino, with Larry himself often playing. In the $400-$800-limit poker game it's easy to take a $25,000 swing in one hour. In the $2,000-$4,000-limit game, where movie stars, former governors, and billionaires play, it's not uncommon for someone to win or lose $250,000 in one night. In these "nosebleed" poker games (the term refers to the altitude of the stakes), strategy, discipline, calculation of the odds, and practiced observation contribute to a game that involves much more skill. Better play wins more hands in the long run.
>
> –*Play Poker Like the Pros* by Phil Hellmuth, Jr., 2003 [34, pages 4–5]

The society of poker players has given us an important concept—the *tell*—not only for poker but for CSIs generally. *Webster's Unabridged Dictionary* finds only two senses for *tell* as a noun. Quoting:

1. something that is told : TALK, TALE, ACCOUNT

   "have a tell with you —Eden Phillpott"

2. [Ar *tall*]

   : HILL, MOUND

   specif : an ancient mound in the Middle East composed of remains of successive settlements — compare TEPE

The *Oxford English Dictionary* is no more helpful. The Wikipedia gets it right. Quoting:

> Tell (poker)
>
> From Wikipedia, the free encyclopedia.
>
> In poker, a tell is a detectable change in a player's behavior that gives clues to that player's hand. Possible tells include leaning forward or back, placing chips with more or less force, fidgeting, changes in breathing or tone of voice, direction of gaze and actions with the cards, cigarettes, or drinks.
>
> For example, a player with a weak hand, hoping to bluff, may throw his chips into the pot forcefully and with a direct gaze at a player he hopes to discourage from calling.

> Tells may be common to a class of players or unique to a single player. A player gains an advantage if she observes another player's tell, particularly if that action is unconscious and reliable. However, better players may fake tells, hoping to lead their opponents into costly traps when they rely on the false information. So the observing, creating, and evaluating of tells can add another level to the play of poker.
>
> Mike Caro has published the most comprehensive information on tells; his *Book of Tells* (ISBN 0897461002) is now a standard reference on the subject.
>
> David Mamet's 1987 movie *House of Games* includes an interesting discussion and visual reference to tells as an essential part of the plot. The 1998 movie *Rounders* contains an even more subtle use of strategy: at one point, "Mike" discovers a tell in his opponent (that he eats cookies in a particular way after he has bet a very strong hand), and after using it once, he reveals to the opponent that he has this tell; although this eliminates the usefulness of the tell itself, it upsets his opponent so much that it affects his later play.
>
> –*Wikipedia,* http://www.wikipedia.org/wiki/Tell_(poker)

The Wikipedia is also better than the dictionaries on *tell* as a noun, not in the context of poker. Quoting:

> Tell
>
> From Wikipedia, the free encyclopedia. A tell (Arabic, or tel, Hebrew) is a mound site formed through successive human occupation over a very long timespan.
>
> The word is used as a term in archaeology, particularly Middle-Eastern archaeology. It is sometimes used in a toponym, that is, as part of a town or city name, the most well known example being the city of Tel Aviv. Often a modern city is located next to an ancient mound with a similar tell name, for example the city of Arad is a few kilometers (miles) away from an ancient mound called Tel Arad.
>
> This article is a stub. You can help Wikipedia by fixing it.
>
> External link
>
> * http://www.webref.org/anthropology/t/tell.htm

> Tell is an English verb meaning "to speak to" or "to talk to"; also "to give an order". For more information on what that is, see talking.
>
> Retrieved from "http://www.wikipedia.org/w/wiki.phtml?title=Tell" This page was last modified 01:47, 19 Jun 2003. All text is available under the terms of the GNU Free Documentation License.

Note the meaning associations and similarities between these two senses of *tell.* The strategic sense of *tell* is lucidly on display in the following passage about the great baseball player and base runner, Rickey Henderson.

> But Kennedy knew how devastating stealing could be: he had been with the San Francisco Giants in the 1989 World Series, when Henderson and the A's swept the Giants in four games and Henderson set a post-season record, with eleven stolen bases.
>
> Henderson agreed to give a demonstration, and there was a buzz as Goodman, Johnson, and the other players gathered around first base. Henderson stepped off the bag, spread his legs, and bent forward, wiggling his fingers. "The most important thing to being a good base stealer is you got to be fearless," he said. "You know they're all coming for you; everyone in the stadium knows they're coming for you. And you got to say to yourself, 'I don't give a dang. I'm gone.' " He said that every pitcher has the equivalent of a poker player's "tell," something that tips the runner off when he's going to throw home. Before a runner gets on base, he needs to identify that tell, so he can take advantage of it. "Sometimes a pitcher lifts a heel, or wiggles a shoulder, or cocks an elbow, or lifts his cap," Henderson said, indicating each giveaway with a crisp gesture.
>
> Once you were on base, Henderson said, the next step was taking a lead. Most players, he explained, mistakenly assume that you need a big lead. "That's one of Rickey's theories: Rickey takes only three steps from the bag," he said. "If you're taking a big lead, you're going to be all tense out there. Then everyone knows you're going. Just like you read the pitcher, the pitcher and catcher have read you."
>
> He spread his legs again and pretended to stare at the pitcher. "O.K., you've taken your lead; now you're ready to find that one part of the pitcher's body that you already know tells you he's throwing home. The second you see the sign, then *boom*, you're gone." [28, page 58]

Hellmuth's book has a great deal of information on Texas Hold 'em, which is generally the most popular form of poker in tournaments and is the variety of poker played at the World Series of Poker each year at Binion's Horseshoe Hotel & Casino in Las Vegas.[7] *Positively Fifth Street* [56] by Jim McManus, a published poet, novelist, and professor, describes the 2001 World Series of Poker and the world around it. Strategic insight abounds in both works.

1.1.5 Business Strategy

The gambit, a term from chess, is a favorable trade, which the opponent may or may not realize is happening. The player offering the gambit offers a comparatively small loss in exchange for a larger gain in position or other form of resource. Here is something very like a gambit played big time in business.

> Analysts called it "Marlboro Friday"—Philip Morris announced on April 2, 1993 that it would reduce the U.S. price of its premium brand of cigarettes by 20%. The tobacco manufacturer also said it would increase the budget for its domestic advertising by a substantial amount. R.J. Reynolds, Philip Morris's biggest competitor, responded by matching the price cut on its own premium brands (Camel and Winston among them) and by pouring more money into its own domestic advertising.
>
> The pricing war that ensued cost both companies tens of millions of dollars. But was the domestic market share the real reason Philip Morris lowered the price of Marlboro cigarettes? Consider that just as R.J. Reynolds had depleted its cash resources trying to keep up with its chief opponent, Philip Morris was expanding aggressively into the Eastern European market, investing $800 million in Russia and other regions that were formerly part of the Soviet Union. R.J. Reynolds was in no position to fight back, having spent so much money to maintain its market share in the United States, and Philip Morris won the battle for Eastern European market share, hands down.

[7] See http://conjelco.com/wsop.html and http://www.binions.com/home.asp. The Wikipedia has an introduction to Texas Hold 'em: http://www.wikipedia.org/wiki/Poker/Texas_holdem. At the World Series of Poker, No Limit Texas Hold 'em is the game. "No Limit" means that the largest bet permitted is the size of the current wealth outside the pot of the poorest player still in the hand. Once a player has bet all of his or her chips, the player is said to be "all in," since the player's wealth is all in the pot. Once a player is all in for a particular hand, other players may call but may not raise.

> –"Global Gamesmanship" by Ian C. MacMillan et al., *Harvard Business Review,* May 2003 [51]

Sometimes you can even make a profit on a gambit:

> One day earlier in his career [Robert] Dall was in the market to buy (borrow) fifty million dollars. He checked around and found the money market was 4 to 4.25 percent, which meant he could buy (borrow) at 4.25 percent or sell (lend) at 4 percent. When he actually tried to buy fifty million dollars at 4.25 percent, however, the market moved to 4.25 to 4.5 percent. The sellers were scared off by a large buyer. Dall bid 4.5. The market moved again, to 4.5 to 4.75 percent. He raised his bid several more times with the same result, then went to Bill Simon's office to tell him he couldn't buy money. All the sellers were running like chickens.
>
> "Then you be the seller," said Simon.
>
> So Dall became the seller, although he actually needed to buy. He sold fifty million dollars at 5.5 percent. He sold another fifty million dollars at 5.5 percent. Then, as Simon had guessed, the market collapsed. Everyone wanted to sell. There were no buyers. "Buy them back now," said Simon when the market reached 4 percent. So Dall not only got his fifty million dollars at 4 percent but took a profit on the money he had sold at higher rates. *That* was how a Salomon bond trader thought: He forgot whatever it was that he wanted to do for a minute and put his finger on the pulse of the market. If the market felt fidgety, if people were scared or desperate, he herded them like sheep into a corner, then made them pay for their uncertainty. He sat on the market until it puked gold coins. *Then* he worried about what he wanted to do.
>
> —*Liar's Poker,* Michael Lewis [46, page 88]

1.1.6 Negotiation

Negotiation exemplifies strategic interaction *par excellence.* After all, there is no point in negotiating if your counter-party's actions don't matter to you. Familiar as negotiation is to everyone, it is useful to be reminded that often negotiation is not explicit, at least not at first. Here is a description of this sort of encounter "in the wild."

> To begin to negotiate the environment does not, of course, mean that you enter the negotiation with a clear-cut goal in mind. A clear-cut goal is not needed even in purely human negotiations. Suppose you pass a stall in a market every week and notice an antique ornament for sale. At first it seems ugly, but as it grows familiar, you catch yourself wondering how it would look on your shelf. One day it rains while you are crossing the market and you take shelter in the stall. the ornament is still there; for something to do you ask its price. Even when a low price is mentioned you automatically snort in contempt, for you have no intention of buying. . . or have you? During the week that follows you decide that the price really was low and think of a friend who has a birthday soon and might like it. Next week you stop and begin to bargain.
>
> When did the negotiation begin? When you started to bargain? Or earlier, when you asked the price? Or earlier still, when you first noticed the ornament among an anonymous heap of others? Pointless to say, as pointless as to say where mind began.
>
> –*Language and Species* by Derek Bickerton [2, pages 234–5]

1.1.7 Coordination, Symbiosis, Mutualism, Cooperation

Contexts of strategic interaction are not all adversarial in the sense that one agent's gain is another's loss (so-called *constant-sum* or equivalently *zero-sum* games). In *coordination games* all players gain if they can arrive at a common outcome and lose if they fail. Think of the game of finding someone you have separated from during a shopping trip. You both wish to meet up again, but did not plan for the separation and have no easy means of communication. Schelling's early treatment of such games is masterful and well worth reading today [68].

Biologists have named and studied several kinds of interactive decision making that—in terms of game theory lingo—is not constant-sum. Symbiosis and mutualism are two of the most important for our purposes.

> Symbiosis (pl. symbioses) is an interaction between two organisms living together in more or less intimate association or even the merging of two dissimilar organisms.
>
> The term host is used for the larger of the two members of a symbiosis. The smaller member is called the symbiont.

> Symbiosis may be divided into two distinct categories: ectosymbiosis and endosymbiosis. In ectosymbiosis, the symbiont lives on the body surface of the host, including the inner surface of the digestive tract or the ducts of exocrine glands. In endosymbiosis, the symbiont lives in the intracellular space of the host.
>
> An example of mutual symbiosis is the relationship between anemonefishes of the genus Amphiprion (family, Pomacentridae) that dwell among the tentacles of tropical sea anemones. The territorial fish protects the anemone from anemone-eating fish, and in turn the stinging tentacles of the anemone protects the anemone fish from its predators (a special mucous on the anemone fish protects it from the stinging tentacles).
>
> The biologist Lynn Margulis, famous for the work on endosymbiosis, contends that symbiosis is a major driving force behind evolution. She considers Darwin's notion of evolution, driven by competition is incomplete, and claims evolution is strongly based on co-operation, interaction, and mutual dependence among organisms. According to Margulis and Sagan (1986), *Life did not take over the globe by combat, but by networking.*
>
> (From: http://www.wikipedia.org/wiki/Symbiosis.)

(See [52] for a recent treatment of this theme by Margulis and Sagan.)

> Mutualism is a interaction in which both organisms in a close relationship derive some degree of benefit. Mutualism is usually temporary or not obligatory.
>
> (From: http://www.wikipedia.org/wiki/Mutualism.)

Lichens—those familiar greenish splotches on trees and rocks—present a most striking example of symbiosis.

> Lichens have been described as "dual organisms" because they are symbiotic associations between two (or sometimes more) entirely different types of microorganism -
>
> - a fungus (termed the mycobiont)

- a green alga or a cyanobacterium (termed the photobiont).[8]

> There are many examples of symbiosis in nature, but lichens are unique because they look and behave quite differently from their component organisms. So, lichens are regarded as organisms in their own right and are given generic and species names. However, for taxonomic purposes the names are actually fungal names: lichens are regarded as a special group of fungi - the lichenised fungi.
>
> There are an estimated 13,500 to 17,000 species of lichens, extending from the tropics to the polar regions. Some of them grow on the bark of temperate trees or as epiphytes on the leaves of trees in tropical rain forests. Others occupy some of the most inhospitable environments on earth, growing on cooled lava flows and bare rock surfaces, where they help in the process of soil formation, and on desert sands where they help to stabilise the surface and enrich it with nutrients (see Cyanobacteria [cf., footnote 8, page 18]). Some other types of lichen grow abundantly on tundra soils, providing a vital winter food source for animals (including reindeer and caribou) in arctic and subarctic regions.Yet other lichens grow on or in the perennial leaves of some economically important tropical crop plants such as coffee, cacao and rubber, where they are regarded as parasites.
>
> All these features make lichens interesting and significant in environmental terms. But lichens also pose challenging scientific problems - how do two or more microorganisms interact at the cellular, genetical and biochemical levels to produce a unique, hybrid organism?
>
> (From: http://helios.bto.ed.ac.uk/bto/microbes/lichen.htm.)

A form of emergence occurs with lichens. Surprisingly, what appears to be, and in many ways is, a single individual is actually composed of, arises through the inter-

[8] Author's note: From the Wikipedia, www.wikipedia.org/wiki/Cyanobacteria: "Cyanobacteria or blue-green bacteria are a group of aquatic bacteria that obtain their energy through photosynthesis. They are often referred to as blue-green algae, even though it is now known that they are not related to any of the other algal groups, which are all eukaryotes. Nonetheless, the description is still sometimes used to reflect their appearance and ecological role. Fossil traces of cyanobacteria have been found from around 3800 million years ago, making cyanobacteria some of the earliest living things known." Implied but not said, the cyanobacteria are prokaryotes, of ancient origin and lacking a cell membrane. It is thought by some that photosynthesizing plants acquired or incorporated the genomes of photosynthesizing bacteria.

actions of, individuals from two distinct biological kingdoms. It is even surprising who first noticed this underlying, symbiotic structure.

> Lichens are unusual creatures. A lichen is not a single organism the way most other living things are, but rather it is a combination of two organisms which live together intimately. Most of the lichen is composed of fungal filaments, but living among the filaments are algal cells, usually from a green alga or a cyanobacterium.
>
> In many cases the fungus and the alga which together make the lichen may each be found living in nature without its partner, but many other lichens include a fungus which cannot survive on its own – it has become dependent on its algal partner for survival. In all cases though, the appearance of the fungus in the lichen is quite different from its morphology as a separately growing individual.
>
> The true identity of lichens as symbiotic associations of two different organisms was first proposed by Beatrix Potter, who is best remembered for her children's books about Peter Rabbit. In addition to her books, she spent time studying and drawing lichens. Her illustrations are still appreciated for their detailed and accurate portrayal of the delicate beauty of these bizarre organisms.
>
> (From http://www.ucmp.berkeley.edu/fungi/lichens/lichens.html.)

(Searching Google's image base on "lichens" turns up an excellent collection of images.)

Next, cooperation is—in its prototypical sense—a human social phenomenon, one that has been much noticed and remarked upon by social scientists, including game theorists. Cooperation, or roughly non-greedy behavior, has been called "the cement of society" [13] (by analogy with causation, which Hume called "the cement of the universe"). Without it, in the pungent phrasing of Thomas Hobbes, there would be

> no place for industry, because the fruit thereof is uncertain; and consequently no culture of the earth; no navigation, nor use of the commodities that may be imported by Sea; no commodious Building; no Instruments of moving and removing such things as require much force; no Knowledge of the face of the Earth; no account of Time; no Arts; no Letters; and which is worst of all, continuall feare, and danger of violent death; And the life of man, solitary, poore, nasty, brutish, and short. (Hobbes, *Leviathon*)

(See, e.g., http://plato.stanford.edu/entries/hobbes-moral/, http://www.philosophypages.com/ph/hobb.htm.) Without cooperation we are lost. How, then, does it arise and how might it be sustained? Hobbes thought that realistically it was necessary to turn power over to a sovereign (king or powerful government)—a leviathon—who would enforce cooperation on society. Others have thought that perhaps cooperation could emerge and be sustained naturally, without a central authority, much as, say, lichens emerge and are sustained naturally. Is this possible? If so, what is required of the games and the players?

1.1.8 Conversation

When we speak we have in mind how others will react to what we say and what we do not say. In this regard, a representative news story—"Official's comments set off euro's surge. U.S. Treasury's Snow said a weaker dollar would help U.S. exports. The dollar fell against the euro." by David McHugh—appeared in *The Philadelphia Inquirer* on May 13, 2003. The first sentence says it all: "The U.S. dollar fell to another four-year low against the euro yesterday, inching closer to its all-time low, after U.S. Treasury Secretary John Snow said a weaker dollar would help U.S. exports." Secretary Snow never said he favored letting the dollar fall, but what he did say, as he no doubt understood, led the markets to infer that he favored a decline in the dollar. This form of strategic interaction is rife in linguistic communication and even has a special name: conversational implicature. Examples abound. A sign at Big Sur Lodge, Pfeiffer State Park, near a food counter:

Stressed?
Spelled
Backwards
Is
Desserts

Translation: Buy a dessert from us; it'll make you feel good. Or the concluding line in Hitchcock's movie, "Frenzy": "Mr. Rusk, you're not wearing your tie." Translation: You're the necktie murderer and I'm placing you under arrest. Or the use of irony, as in "Rick, Major Strasse is one of the reasons the German Reich enjoys the reputation it has today," from the movie "Casablanca." Translations: (to Strasse) The Reich is an impressive accomplishment and you are a big part of it; (to Rick) Watch out, this guy Strasse is a very bad man. Or the dialog-less eating scene in the movie "Tom Jones" with Albert Finney. Translation: This is just foreplay foreplay; the best is yet to come. See Paul Grice's "Logic and Conversation" [30] for the original treatment, still worth reading.

1.1.9 Games against Yourself

The long and justly celebrated story from the *Odyssey* of Ulysses and the Sirens continues to enchant and inform us. (Jon Elster has even written an entire book relating the story to modern social science [11].) From the perspective of strategic interactions, the story may be interpreted as a game played by Ulysses at one time against Ulysses at another time. At t_0, before approaching within earshot of the Sirens, Ulysses foresees that Ulysses at t_1, within earshot, will have preferences and inclinations quite at variance from Ulysses at t_0 and from Ulysses at t_2, post the encounter with the Sirens (if he should live that long). So Ulysses at t_0 cleverly prevents Ulysses at t_1 from acting as Ulysses at t_1 would prefer. He hears the Sirens and lives to tell the tale.

Robert Louis Stevenson's familiar story, *Dr. Jekyll and Mr. Hyde,* carries a similar theme. Thomas Schelling tells a fable about a man who is struggling to quit smoking. A friend who smokes arrives at his house, converses, and leaves without incident. The friend, however, forgets his jacket and our protagonist notices the jacket contains a package of cigarettes. Not having an immediate compulsion to smoke and knowing the friend will return tomorrow, he puts the jacket away. Later, upon reflection, he recovers the jacket, removes the cigarettes, and destroys them.

1.1.10 Confidence Games

The con man (or woman) first gets your trust, your confidence, and then abuses it for profit. "Take the money and run" is the operating creed. Confidence rackets are celebrated in literature, theater, and film. Examples include Herman Melville's novel *The Confidence Man,* Thomas Mann's novel *Confessions of Felix Krull, Confidence Man: The Early Years,* Sinclair Lewis's novel *Elmer Gantry* (movie with Burt Lancaster and Jean Simmons), Jim Thompson's novella *The Grifters* (movie with Anjelica Huston, John Cusack, and Annette Bening), Guy Owen's short story "The Flim-Flam Man" (now published as *The Ballad of the Flim-Flam Man,* Coastal Carolina Press, May 2000; movie with George C. Scott and Sue Lyon), N. Richard Nash's play *The Rainmaker* (movie with Burt Lancaster and Katherine Hepburn), David Mamet's movie "House of Games" (with Lindsay Crouse and Joe Mantegna), and Meredith Wilson's Broadway musical *The Music*

Man (movie with Robert Preston and Shirley Jones). This is from a Penn Web site, August 2003:[9]

```
6:30 pm - 8:30 pm     Confidence Games at the GSC

The GSC shows films about con artists:
Catch Me If You Can on 7/31;
The Thomas Crown Affair on 8/7;
The Spanish Prisoner on 8/14; and
The Grifters on 8/21.

Location:  Graduate Student Center, 3615 Locust Walk
Category:  Film
More info:
http://www.upenn.edu/gsc/programs/film.htm#con
```

Con games lie at the core of much detective fiction and fact, as well as recently popular email scams. There is a confidence business, indeed an industry, with its own lessons and skills. (This takes us beyond the scope of the book. Those wishing to go further might consult such works as *How to Become a Professional Con Artist,* by Dennis M. Marlock.)

1.1.11 Statesmanship

Ending this list on a less cynical note, George Washington is understood to have been a politically ambitious man throughout his life. He actively, deliberately sought and schemed for the power, influence, and adulation he ultimately received. Washington notoriously wore his military uniform during the deliberations on the Declaration of Independence, just to remind the other delegates of his availability for command. In pursuing his ambitions Washington consistently and consciously followed a strategy of seeking rewards by actually deserving to get them. Resigning from the army at the end of the Revolution, an unexampled act, was a move calculated to make him fit for political leadership in a democracy. Declining to run for a third term as president was a move calculated to secure the success of the new country and of Washington's legacy.

Napoleon on his deathbed and in prison lamented that "They expected me to be another Washington."

[9] See http://www.gametheory.net/ for yet another list of game-related movies, as well as lots of useful material on game theory.

1.2 Why Study Games?

Why are we interested in contexts of strategic interaction (CSIs)? The examples just discussed will, I hope, make it plain that games are interesting because war, diplomacy, poker, business strategy, and so on are each interesting in themselves and each are examples from a larger pool of important phenomena meriting attention. Besides interdependent, interactive choice—the characterizing feature of games—we see the interplay of reasoning, calculation and reckoning, deception, skill, bluffing, power, adaptation, flexibility, cooperation, learning, arbitrage, coordination, norms, communication, markets, social organization, and much else that is pervasive in, and fundamental to our understanding of, the social order. (And belonging to the AGE society of ideas.) Games are interesting because they are vortices of many interesting phenomena. Social phenomena manifest themselves—play themselves out—in games.

What would we like to know about games? As in any field of science we seek to describe, explain, predict, and intervene. We wish to describe and classify games in the wild systematically. The previous section is merely a hint at a much-needed natural history of games. We wish to explain and predict game outcomes. This often called *solving the game* in the classical literature. We also wish to understand—explain and predict—how it is outcomes are reached. How do agents of various sorts (experienced humans, naïve humans, monkeys, rats, lichens, organizations, artificial agents) find and implement their strategies of play? How does play unfold over time (and over space when geography is relevant)? Finally, we seek understanding of games in order to intervene in the world. We might hope to improve our own play in strategic contexts, or to design better social institutions (such as markets for electrical power that resist manipulation—"gaming"—as in the Enron affair [79]), or to field artificial agents that labor on our behalves (perhaps for negotiation or purchasing over the Internet) . The scope of potential investigation is both magnificent and beyond our means. We should be content with modest progress, while keeping ourselves reminded of the larger issues. That, at least, describes my aims in this essay.

1.3 Methods of Study

Games in the wild, we must always remind ourselves, are the primary phenomena that motivate study of contexts of strategic interaction. The games we make up or develop as abstract models are ultimately interesting only because they contribute to understanding games in the wild. How, in particular then, can we study strategic interaction? Various ways are open to us:[10]

1. *a priori.* CSIs or games in the wild may be abstracted and reduced to formal models, then studied mathematically, typically upon assumption of axioms of rationality. From this perspective, the theory of games is a branch of mathematics. Much of classical game theory proceeds in this mode. Standard textbooks and reference works include [3, 22, 44, 49, 73].

2. *in vivo.* Games, or strategic situations, may be studied *in situ,* as they (more or less) naturally occur. This is an historical—"natural history"—mode of investigation, but of course the history may be contemporary and the means of study may use techniques from anthropology, sociology, or journalism. Pioneers of this approach include Thomas Schelling (e.g., [68, 66, 67, 69]), and Jon Elster (e.g., [11, 13]).

3. *in vitro.* We can study games by doing experiments with real ("wet") agents, including humans (e.g., [15, 37]), monkeys (e.g., [16]), even blue jays (e.g., [77]). And why not lichens and bacteria? The literature uses such names as *behavioral game theory* and *experimental economics* to refer to these kinds of investigations.

4. Algorithmic or *in silico.* There is much to be learned about games by representing agents as decision algorithms that choose their plays, and then studying the behavior of the resulting system. Fundamentally, this method of simulation or experimental mathematics is a variant of the *a priori* method. Let us call it *algorithmic game theory.* By allowing ourselves to use computational methods (instead of purely analytic mathematics) we may greatly extend the range, scope, and realism of models addressed, and concommitantly reduce the stringency of the assumptions required.

In what follows, we shall draw upon each of these methods. Our main focus methodologically, however, will be on algorithmic (or *in silico*) studies of artificial agents. Such studies may be, and have been, conducted from a variety of perspectives. Agents may be modeled as naked strategies (what we call *identity-centric* agents), possibly reactive or adaptive strategies, that play in tournaments (e.g., Axelrod's original and seminal study [1]) or that play in a populated ecology which evolves under the replicator dynamic (e.g., [1, 25, 74]) or that play in a differentiated geography (aka: spatial games; e.g. [14, 31]).

[10] I am grateful for the discussion in [58].

Again, we shall draw upon these and related studies but focus our efforts elsewhere. That focus has four main aspects:

1. *Finite, non-ideal* contexts of strategic interaction. Agents are finite beings. Their rationality, their abilities to reckon and foresee, are limited. The algorithms with which we model these agents must be computable[11] and computable without exorbitant use of resources. Play unfolds in a finite population, for a finite time, and in a finite space. A major theme will be to compare and contrast results under finite, non-ideal and infinite, ideal regimes of play. Classical game theory employs what philosophers call an *externalist* theory of rationality. Here we are asking different questions and shall be focusing on *internalist* notions of rationality. The upshot of this point will emerge as we proceed.

2. *Identity-centric* more so than strategy-centric agents. Humans, and indeed monkeys and blue jays, in contexts of strategic interaction may be said to *have* strategies (rather than to *be* strategies), and to be capable of changing them in response to experience. These players, and most of the agents we shall consider, may meaningfully be said to have identities distinct from the strategies they employ at any given time. They are more than naked strategies. In particular, they are

3. *Exploring, probing* agents, not merely reactive agents. Humans, monkeys, blue jays, and most of our agents face the exploration–exploitation dilemma/-tradeoff, addressed throughout the machine learning literature.

   Finally, the strategic contexts we will focus on will be

4. *Chronic* and *social* more so than acute and singular, *ongoing* and *widespread* more so than unique. The games may be *repeated* or *iterated* or be like other games that will be played, rather than being unique, non-repeatable events.

A word of elaboration and justification for this last aspect of our focus. Contexts of decision, or choice, may be distinguished into *strategic* (or game-theoretic, the principal subject of this book) and *parametric* (not strategic, the principal subject of the field of decision analysis).[12] Further, contexts of decision or choice may

[11]Technically, effectively computable, which I assume to be coterminus with the partial recursive functions. We are only interested in algorithms that implement partially recursive functions.

[12]See Paul Bloom's *Descartes' Baby* [4] for an accessible presentation of evidence that human mentality innately is organized in recognition of the distinction between parametric and strategic decisions.

be distinguished into *acute* ("one-shot" or once-only) and *chronic*. Herrnstein's name for chronic choices [35]—*distributed*—is an apt description, and I shall use it. Chronic decisions or choices are distributed, usually in time. We may think of habits as chronic decisions that are, or become, more or less settled. The following table, then, summarizes this framework:

	Acute (one-shot)	Chronic (distributed)
Parametric	decision analysis	decision analysis
Strategic	classical game theory	this essay

Table 1.1: Framework categorizing decision/choice contexts

Further, as I have noted, decision contexts may be distinguished into *individual* and *social*, although this distinction is perhaps more applicable to strategic than to parametric contexts. In any event, there will be much emphasis in what is to follow on *social* aspects of distributed strategic choice.

We often *are* inclined to think of games in terms of acute, dramatic points of decision. This is captured in the penultimate stanza of "Casey at the Bat" (Ernest L. Thayer, alias Phin, page 4 of the San Francisco *Daily Examiner,* June 3, 1888).

> The sneer has fled from Casey's lip, the teeth are clenched in hate.
> He pounds, with cruel violence, his bat upon the plate.
> And now the pitcher holds the ball, and now he lets it go,
> and now the air is shattered by the force of Casey's blow.

Of course the final stanza is

> Oh, somewhere in this favored land the sun is shining bright.
> The band is playing somewhere, and somewhere hearts are light.
> And, somewhere men are laughing, and little children shout,
> but there is no joy in Mudville – mighty Casey has struck out.

But first, many games, many contexts of strategic interaction, *are* distributed or chronic, or approximately so. Agents do business with a particular merchant, doctor, lawyer, restauranteur repeatedly. Agents have friends, partners, lovers, spouses, colleagues they encounter more than once. Agents have competitors in the market for more than a day. Agents are embedded in societies. Very often indeed, strategic contexts cannot be separated from the future or the past.

And second, is Casey's situation really unique, even for Casey? True enough, Casey is in a zero-sum game in the sense that only one team can win. It is also true that in any given at-bat the pitcher in baseball has the advantage; anyone can strike out. Most likely, however, there will be another game tomorrow or the next day. Casey's interest lies in maximizing the expectation of his future contributions to the team. Getting angry, pounding the bat, focusing exclusively on this game and this moment is, perhaps, not the wisest of moves on Casey's part. Better to take the long view. Better to have the pitcher strike you out than for you to strike yourself out. Perhaps the long view can inform the acute. Perhaps, at least sometimes, learned policies of play in the chronic case should drive or at least inform play in the acute case. What follows has among its aims the investigation of such conjectures and their ramifications.

Chapter 2

Four Themes

My purpose of this chapter is to introduce four topics that figure prominently, four themes that recur frequently, in the remainder of this book. Very briefly and quite approximately they are:

1. The Social Order

   The social order is the regular, systematic patterns of behavior evidenced by a particular society (or collection of individuals). Trust is one such behavior that will engage us. It is commonly described as the "cement of society," the glue that makes social order possible. For example, among a group of individuals conducting business, handshakes and verbal commitments will often suffice instead of written, witnessed agreements closely supervised by legal authorities. What accounts for this? When are more stringent measures required and why?

2. Self-Organization and Emergence

   We often find in complex systems *self-organizing* behavior: *spontaneous order* [3, page 393] emerges without centralized direction or control. Any property (including behavior) of a larger object is an emergent property if it arises through a decentralized process, "from the bottom up" (rather than centrally, "from the top down"), conducted by smaller, constituent objects. For example, the theory of evolution by natural selection views species (larger objects) as "emerging" from competition in the "struggle for life" among individuals (smaller objects). The process is decentralized and not, according to the theory, designed and directed from above.

3. Combinatorial Complexity

   From a small alphabet we may combine letters to form a large number of words. By combining in different ways a modest number of words (just a few thousand) we can compose not only millions of meaningful sentences, but millions of coherent books. The possible ways of combining a number of distinct objects is called a *combinatorial space* for those objects. In investigating agents, games, and evolution, we routinely encounter combinatorial spaces of enormous size (aka: complexity) and must face the problem of understanding how agents and evolution can search and navigate these spaces.

4. Rationality

   As we shall see, rationality is a contested concept. Superficially at least, *rational* and *rationality* commonly refer to several rather different ideas. For example, a rational belief is often said to be a belief for which the believer has good reasons. But also, a rational choice is often said to be one that is consistent with a body of preferences. We shall need a concept of rationality applicable to agents in games, agents that need not be possessed of human intelligence, agents whose rationality is real even if less than ideal.

The remaining sections of this chapter elaborate more thoroughly, but still quite incompletely, upon these topics. This serves as a start for further elaboration, which is continued throughout the book.

2.1 Problems of the Social Order

Societies are collections of individuals, sustained over time and exhibiting systematic behavior. They offer even the casual observer an inexhaustible supply of interesting questions and puzzles. Why do animals live in groups? What explains the sizes and behavioral orderings of these groups? Why do animals sometimes not live in groups? Among humans, why and when do voluntary associations form? What maintains them and what destroys them? Why are territories established and what explains the degrees of effort made to defend them? When does cooperation arise? How is it destroyed? How is cooperation to be distinguished from mutual parasitism and when does one occur rather than the other? Is trust to be distinguished from cooperation? If so, what are its conditions and dynamics? Can the unrestrained actions of purely selfish individuals yield a productive and stable social order? If so, under what conditions? Under what conditions does genuine altruism arise? How is the social order affected by incentives given the participating agents? How can under- and over-incentivizing be recognized? How can optimal levels be determined? How do intelligence and memory affect strategic outcomes and social structure? How stable are the social structures we see about us? What is most likely to make them unstable and what forms of organization would succeed them?

These are just a few examples of the important problems of the social order. More generally, we can ask: for a given society, Why are things the way they are? and What would happen if a particular intervention were to occur? These and the questions above are questions and issues that arise in attempting to understand the formation and behavior of collections of individual agents, be the agents human or not, be the collections communities, societies, entire ecologies or any other sustained form of interaction among agents exhibiting at least minimal forms of behavior. These problems are addressed throughout the social sciences, in much of biology (behavioral ecology is entirely devoted to these problems, cf., [42]), and in much of philosophy (especially, but not exclusively, political and moral philosophy). Each of these disciplines recognizes, to some degree or other, the relevance of strategic decision making and strategic interaction (games) for understanding the social order.

Certain questions, themes, and modes of explanation have figured prominently in the science of the social order, regardless of academic discipline. We are well advised to begin among them. Exhibit 1 is a famous passage from Adam Smith, *The Wealth of Nations*, Book I Chapter II (1776).

> It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from their regard to their own interest. We address ourselves, not to their humanity but to their self-love, and never talk to them of our necessities but of their advantages.

As in many a folk saying—think: “It ain’t over ’til it’s over”—Smith is stating a truism for the sake of making a point. What is the point? We can safely avoid scholarly exegesis in this case. Two ideas lurk. The first is that individually self-interested behavior can lead to socially good results. To get one’s dinner one appeals to the self-interest of the butcher and company by paying them. As the folk saying says, “Every trade has two sides.” The suggestion in both cases—Smith and the folk saying—is that these are transactions in which all parties benefit. Acting from self-interest is sufficient to explain both obtaining dinner and sustaining the butcher, the brewer, and the baker in their livelihoods. To some it will be tempting to strengthen this first idea to the claim that myopically self-interested behavior always or nearly always leads to socially good results. While the weaker claim seems unassailable as an observation, let us leave this second claim as something to be decided by scientific investigation. We shall address it often in the sequel.

The second idea lurking within Smith's passage has the modern terms *emergence* and *self-organization* associated with it. The notion is that (some) macro-phenomena appear as a result of micro-processes that neither resemble the macro-phenomena nor involve intentions to produce the macro-phenomena. The suggestion to hand is that the orderly conduct of business and the sustaining of commercial activity is *emergent.* It is a by-product of a self-organizing system in which large numbers of small transactions are conducted only with myopic self-interest in mind. It is not the product of centralized planning and direction. Smith's name for this idea is a name we continue to use: *the invisible hand*. Here is the single passage in *The Wealth of Nations* (Chapter II of Book IV) in which Smith uses the term.

> As every individual, therefore, endeavours as much as he can both to employ his capital in the support of domestic industry, and so to direct that industry that its produce may be of the greatest value; every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it. By preferring the support of domestic to that of foreign industry, he intends only his own security; and by directing that industry in such a manner as its produce may be of the greatest value, he intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention. Nor is it always the worse for the society that it was no part of it. By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it.

Note that, contrary to many popular accounts, Smith is *not* claiming that emergent properties, produced by the invisible hand, are always or even usually socially, or even individually, good. Often, yes, but he offers no stronger claim, certainly nothing like the "Greed is good" ideology articulated in the movie *Wall Street* (1987, directed by Oliver Stone) and on many blogs and pundit shows these days. Smith is not espousing social Darwinism. Nor is Smith holding a brief for outsourcing or globalization. It is "support of domestic to that of foreign industry" that promotes the interest of society.

Adam Smith's account of economic social order is generally one of good cheer. Self-interested individuals engage in transactions that make them better off and social gains emerge as well. The pattern is far from universal, yet occurs often enough. Although Smith was an acute observer of emergent phenomena, he was hardly the first. Thomas Hobbes was among the earliest and remains important. Here is the signature passage from his *Leviathan* (1651, CHAPTER XIII OF THE NATURAL CONDITION OF MANKIND AS CONCERNING THEIR FELICITY AND MISERY).

> Hereby it is manifest that during the time men live without a common power to keep them all in awe, they are in that condition which is called war; and such a war as is of every man against every man. For war consisteth not in battle only, or the act of fighting, but in a tract of time, wherein the will to contend by battle is sufficiently known: and therefore the notion of time is to be considered in the nature of war, as it is in the nature of weather. For as the nature of foul weather lieth not in a shower or two of rain, but in an inclination thereto of many days together: so the nature of war consisteth not in actual fighting, but in the known disposition thereto during all the time there is no assurance to the contrary. All other time is peace.
>
> Whatsoever therefore is consequent to a time of war, where every man is enemy to every man, the same consequent to the time wherein men live without other security than what their own strength and their own invention shall furnish them withal. In such condition there is no place for industry, because the fruit thereof is uncertain: and consequently no culture of the earth; no navigation, nor use of the commodities that may be imported by sea; no commodious building; no instruments of moving and removing such things as require much force; no knowledge of the face of the earth; no account of time; no arts; no letters; no society; and which is worst of all, continual fear, and danger of violent death; and the life of man, solitary, poor, nasty, brutish, and short.

Hobbes's view was that in the absence of a controlling power—the Leviathan—the war of all against all would prevail and nothing would emerge except "continual fear, and danger of violent death; and the life of man, solitary, poor, nasty, brutish, and short."

We can fairly say that subsequent experience has confirmed both Hobbes and Smith to a degree. Trade flourishes, life is often sweet, yet wars, horrific cruelty and destruction continue. Whether we see peace as the interim between wars or war as the interim between periods of peace, we have to admit the continued presence of both. (Hobbes wrote at a time of horrific civil wars and Smith at the beginning of the Industrial Revolution.) It is a problem of the social order to explain and understand why. Note in particular that in Hobbes's world, "where every man is enemy to every man," there is no trust or cooperation. Note further that in Smith's more felicitous world trust and cooperation are required, at least oftentimes. Explaining when and why trust and cooperation will arise and be sustained is a central concern in understanding the social order.

2.2 Self-Organization and Emergence

It was Hobbes's view that trust and cooperation are not, and for the most part cannot be, emergent properties. They cannot arise by self-organization, emerging socially "from the bottom up"—or in Hobbes's phrase from "the state of nature"—as a result of decentralized, microbehavior by numbers of individuals. Instead, according to Hobbes, trust and cooperation—as macroproperties of a society—will only occur among a group of people if there is "a common power to keep them all in awe." Hobbes had in mind a strong monarch. Deferring for the moment the question of whether, and under what conditions, Hobbes was right about the (lack of) emergence of trust and cooperation, there are plenty of credible examples of emergence.

Darwin ends *On the Origin of Species* (first edition, 1959) with this wonderful and famous paragraph, summarizing the general mechanism by which species emerge by a self-organizing (decentralized, bottom-up) process :

> It is interesting to contemplate an entangled bank, clothed with many plants of many kinds, with birds singing on the bushes, with various insects flitting about, and with worms crawling through the damp earth, and to reflect that these elaborately constructed forms, so different from each other, and dependent on each other in so complex a manner, have all been produced by laws acting around us. These laws, taken in the largest sense, being Growth with Reproduction; Inheritance which is almost implied by reproduction; Variability from the

> indirect and direct action of the external conditions of life, and from use and disuse; a Ratio of Increase so high as to lead to a Struggle for Life, and as a consequence to Natural Selection, entailing Divergence of Character and the Extinction of less-improved forms. Thus, from the war of nature, from famine and death, the most exalted object which we are capable of conceiving, namely, the production of the higher animals, directly follows. There is grandeur in this view of life, with its several powers, having been originally breathed into a few forms or into one; and that, whilst this planet has gone cycling on according to the fixed law of gravity, from so simple a beginning endless forms most beautiful and most wonderful have been, and are being, evolved.

To what extent can social order also be explained as emergent from self-organizing, microbehavior? What about failures of the social order, in particular what are called social dilemmas? These

> ... are defined by two simple properties: (a) each individual receives a higher payoff for a socially defecting choice (e.g. having additional children, using all the energy available, polluting his or her neighbors) than for a socially cooperative choice, no matter what the other individuals in society do, but (b) all individuals are better off if all cooperate than if all defect. [10]

These are pervasive and

> While many thinkers have simply pointed out that our most pressing societal problems result from such dilemmas, most have addressed themselves to the question of how to get people to cooperate. Answers have ranged from imposition of a dictatorship (Leviation) to "mutual coercion mutually agreed upon," to appeals to conscience.

What is the scope of behavior for emergence and self-organization in the context of social dilemmas? Is centralized control needed? If so, under what conditions? What are the risks and pathologies of the various kinds of structures for relieving social dilemmas?

2.3 Combinatorial Complexity

To a first approximation, it may be said that the task of micro-evolution is to find genetic codings for felicitous proteins, which are used to catalyze chemical reactions. Proteins are linear sequences of amino acids (again, I'm approximating). Life forms on earth use 20 amino acids. So there are 20^n possible proteins of length n. Naturally occurring proteins may easily be 1000 amino acids long. The number of possibilities—20^{1000}—is vastly greater than can be searched exhaustively. The estimated total number of atomic particles in the universe is only on the order of 10^{80}. Most of the 20^{1000} possibilities are thought to be useless. How then is successful search by evolution possible?

The number of species used for food by we humans is small compared to the total number of species, yet the number of possible recipes—the combinatorial space of recipes—is enormous. Restricting ourselves only to herbs and spices, Harold McGee in his magnificent work, *On Food and Cooking,* [55] lists more than 100 herbs and spices commonly used in cooking. Many of these come in multiple varieties (for example, parsley has just one entry). Neglecting varieties (Italian or small leaf parsley?) and quantities (How much parsley?), and counting only whether a spice or herb is present or not in a recipe, there are thus more than 2^{100} or about 10^{30} possible combinations of spices. With fewer than 32×10^6 seconds per year, trying one recipe/combination per second would take more than 10^{21} years. But there are fewer than 10^{11} years since the Big Bang. Why, we might wonder, are chefs able to produce new and tasty recipes year and after year? Are happy combinations easily found by purely random search, or is something else going on? Cloves with cumin and ginger anyone?

The number of strategies possible in iterated games is similarly explosive. Consider a 2×2 game played repeatedly from player R's perspective. Played once, there are 2 possible strategies of play. Played twice, there are 2×2^4 possible (pure) strategies. Played n times the number (of pure strategies) is $2^{4^0} \times 2^{4^1} \times 2^{4^2} \times \ldots \times 2^{4^{n-1}}$. For even modest n, we're better off cooking. How is it possible for players to arrive at sensible, well-founded strategies of play in iterated 2×2 games? What about larger games?

2.4 Concepts of Rationality

We use the terms *rational, rationality* and their ilk in ordinary language, in informal but carefully considered theories, and in formal, rigorously articulated theories. Consequently, several distinct senses of these words can be found. In ordinary language, in a commonsense but somewhat vague sense, to be rational is roughly to pursue one's goals efficiently, sensibly, intelligently. To be irrational, or not rational, is to be possessed of self-destructive, or at least self-frustrating, impulses that one cannot control.

WordNet, the electronic dictionary, finds four senses of "rational" as an adjective (`http://wordnet.princeton.edu/`):

- S: (adj) rational (consistent with or based on or using reason) "rational behavior"; "a process of rational inference"; "rational thought"
- S: (adj) intellectual, rational, noetic (of or associated with or requiring the use of the mind) "intellectual problems"; "the triumph of the rational over the animal side of man"
- S: (adj) rational (capable of being expressed as a quotient of integers) "rational numbers"
- S: (adj) rational (having its source in or being guided by the intellect (distinguished from experience or emotion)) "a rational analysis"

and two for "rationality" as a noun:

- S: (n) rationality, reason, reasonableness (the state of having good sense and sound judgment) "his rationality may have been impaired"; "he had to rely less on reason than on rousing their emotions"
- S: (n) rationality, rationalness (the quality of being consistent with or based on logic)

No doubt correct, and quite consistent with other sources, WordNet's definitions nonetheless tell us little about what constitutes rationality. The Wikipedia (`http://en.wikipedia.org/wiki/Rationality`, accessed 22 August 2005) is also explicit in recognizing different senses of, and concepts for, the word rationality, some of which point usefully towards operational theories, e.g., "A logical argument is sometimes described as rational if it is logically valid," and "In economics, sociology, and political science, a decision or situation is often called rational if it is in some sense optimal, and individuals or organizations are often called rational if they tend to act somehow optimally in pursuit of their goals," and "Rationality is a central principle in artificial intelligence, where a rational agent is specifically defined as an agent which always chooses the action which maximises its expected performance, given all of the knowledge it currently possesses."

Others might be added. Many philosophers, for example, would say that someone—say a believer—is rational regarding something—say a belief—if the one has good reasons for the thing. Notoriously, there are many people who often find (correctly or not) that good reasons (and that type of rationality) conflict with self interest (and that type of rationality).

We could go on. We could add new senses, we could refine and more deeply articulate existing senses. Others have; we needn't and won't.

Given such a variety of concepts of rationality, and corresponding senses of the word rational, it is temping to ask which is right. What is it *really* to be rational? These are questions I wish deliberately and forthrightly to evade. I see no good reason why there should be just one concept that we, in ordinary speech or even in carefully wrought theories, attach to the word rational. Let a 100 senses bloom. There may be many ways of being rational, ways whose discovery and application should not be blocked definitionally. This risks confusion, of course, if one sense is meant and another understood. But forewarned is forearmed. Many words have multiple senses. If it matters, speech and communication may be clarified. For similar reasons, Amartya Sen has embraced vagueness with regard to rationality [71, page 4]:

> Rationality is interpreted here, broadly, as the discipline of subjecting one's choices—of actions as well as of objectives, values and priorities—to reasoned scrutiny. Rather than defining rationality in terms of some formulaic conditions that have been proposed in the literature (such as satisfying some prespecified axioms of "internal consistency of choice," or being in conformity with "intelligent pursuit of self-interest," or being some variant of maximizing behavior), rationality is seen here in much more general terms as the need to subject one's choices to the demands of reason.

Present purposes call for a concept of rationality appropriate for decision making in contexts of strategic interaction (games). It is useful to distinguish two such concepts. The first—type A—is well articulated and explored in classical game theory. The second—type B—is the kind of strategic rationality that is the focus of this book. We can get a sense, good enough for present purposes, of what these two concepts are and how they differ by illustrating their application to four games.

Our first game is the well-known Standard Prisoners' Dilemma (SPD), presented in Figure 2.1 in strategic form.

	c_1	c_2
r_1	(3, 3)	(0, 5)
r_2	(5, 0)	(1, 1)

Figure 2.1: Standard Prisoners' Dilemma (SPD). Player R chooses between strategies r_1 and r_2. Player C chooses between c_1 and c_2.

The interpretation of a game in strategic form is straightforward. There are two players,[1] the row player, R, and the column player, C. The row player must choose one of its available strategies, either r_1 or r_2 in figure 2.1, and column must choose one of its strategies, either c_1 or c_2 in the present case. When, as in the Standard Prisoners' Dilemma game, there are two players each with two strategies, we say the game is a 2×2 game: two players, two strategies each. For games in strategic form, we stipulate that each player picks its strategy without observing the other player's choice. Similarly, the players can make no enforceable agreement about which strategies to pick. Once the strategies are picked, the payoffs to the players are determined, as shown by the cells in the figure. If, for example, row chooses r_2 and column chooses c_1, then row's payoff is 5 and column's is 0. In a 2×2 game, there are four possible outcomes—$(r_1, c_1), (r_2, c_1), (r_1, c_2), (r_2, c_2)$—and each outcome has a payoff for each of the players. The payoffs for SPD are given as the entries in figure 2.1.

Our second example game is a *constant sum* game: the total payoff is the same for every outcome. This makes the game one of pure conflict. R's gain is C's loss, and vice versa. Because the game happens to appear on page 90 of Federic Schick's *Making Choices* [70, page 90], I'll call this 2×3 game Schick90. See figure 2.2.

	c_1	c_2	c_3
r_1	(3, 7)	(9, 1)	(1, 9)
r_2	(5, 5)	(7, 3)	(6, 4)
r_3	(4,6)	(2,8)	(8,2)

Figure 2.2: Schick90: A game of pure conflict

Our third example game, Standard Stag Hunt (SSH), is also well-known. It is presented in figure 2.3. We will have occasion to discuss both Stag Hunt and Prisoners' Dilemma at length in the sequel.

[1]More general formulations are possible, but are not needed for present purposes.

	c_1	c_2
r_1	(3, 3)	(0, 2)
r_2	(2, 0)	(1, 1)

Figure 2.3: Standard Stag Hunt (SSH). Player R chooses between strategies r_1 and r_2. Player C chooses between c_1 and c_2.

Our fourth game is an ancient one. I'll call this version of it *One-Two-Twenty*. Two players take turns placing either 1 or 2 tokens on a table, starting from an empty table. The player placing the twentieth token on the table wins the game.

Here now are two approaches—type A and type B—to analyzing these games.

2.4.1 Type A Setup and Analysis

For type A analysis of a game (or a type A game), we need to specify the following items as constituting the setup of the game:

1. The players.

   For the examples to hand there are two players, R and C (think row or red or Robert, and column or cyan or Cynthia). In general there may be any finite number of players.

2. The pure strategy sets, Σ^i for each player, i.

   A strategy (for player i) is a complete set of instructions for play of the game (by player i). In the Standard Prisoners' Dilemma game, $\Sigma^R = \{r_1, r_2\}$, and $\Sigma^C = \{c_1, c_2\}$. Σ for a player is its set of *pure* strategies. When the game is presented in strategic form, as in figures 2.1, 2.2 and 2.3, the pure strategies for the row (column) player are the rows (columns) in the table. In addition to its pure strategies, each player also has *mixed strategies*. These are the probability-weighted combinations of the pure strategies. We denote mixed strategies with a tilde. For example, Σ^C denotes player C's set of pure strategies and $\tilde{\Sigma}^C$ denotes C's mixed strategies. Since the pure strategies are a special case of probabilistic combination of pure strategies (one has a weight of 1, the others have 0), $\tilde{\Sigma}^C$ denotes all of C's strategies.

3. For each outcome, a payoff vector giving payoffs in that outcome for each player.

An *outcome* of a (type A) game is a strategy vector, giving the played strategy of each player.[2] Thus for Standard Prisoners' Dilemma, there are four possible outcomes: (r_1, c_1), (r_1, c_2), (r_2, c_1), and (r_2, c_2). The payoff vectors, $\omega(\cdot)$, for these outcomes are: $\omega(r_1, c_1) = (3, 3)$, $\omega(r_1, c_2) = (0, 5)$, $\omega(r_2, c_1) = (5, 0)$, and $\omega(r_2, c_2) = (1, 1)$. Our convention is that in payoff vector (x, y), R gets x and C gets y.

4. Rules of play for the game.

   Standardly, "The rules of a game must tell us *who* can do *what* and *when* they can do it. They must also indicate who gets *how much* when the game is over." [3, page 25]. I am handling the *how much* aspect of a game separately, as the payoff vectors ω (previous item).

   In the general case of a 2×2 game in strategic form, such as Standard Prisoners' Dilemma in figure 2.1, each player picks a strategy from its strategy set and this determines the outcome (and the payoffs). Players pick without observing each other's choices of strategy.

In addition to the game setup, type A games assume that the players have individual preference orders on the payoffs for the game. I'll illustrate with Standard Prisoners' Dilemma. The game has four possible payoff vectors: $\Omega = \{(3, 3), (0, 5), (5, 0), (1, 1)\}$. For player R his *payoff* for a payoff vector (x, y) is x and because R prefers higher payoffs to lower payoffs, R, let us assume, has the following *preference ordering* on Ω: $(5, 0) \succ_R (3, 3) \succ_R (1, 1) \succ_R (0, 5)$. If an agent a prefers x to y or is indifferent between x and y, we write $x \succeq_a y$ (or just $x \succeq y$ if it is clear who the agent is). If an agent prefers x to y strictly (the agent is not indifferent between them), we write $x \succ y$. An agent is said to be *fundamentally rational* (with respect to a set of payoff vectors Ω) if the agent has a preference ordering, $\succeq$, on Ω such that for every $a, b \in \Omega$:

fundamental rationality

1. The *totality* condition obtains:

   $a \succeq b$ or $b \succeq a$ (or both, in which case we say that the agent is indifferent between a and b and we write $a \sim b$), and

2. The *transitivity* condition obtains:

   If $a \succeq b$ and $b \succeq c$ then $a \succeq c$.

[2]My terminology here is nonstandard. Usually, *outcome* is used for what I am calling the payoff vector. Also, see below for the distinction between the *play* of a game and the *outcome* of play.

Clearly, R's assumed preference ordering on the outcomes qualifies as fundamentally rational in this sense. For the sake of the example, we also assume that R's counter-player, C, has a different preference ordering on the outcomes: $(0,5) \succ_C (3,3) \succ_C (1,1) \succ_C (5,0)$. It too is fundamentally rational.

Given a set of payoff vectors, we say that an agent's choice for or decision of $\omega \in \Omega$ is *rational* or *consistent* with respect to the rational preference ordering $\succeq$ on Ω if for all $\omega' \in \Omega$, $\omega \succeq \omega'$. In short, a choice is rational or consistent with regard to a fundamentally rational preference ordering, if there is no better choice available.

I am *not* suggesting that this is *the* definition of rationality, although economists and game theorists normally treat it that way (e.g., [64, page 19]). Because of its pervasive use and acceptance in this literature, I will honor the term. At times it will serve the purposes of clarity to refer explicitly to *fundamental rationality*. When confusion is unlikely, I will use *rational* and *rationality* either for this specific kind or in a general, unspecified sense (recall the passage from Sen, above) or for the specific kind under discussion.

Agents in games, however, do not get to choose payoff vectors, elements of Ω, directly. Instead they get to choose strategies, elements of their $\tilde{\Sigma}^i$s. Type A analysis of games is about rational choice among strategies, assuming all players are fundamentally rational regarding the payoff vectors, Ω. We thus need to define *type A rationality* in terms of choice among strategies. It is natural to define this sort of rationality much as we defined fundamental rationality with respect to Ω. The complication, of course, is that in a game the strategy choices of all the players must be taken into account. A further complication is that we must allow for play of *mixed strategies*. We need to discuss this last complication first. A little notation, mostly. Nothing terribly complicated.

Each agent has its Σ, a set of basic or *pure strategies*. To repeat: for the Stag Hunt game, R's pure strategies are r_1 (hunt stag) and r_2 (hunt hare), so we have $\Sigma^R = \{r_1, r_2\}$. In addition to the strategies in its Σ, an agent may also form a *mixed strategy* by probabilistic combination of its pure strategies. For the Standard Stag Hunt game, if R plays r_1 with probability $\frac{1}{2}$ and r_2 with probability $\frac{1}{2}$ we can write $\tilde{r} = (r_1, \frac{1}{2}; r_2, \frac{1}{2})$. I'll denote a mixed strategy by using this tilde notation, e.g., $\tilde{r}$ is a mixed strategy. Note that even if Σ is finite, $|\tilde{\Sigma}|$, the number of possible mixed strategies an agent can form from it is (uncountably) infinite.

Now the terminology. We need to distinguish the *play* of a game, from the *outcome* of (play of) a game, from the *payoff* resulting from the outcome of a game. The play of a game, *Play*, is the vector of strategies chosen by each player

Play vs. *Outcome* vs. *Payoff*

for playing the game. If every player plays a pure strategy, then the outcome of play, *Outcome*, is identical to the play. If at least one player plays a mixed strategy, however, chance must resolve the game into some pure strategy, and it is the resulting pure strategy that belongs to the outcome.

The following example illustrates this simple framework. Let the game be Standard Stag Hunt (SSH, figure 2.3). Suppose that *Play* = $(\tilde{r}, c_2)$. That is, R chooses to play the mixed strategy $\tilde{r} = (r_1, \frac{1}{2}; r_2, \frac{1}{2})$, defined above. C decides to play her pure strategy, c_2. Given *Play*, chance now resolves it by instantiating any mixed strategies. Let us say that the coin is flipped and R's $\tilde{r}$ gets resolved to r_1. Then the *Outcome* = (r_1, c_2). Then the *Payoff* = $(0, 2)$, as indicated in figure 2.3, page 40. Note that R's realized payoff is $\frac{1}{2} \cdot 0 = 0$. On average, however, R's expected payoff (given that C plays c_2) is $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$. We can denote this compactly by writing E*Play*$(R, (\tilde{r}, c_2)) = \frac{1}{2}$. In words, the expected value, E, to R of the play $(\tilde{r}, c_2)$ is $\frac{1}{2}$.

We are now in position to define rationality for type A games. Let $\tilde{s}^i \in \tilde{\Sigma}^i$, that is, $\tilde{s}^i$ denotes a (possibly mixed) strategy available to agent i. Numbering the players from 1 to n, we can denote the play of a game by *Play* = $(\tilde{s}^1, \tilde{s}^2, \ldots, \tilde{s}^i, \ldots, \tilde{s}^n)$. We say that agent i is *individually economically rational* (IER) in the play of a game, if there is no strategy other than the one played by i that would yield a superior payoff for i, assuming the play is otherwise unchanged. Formally, given *Play* = $(\tilde{s}^1, \tilde{s}^2, \ldots, \tilde{s}^i, \ldots, \tilde{s}^n)$, agent i is IER if there is no $\tilde{t}^i \in \tilde{\Sigma}^i$ such that E*Play*$(i, (\tilde{s}^1, \tilde{s}^2, \ldots, \tilde{t}^i, \ldots, \tilde{s}^n) \succ$ E*Play*$(i, (\tilde{s}^1, \tilde{s}^2, \ldots, \tilde{s}^i, \ldots, \tilde{s}^n)$. Put otherwise, given i played $\tilde{s}^i$, then i is IER if i has no better strategy, given how the other players played. Again, i is IER if the strategy i played is on average a *best response* to what the other players played.

IER: individually economically rational

An agent that is individually economically rational (IER) is also said to be *consistent* with its fundamental preference ordering on Ω. Individually economically rational agents are said to maximize or optimize, given their fundamental preferences and the choices made by the counter-players in the game. In ordinary discourse, there is a sense of "Agent i is maximizing" that is equivalent to "Agent i is attempting to maximize" or "trying to maximize." This is *not* the sense employed in the present context. To the contrary, to say that an agent maximizes or is individually economically rational or is consistent with its fundamental preferences on Ω *means* that the agent actually succeeds in playing a strategy that is a best strategy, given the actual play by the other players. How the agent might know which strategy to play, given that the strategy choices by the other players are hidden, is an entirely separate matter—in the type A setup—for the sake of determining whether the agent is individually economically rational or not.

Nash equilibrium

For a given play, if (and only if) all of the players are individually economically rational (in this type A sense), we say that the play is a *Nash equilibrium* and that the outcome resulting from the play is *supported by* the Nash equilibrium play. Put equivalently, in a Nash equilibrium no player *individually* has incentive to change its chosen strategy. It is not precluded by the Nash equilibrium concept that two or more players might together change their strategies in such a way that both (or all) are better off. The concept of a Nash equilibrium is tied essentially to that of the possibilities for individuals acting alone, one at a time. Note again that individually economically rational outcomes and Nash equilibrium outcomes are defined without reference to how they may be arrived at or discovered. While analysts of a finite game may discern all of the solutions and test for individual rationality and Nash equilibrium, players in the game may or may not have sufficient means to hand to make these discoveries. Finally, we say that a *solution to a type A game setup* is a Nash equilibrium.

With these notions to hand, we can now treat the example games from the type A perspective.

A type A story (or analysis) for the Prisoners' Dilemma game is especially straightforward. There is exactly one play that is individually economically rational for both players (and hence is a Nash equilibrium): (r_2, c_2). Note that the associated payoff vector is $(1, 1)$ and that both players would do better if the play were (r_1, r_1). Hence the dilemma. Type A analysis predicts the Nash equilibrium as the play of this game.

There is also an attractive line of reasoning that explains how the players might reach the Nash equilibrium by reasoning individually from their knowledge of the game. Player R has two pure strategies. The second, r_2, is said to *dominate* the first because no matter which (mixed or pure) strategy the counter-player, C, plays, R gets an outcome he (I'll use "she" for C) prefers more if he plays r_2 rather than r_1. So, if R is to be consistent with his rational preference ordering on his Ω, R must choose to play strategy r_2. A completely analogous story applies to C. She must play c_2, her dominant strategy, if she is to be consistent with her preference ordering over her Ω. Game solved.

Our second example game is Schick90, figure 2.2 page 39. Each player for this game of pure conflict has a plausibly attractive decision rule for finding a strategy: play a *maximin* strategy, a strategy that maximizes its minimum possible payoff. Taking row's perspective, playing r_2 guarantees R a payoff of at least 5. Since the minimum payoff for r_1 is 1 and the minimum for r_3 is 2, r_2 is uniquely the strategy for R that maximizes the player's minimum payoff. Similar reasoning identifies c_1 as C's maximin strategy. Play of (r_2, c_1) is in consequence individually economically rational for both players and is a—indeed the—Nash equilibrium for this game. Game solved.

Note a subtle difference between Standard Prisoners' Dilemma (SPD) and Schick90. In SPD each player is able to reason by elimination of dominated strategies and determine a uniquely attractive strategy of play completely independent of how the counter-player will play. No matter what row does, column is better off playing c_2. The situation is different in Schick90. When both players are IER each gets a payoff of 5. The reasoning and justification for r_2 (and c_1) relies on the fact that these are "safety strategies". No matter what the counter-player does, these strategies guarantee to their players a maximal minimum. No other strategy for each player is guaranteed to produce more than 5. If, however, one player knows, or has a reasonable amount of evidence, that the counter-player will not play its maximin strategy, then the player might be able to do better with a different strategy. For example, if C is certain that R will play r_1, then C should play c_3 for a payoff of 9. Generally and in distinction to Prisoners' Dilemma, which strategy yields the highest payoff for a player now depends on which strategy is played by the counter-player. Reasoning by an individual player leading to a Nash equilibrium play will normally require the assumption of IER play by the counter-player(s).

The type A analysis of Stag Hunt is straightforward, but a bit problematic. Plays (r_1, c_1) and (r_2, c_2) are both Nash equilibria. There is in addition a third play, involving *mixed strategies*. If an agent plays x with probability p and y with probability $(1-p)$ we write this mixed strategy as $(x, p; y, 1-p)$. For the Stag Hunt game in figure 2.3 the play in mixed strategies, $(\tilde{r} = (r_1, \frac{1}{2}; r_2, \frac{1}{2}), \tilde{c} = (c_1, \frac{1}{2}; c_2, \frac{1}{2}))$ is the third Nash equilibrium. Players playing at this equilibrium can expect a payoff of 1.5 each.

Two things make the Stag Hunt problematic in type A analysis. First, which play will prevail and what will be the distribution of plays when the game is surveyed across many plays? The Nash equilibrium concept by itself cannot discriminate among the three Nash equilibria. It is possible, of course, to single out one or another of the equilibria as favored in virtue of properties it has in addition to being a Nash equilibrium. Game theorists have tended to favor (r_2, c_2) because it poses the least risk of a 0 payoff to any player, but there is not general agreement on this. There is a worry, moreover, that the properties so identified will not generalize to other games with multiple equilibria. Although principled selections may be made in specific cases, the problem of multiple equilibria for type A analysis remains recalcitrant.

The second problem for type A analysis presented by the Stag Hung game is related to the first, and is perhaps but an aspect of it. This is the problem of specifying a procedure for arriving at an equilibrium. Sticking to just the two equilibria in pure strategies, r_1 is R's strategy for one of the equilibria, while r_2 is the strategy for the other. But R can't by himself pick one of the equilibria. If he prefers (r_1, c_1) he can play r_1, but if C prefers (r_2, c_2) and she plays c_2 then the play is not a Nash equilibrium. Further, although $(\tilde{r}, \tilde{c})$ is a Nash equilibrium, plays combining mixed and pure strategies are not, e.g., $(\tilde{r}, c_2)$. If there is no procedure or path of reasoning by which the players coordinate on a single equilibrium, how is it that play is at an equilibrium?

The type A story for One-Two-Twenty is a bit complex, but not especially difficult. I will give it only in outline. A *state of the game*, $e = (i, n)$, is specified by which player, i, has the next play and how many stones, n, are on the board. Since there may be $0, 1, \ldots, 20$ stones on the board and at any time it may be either player's turn, there are 42 possible states. This number is reduced once we specify who goes first, but the details are not important for us. Let us say that C goes first. It is easy to see that there is a winning strategy for C. Let the value of state e to player i, $V^i(e)$, be 1 if once the game is in that state player i can be guaranteed of a win. Clearly $V^C(C, 19) = 1 = V^C(C, 18)$, for C can add either 1 or 2 stones to produce 20 on the board and thereby win the game. Consequently $V^C(R, 17) = 1$, since R can then produce only states $(C, 18)$ and $(C, 19)$, which have a value to C of 1. Continuing to reason backwards in a similar manner we find that $V^C(R, 14) = 1 = V^C(R, 11) = \ldots = V^C(R, 2)$. Consequently, $V^C(C, 0) = 1$. C can begin the game by placing 2 stones on the table, thereby producing state $(R, 2)$. If R produces state $(C, 3)$, then C puts 2 stones down, producing state $(R, 5)$; otherwise, R produces state $(C, 4)$ and C puts 1 stone down, again producing state $(R, 5)$. Play continues in this fashion until C wins the game. C's strategy, combined with *any* play by R is a Nash equilibrium. Note that if C deviates from this strategy, say by producing state $(R, 6)$, then there is a strategy available to R for winning the game. In the example, R would produce $(C, 8)$ and be in position to win the game. Thus, in One-Two-Twenty rational (type A) play as specified by the Nash equilibrium accords well with what we would atheoretically expect rational players to do. The game favors whoever goes first. That player wins if the outcome is a Nash equilibrium.

2.4.2 Type B Setup and Analysis

The setup for, or description of, a type B game includes the following elements:

0. The supergames.

   A type B game consists of one or more supergames. Each supergame comprises many (2 or more) subgames. In a simple case, the subgame (aka: stage game) would be Standard Prisoners' Dilemma and the supergame would be 25 rounds of play of the subgame between two fixed players.

1. The players.

   Every game, including type B games, has a least 2 players. If there are only 2, I will continue to call them R and C; otherwise they will be labeled $P_1, P_2, \ldots, P_i, \ldots$.

2. The policy sets, Π^i, for each player, i.

   A policy is a strategy (complete set of instructions) for playing a subgame. Policies are defined in such a way that a player may during the course of a supergame play under one policy for part of the supergame and under another policy for a different part. For example, if the supergame consists of 25 rounds of play of Standard Prisoners' Dilemma, a player might cooperate for rounds 1-11 and defect for rounds 12-25.

   A main difference between games of type A and games of type B, is that in the former we view players as choosing among strategies, while in the latter we view players as choosing (directly) among policies. They choose strategies only indirectly, as emerging from their policy choices.

3. For each outcome of every atomic subgame, a payoff vector specifying payoffs for that outcome for each (involved) player.

   A subgame is atomic if all of its outcomes are associated with elements of a relevant Ω. A nonatomic subgame may be composed of atomic subgames or may have as payoffs the rewards from playing in other subgames.

4. The adaptation regime(s) used by the players, ρ (or indexed , e.g., ρ_i if there is more than one).

   Players play by following policies for play during a supergame. A particular action in a particular subgame is determined by the policy in effect for the player in question. The player's adaptation regime determines which policy will be in effect at any given time. Necessary for games of type B, a game description of type A lacks this element entirely. I will discuss examples shortly.

5. Rules of play for the (super)game(s).

   As in type A game descriptions, the rules of play govern the sequencing and other conditions under which the players make their decisions.

The key differences, then, between a type B game setup (description, model) and one of type A are that (i) type B games are always supergames, consisting of multiple subgames, (ii) players directly choose policies rather than strategies, with the policies in turn determining play in subgames,[3] and (iii) players have adaptation regimes which produce their choices of policies. The salient feature of type B game setups is that players try policies, receive feedback from play of subgames, invoke their adaptation regimes, and either try new policies or continue on, depending on direction from their adaption regimes. This is a process of adapting (and perhaps learning) in policy space.

Our example games can be used to illustrate type B game setups. Standard Prisoners' Dilemma (SPD) first. The supergame for this example consists of iterated play of SPD as a stage game. After each round of play the supergame halts with probability 0.02; otherwise another round is played. Players R and C have identical consideration sets of policies for play, $\Pi = \{$ALWAYS DEFECT, ALWAYS COOPERATE, TIT FOR TAT$\}$. Under the ALWAYS DEFECT policy, if the player is R he plays r_2 whenever he has the policy in force and if the player is C she plays c_2 whenever she has the policy in force. Similarly, they play r_1 and c_1 if ALWAYS COOPERATE is in force. Finally, under TIT FOR TAT, the player playing it cooperates (r_1 or c_1, depending on the player) in the first play for which the policy is in force. After that, so long as the policy is used, the player mimics the play (cooperative or not) of the counter-player in the previous round of play. Players independently pick policies and play them for a number of rounds of play. Each player keeps track of the performance of, the returns from, play with its policies, and uses this information when selecting a new policy for play. What will happen? Will ALWAYS DEFECT win out as the best of the available policies? Will a different policy win out or will there be no settling down of the process? Later we shall see.

Our example model for Standard Stag Hunt is a *gridscape* model. Players are arrayed on a regular network. Think of a chessboard, but one that "wraps" around so that each cell has 8 neighbors. This is a model for a simple society. Agents are either stag hunters or hare hunters. Agents in parallel play all of their 8 neighbors

[3] Policies need not be deterministic. They may involve randomized decisions. No special notation—such as use of the tilde in type A games, $\tilde{s}$—will be needed.

and count up their points. Each agent then looks to its neighbors and sees if any have obtained more points by using a different strategy. If so, the agent adopts the strategy of a neighbor whose achieved points is highest and play continues. Figure 2.4 is a screen shot from the program `m1-symmetric-2x2.nlogo`, set up to run this model on a gridscape with 49,933 cells. The cells have been initialized randomly, with a strong bias towards hare hunting. Black cells are occupied by stag hunters, which are very much in the minority. How will the gridscape evolve when play commences? Will one or another policy (HUNT HARE, HUNT STAG) take over the gridscape or will a mixture be present indefinitely?

For either One-Two-Twenty or Schick90 (both of which are games of pure opposition) imagine again that two players R and C play the game repeatedly and that in the case of One-Two-Twenty C is given the first move each time the game is played. Neither R nor C, let us assume, have the capacity to analyze the game as we did above and to figure out an optimal strategy. Are there ways that simpler agents (simpler than us) might figure out the game? Clearly yes. An agent with a bit of memory and an elementary ability to reason backwards could surely learn by experience in iterated play of the game and achieve optimal, or at least high quality, play. In the case of One-Two-Twenty, the value of later states could be learned first by trial and error, followed by the value of neighboring earlier states. Eventually the entire game could be learned. How quickly might this be done? How well does the approach generalize to more complex games, such as chess and checkers?

2.5 Discussion: Accessibility & Games

A few terminological stipulations will facilitate discussion here and in the sequel.

Suppose that algorithm (or procedure or rule) α accepts inputs β and produces γ. Let us then say that γ *is accessible from* β *via* α. If needed, it is possible to give a more formal, rigorous definition of accessibility, but that requirement is not to hand. Examples can carry the burden of clarification: (1) γ is $\neg P$, β is $P \rightarrow Q, \neg Q$, and α is *modus tollens.* (2) γ is 27, β is $x = 3$, and $\alpha = x^3$. I deliberately leave open what sorts of things γ and β may be (e.g., numbers, statements, formulas, etc.). α is correspondingly open; it is any procedure—deterministic or randomized—for producing γ from β. γ itself may be definite—*Rain tomorrow*—or probabilistic—*Chance of rain tomorrow greater than 0.7*.

Let us also say that γ *is accessible from* β if there is some (not necessarily specified) algorithm α such that γ is accessible from β via α.

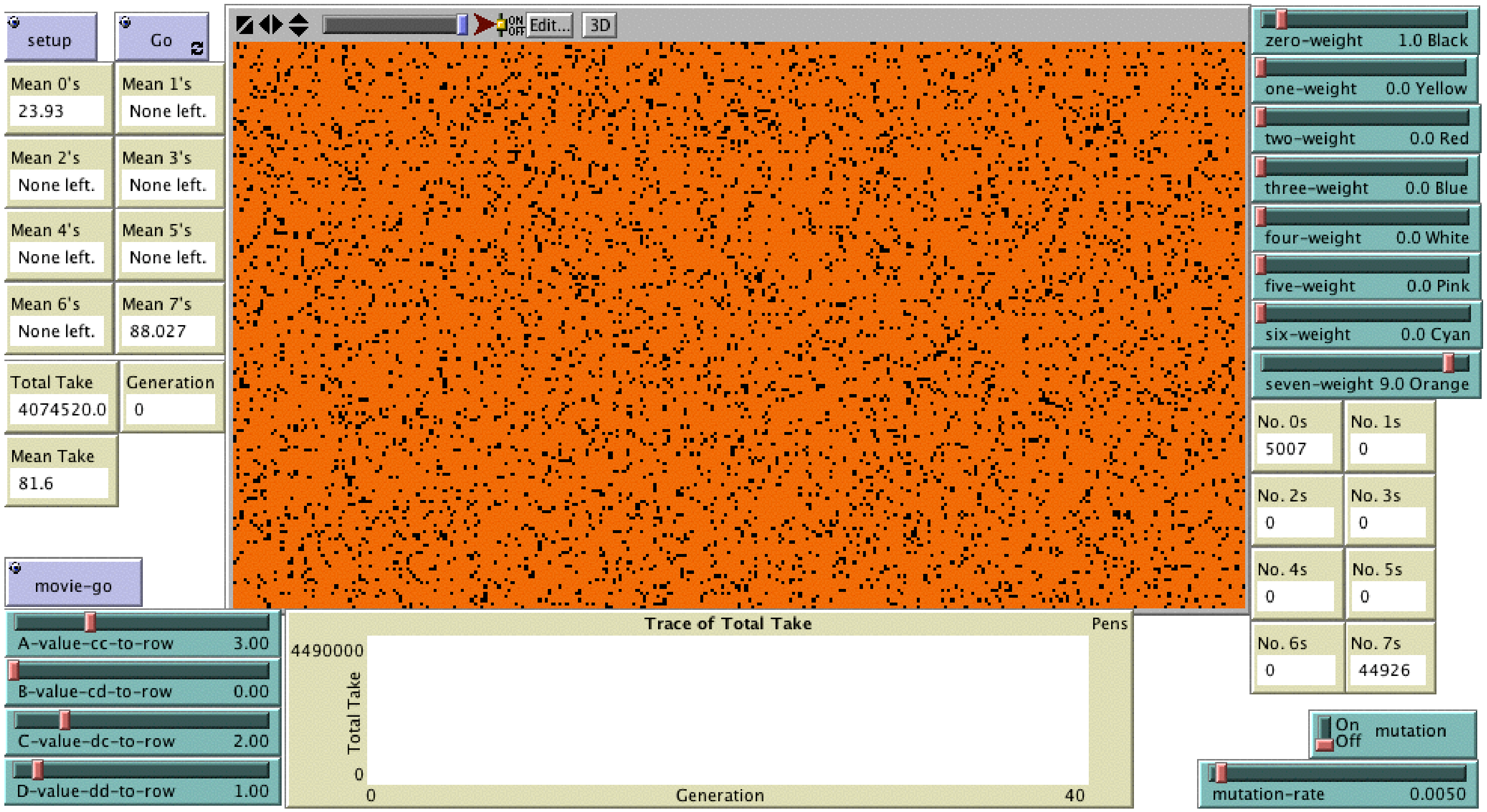

Figure 2.4: Setup for Iterated Play of Standard Stag Hunt on the Gridscape. Program: `m1-symmetric-2x2.nlogo`.

Under what conditions is something *not* accessible? Let us say that γ *is practicably accessible from* β *via* α if γ is accessible without undue cost or delay from β via α. Quite clearly, much that is accessible (in principle) is not practicably accessible. practicable accessibility

One way for absolute inaccessibility to occur is when applying α to β yields more than one result. Then we have to say that any single result is not accessible from β via α. Equations with more than one root (solution) will perhaps be the most familiar example. Suppose that y, the position of an object, equals t^2 in minutes, with noon today set as $t = 0$. At what time is y at 16? The equation allows two answers: 4 minutes before noon and 4 minutes after noon. So here we can say that γ is *t = 4 minutes before noon or t = 4 minutes after noon* is accessible from the equation via mathematical solution. It would be incorrect to say that γ is *t = 4 minutes before noon* is accessible from the equation via mathematical solution, even though γ = *t is 4 minutes before noon* is consistent with the equation via mathematical solution. Accessibility resembles visibility under a microscope. A set (organelle) may be accessible (visible under the microscope) without its elements (components) being accessible (visible).

Back now to games. Let β be a type A game with the assumption of fundamental rationality for all players. Let α be a procedure that selects player R's strategy in every Nash equilibrium. Then in general neither γ as R *plays strategy* r_1 nor γ as R *does not play strategy* r_1 is accessible (from β via α). The Stag Hunt game illustrates the point. In some equilibria R plays r_1, in some r_2, and in some a mixture. The Nash equilibrium concept does not reveal which equilibrium will obtain. Because player R has more than one strategy involved in the equilibria, only a set (larger than 1) is accessible.

Practicable accessibility is also an issue for type A games. Under Zermelo's theorem (`http://www.ams.org/featurecolumn/archive/games3.html`, accessed 17 December 2005) any (type A) game that is finite, played by two players under perfect information (each player knows all the moves so far from the other player), and is strictly competitive is such that either the first player can force a win or a draw or the second player can force a win or a draw. Finding such an equilibrium, however, is another matter. Chess, checkers and many other board games qualify under the theorem, but their sizes and complexities preclude completion of the analysis. The equilibria of such a game are practicably inaccessible via the Nash equilibrium procedure (here, Zermelo's backwards induction procedure).

Some but not all type A games are not practicably accessible because of computational complexity. Some but not all are not accessible at all because of multiple equilibria. The situation is quite different for type B games. By hypothesis, the players each have adaptation regimes that select policies for play and these produce outcomes in atomic subgames. Convergence to a stable outcome or even stable distribution of outcomes may or may not occur, let alone convergence to an equilibrium. Because conditions of play may be stochastic, different runs of play may produce different results. We may think of γ in the context of type B games (β) as a stream of outcomes and α as the adaptation regimes assigned to the several players. Conceived this way, something, some γ, is always accessible, practicably accessible, in type B games. They are designed to be that way.

At bottom, the distinction between type A and type B games is largely one of perspective or stance, of attitude we bring to the subject. In a type A model we are mainly interested in the Nash equilibria and are less concerned with accessibility issues.[4] The theory for type A games may be called *equilibrium game theory.* In a type B model we endow the players with policy spaces and adaptation regimes, which they deploy in conducting the game and from which their strategies emerge. We may call the theory for these games *constructive game theory*. It investigates how game results emerge from the interplay of policy spaces and adaptation regimes.

constructive game theory

constructive rationality

Whatever rationality type B agents have is a *constructive rationality*; play, γ, is produced by a policy in force, α, typically relying on a history of play, β. I will have much to say later about constructive rationality in games. For now, this observation. The perspective of constructive game theory is likely to be apt under a number of conditions, including these:

1. When fundamental rationality cannot be assumed.

   Note that economic rationality is easily violated if agents have to rely on realistic perceptual mechanisms. Letting $a \sim b$ be interpreted as "a is not perceptually distinguishable from b, " we can easily have $a \sim b$, $b \sim c$ and $a \succ c$. The relation $a \succ c$ might be used for such perceptual modalities as "is at least as hot as" and perhaps even "is worth at least as much as."

   Constructive game theory (type B) representations may or may not assume fundamental rationality. In either case, the games will play out, agents will adapt and an evaluatable pattern of play will result.

2. When a type B game is a natural model of the system under examination, regardless of whether a type A representation is useful.

[4] As usual, by "Nash equilibria" I mean Nash equilibria or refinements thereof.

The one-shot Prisoners' Dilemma is more naturally modeled as a type A game, while the indefinitely repeated version is usefully addressed from the perspective of constructive game theory.

3. If at least some players are epistemically limited (nonhuman animals surely count) and cannot plausibly be assumed to be individually economically rational.

4. If the underlying game is too complex for meaningful type A (equilibrium) analysis.

   Equilibrium analysis for tic-tac-toe and One-Two-Twenty, constructivist analysis for chess and checkers.

5. If the underlying game has multiple equilibria after excluding implausible equilibria.

   The Folk Theorem (a real theorem; see standard texts such as [3, 43] for proofs and discussion) tells us that in indefinitely repeated games the equilibria proliferate.

6. If a procedure of play is sought by which the players can operate and with which explanations, predictions, and interventions may be made.

   There are special cases, such as when elimination of dominated alternatives produces a unique outcome, in which procedures are available for producing equilibria in type A games. With complexity or repetition, however, such cases are rare.

The main focus of this book is constructive game theory (and constructive rationality), in distinction to equilibrium theory (and economic rationality). I will throughout, however, honor and draw upon the concepts and insights of that approach. Before directly taking up constructive analysis of games and their modeling with agent-based systems, a word on evaluation criteria. Details will be articulated during the analyses that follow.

In a constructive (type B) game setup, we model players as agents with specific, albeit quite limited, powers of thought. Agents have a cognitive apparatus, (Π, ρ), consisting of a consideration set of policies, Π, and an adaptation regime, ρ. Under equilibrium analysis there is only one criterion of evaluation: What are the individually rational outcomes? Such parsimony is not available to us in the case of constructive games. Instead, the list of criteria is best left open. Here is a list that serves to begin a fruitful discussion.

1. Performance against self.

   Does the apparatus do well against itself? If all agents use the apparatus, do the agents as a group prosper relatively well in the ambient environment?

2. Performance against others.

   Does the apparatus do well playing against other regimes that do well against themselves?

3. Exploitability.

   Is the apparatus catastrophically exploitable? Does it have weaknesses that may be discovered by another apparatus?

4. Robustness.

   Is the apparatus robust under perturbations of its parameters? Does the apparatus perform well against a field of others?

5. Learnability.

   Can the apparatus be parameterized in such a way that an agent can (easily) learn profitable, well-performing settings?

6. Computational cost.

   Is the apparatus computationally tractable? Is it simple or does it require excessive computational resources from the agent?

7. Informational requirements.

   Does the apparatus rely on plausibly available information? Or does it require information not likely to be available in the actual system being modeled?

The foregoing is enough to get us properly started. Let the games begin.

2.6 Bibliographic Notes

2.6.1 Social Order

Wikipedia has a useful entry for *social order*, `http://en.wikipedia.org/wiki/Social_order`. Here is an exerpt:

> Social order is a concept used in sociology, history and other social sciences.
>
> It refers to a set of linked social structures, social institutions and social practices which conserve, maintain and enforce "normal" ways or relating and behaving.
>
> Thus, a "social order" is a relatively stable system of institutions, pattern of interactions and customs, capable of continually reproducing at least those conditions essential for its own existence. The concept thus refers to all those facets of society which remain relatively constant over time.
>
> These conditions could include both property, exchange and power relations, but also cultural forms, communication relations and ideological systems of values.

My use here of *social order* is not tied to any particular doctrine or theory of society. Instead, I use the expression to refer generally to the regular, systematic patterns of behavior evidenced by a particular society (or collection of individuals).

Dawes [10], Hardin [32], and Elster [11, 12, 13] are excellent on connecting the study of strategic interaction with problems of the social order.

2.6.2 Emergence

The Wikipedia entry for emergence is quite good: `http://en.wikipedia.org/wiki/Emergence`.

NetLogo (`http://ccl.northwestern.edu/netlogo/`) comes with a library of models, many of which pertain to social order, emergence, or strategic interaction. The following NetLogo models, from the model library that comes with installation, are especially pertinent to the subject of emergence:

- `Flocking.nlogo`
- `Slime.nlogo`
- `Traffic Basic.nlogo`
- `Segregation.nlogo`
- `RumorMill.nlogo`

Rodney Brooks on "situated agents" [6], Robert Frank's *Luxury Fever* [20], and John Holland's *Emergence* [36] all make stimulating reading. There is a terrific book on biology and emergence [7], and I would be remiss not to mention the delightful classic, *Vehicles* [5].

2.6.3 Rationality

Binmore [3] and Kreps [43] are good textbook treatments of economic rationality and game theory. Sen's essay, "Rational Fools," [72] is an indispensable antidote.

On the constructive side, David Fogel's *Blondie24* [19] is a very readable account of his successful construction of a computer program that learned to play checkers at a high level.

Chapter 3

Preview Study: Trust and the Stag Hunt Game

3.1 Introduction

Can trust arise spontaneously—by an invisible hand as it were—among strategically interacting individuals? If so, under what conditions will it arise? When will it be stable and when will it not be stable? What interventions might be effective in promoting or undermining stability? If trust is established, under what conditions will it be destroyed? What are the rôles of social structure, game structure, and cognition in establishing or disestablishing trust? These questions belong to a much longer list of important and challenging issues that the problem of trust presents. Answering them fully and adequately constitutes a research program to challenge a community over a period of many years.

I aim in this chapter to contribute in two ways to that program. In the end, although progress will be made, more work will have been added. First, I shall present findings, focusing on the Stag Hunt game, that bear more or less directly on at least some of these questions. I shall focus on the Stag Hunt game for several reasons. The game does capture well and succinctly certain aspects of the problem, the dilemma, of trust. It has the happy virtue of not being the (overworked but still worthwhile) game of Prisoners' Dilemma. Also, it has fruitfully received new attention of late (e.g., [76]), so that what I will add here will, I hope, enrich a topic that is very much in play.

The second way in which I aim to contribute to the trust research program is more indirect. Trust is a problem or a puzzle, even paradox, in part because there seems to be more of it naturally occurring than can be explained by received theory (classical game theory). I shall say little by way of documenting this claim because space is limited and I take it that the claim is widely accepted. (Chapter 3, “Mutual Aid,” of [74] is a discussion of how and why nature is *not* “red in tooth and claw.” Also behavioral game theory, reviewed in [8] amply documents the imperfect fit between theory and observation in this domain. Those with a taste for blunt talk might consult [26].) Instead, I hope to say something about how the puzzle of trust might be investigated. I will submit that the puzzle of trust arises, at least in part, because of a presupposed account of agent rationality. This account goes by different names, among them *expected utility theory* and *rational choice theory*. I want to propose, in outline form, a very different approach to conceiving of rationality. By way of articulating this general approach, which I shall call a *theory of exploring rationality*, I shall present a model of agent behavior which, in the context of a Stag Hunt game (as well as other games), explains and predicts the presence of trust.

3.2 Stag Hunt and a Framing of the Program

The Stag Hunt game (also known as the Assurance game [23]) gets its name from a passage in Jean Jacques Rousseau’s *A Discourse on the Origin of Inequality,* originally published in 1755.

> Was a deer to be taken? Every one saw that to succeed he must faithfully stand to his post; but suppose a hare to have slipped by within reach of any one of them, it is not to be doubted but he pursued it without scruple, and when he had seized his prey never reproached himself with having made his companions miss theirs. [65, Second Part]

Here is a representative summary of the Stag Hunt game.

> The French philosopher, Jean Jacques Rousseau, presented the following situation. Two hunters can either jointly hunt a stag (an adult deer and rather large meal) or individually hunt a rabbit (tasty, but substantially less filling). Hunting stags is quite challenging and requires mutual cooperation. If either hunts a stag alone, the chance of success is minimal. Hunting stags is most beneficial for society but requires a lot of trust among its members. [24]

This account may be abstracted to a game in strategic form. Figure 3.1 on the left presents the Stag Hunt game with payoffs that are representative in the literature.[1] Let us call this our *reference game*. On the right of figure 3.1 we find the Stag Hunt game presented in a generic form. Authors differ in minor ways. Often, but not always, the game is assumed to be symmetric, in which case $R=R'$, $T=T'$, $P=P'$, and $S=S'$. I will assume symmetry. It is essential that $R>T>P\geq S$.[2]

	Hunt stag (S)	Chase hare (H)
Hunt stag (S)	4 4	3 1
Chase hare (H)	1 3	2 2

	Hunt stag (S)	Chase hare (H)
Hunt stag (S)	R′ R	T′ S
Chase hare (H)	S′ T	P′ P

Figure 3.1: Stag Hunt (aka: Assurance game)

Thus formalized, the Stag Hunt game offers its players a difficult dilemma, in spite of the fact that their interests coincide. Each does best if both hunt stag (S,S). Assuming, however, that the game is played once and that the players lack any means of coming to or enforcing a bargain,[3] each player will find it tempting to "play it safe" and hunt hare. If both do so, (H,H), the players get 2 each in our reference game, instead of 4 each by playing (S,S). Both of these outcomes—(S,S) and (H,H)—are Nash equilibria. Only (S,S), however, is Pareto optimal.

[1] For example, although it is not called the Stag Hunt, the game with these payoffs is discussed at length in [62], where it is simply referred to as game #61.

[2] Usually, and here, $P>S$. Some authors allow $T\geq P$ with $P>S$. None of this matters a great deal for the matters to hand.

[3] In the jargon of game theory, this is a *noncooperative* game.

There is a third Nash equillibrium for Stag Hunt: each player hunts stag with probability $\frac{P-S}{((R+P)-(T+S))}$ For our reference game, this amounts to a probability of $\frac{1}{2}$ for hunting stag (and $\frac{1}{2}$ for hunting hare). At the mixed equilibrium each player can expect a return of $2\frac{1}{2}$. Notice that if, for example, the row player hunts hare with probability 1, and the column player plays the mixed strategy, then the row player's expected return is $2\frac{1}{2}$, but the column player's expected return is $1\frac{1}{2}$. Uniquely, the safe thing to do is to hunt hare, since it guarantees at least 2. Hunting hare is thus said to be *risk dominant* and according to many game theorists (H,H) would be the predicted equilibrium outcome.[4]

We can use the Stag Hunt game as a model for investigation of trust. A player hunting stag trusts the counter-player to do likewise. Conversely, a player hunting hare lacks trust in the counter-player. Deciding not to risk the worst outcome (S) is to decide not to trust the other player. Conversely, if trust exists then risk can be taken. There is, of course, very much more to the subject of trust than can be captured in the Stag Hunt game. Still, something is captured. Let us see what we can learn about it.

Before going further it is worth asking whether Rousseau has anything else to say on the matter to hand. Typically in the game theory literature nothing else in this *Discouse* or even in other of Rousseau's writings is quoted. As is well known, Rousseau wrote in praise of the state of nature, holding that people were free of war and other ills of society, and on the whole were happier. That needn't concern us here. What is worth noting is that Rousseau proceeds by conjecturing (his word, above) a series of steps through which man moved from a state of nature to the present state of society. Rousseau is vague on what drives the process. The view he seems to hold is that once the equilibrium state of nature was broken, one thing led to another until the present. Problems arose and were solved, one after the other, carrying humanity to its modern condition. He laments the outcome, but sees the process as more or less inevitable. With this context in mind, the passage immediately before the oft-quoted origin of the Stag Hunt game puts a new light on Rousseau's meaning. He is describing a stage in the passage from the state of nature to civil society.

> Such was the manner in which men might have insensibly acquired some gross idea of their mutual engagements and the advantage of fulfilling them, but this only as far as their present and sensible interest required; for as to foresight they were utter strangers to it, and far

[4] I'm using the term risk dominant in a general way, since adverting to its precise meaning would divert us. See [33] for the precise meaning.

> from troubling their heads about a distant futurity, they scarce thought of the day following. Was a deer to be taken? . . . [65, Second Part]

Rousseau is describing behavior of people not far removed from the state of nature. Language, for example, comes much later in his account. These people end up hunting hare because "as to foresight they were utter strangers to it, and far from troubling their heads about a distant futurity, they scarce thought of the day following." If we *define* the game to be one-shot, then there is no future to worry about. Rousseau is right: if there is no future or if the players cannot recognize a future, then stag will roam unmolested. Rousseau is also right in presuming that in the later development of civil society the future matters, agents can recognize this, and much coordination and hunting of stag occurs. Rousseau is *not* agreeing with contemporary game theorists in positing hunting of hare as the most rational thing to do in the Stag Hunt game.

More generally, trust happens. Our question is to understand how and why. We assume that there is a future and we model this (initially) by repeating play of a basic game, called the *stage game*, here the Stag Hunt. Given repeated play of a stage game, there are two kinds of conditions of play that call out for investigation. The first is condition is the *social aspect* of play. We investigate a simple model of this in §3.3. The second condition might be called the *cognitive aspect* of play. How do learning and memory affect game results? We discuss this second aspect in §3.4.

3.3 The Gridscape: A Simple Society

We shall work with a very simple model of social aspects of strategic interaction, called the *gridscape*. The gridscape is a regular lattice—think of a checkerboard—which we will assume is two-dimensional and wraps around on itself (is technically speaking a torus). Agents or players occupy cells on the gridscape and each has 8 neighbors. Figure 3.2(a) illustrates. Cell (3,2) has neighbors (2,1), (2,2), (2,3), (3,1), (3,3), (4,1), (4,2), (4,3).[5] Every cell has eight neighbors. Thus, the neighbors of (1,1) are (6,6), (6,1), (6,2) (1,6), (1,2) (2,6), (2,1), and (2,2). With the gridscape as a basis it is now possible to undertake a variety of experiments. We'll confine ourselves to a simple one.

[5]This is called the *Moore neighborhood*. The *von Neumann neighborhood*, consisting of the four neighbors directly above, below, to the left, and to the right, is also widely studied. The results we report here are not sensitive to which of the two neighborhood definitions is in force.

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

(a) Labeled 6×6 gridscape

Generation	Ss	Hs	Total Points
0	7167	7233	344960.0
1	8715	5685	451224.0
2	12288	2112	599424.0
3	13891	509	668312.0
4	14306	94	686720.0
5	14392	8	690712.0
6	14400	0	691200.0

(b) Representative run on a 120×120 gridscape. R=3, T=2, P=1, S=0.

	1	2	3	4	5	6
1	S	S	H	S	S	S
2	S	S	H	S	S	S
3	S	S	H	S	S	S
4	S	S	H	S	S	S
5	S	S	H	S	S	S
6	S	S	H	S	S	S

(c) Stable 6×6 gridscape, for R=3, T=2.5, P=1, S=0

Figure 3.2: Labeled 6×6 gridscape

Protocol 1 (Basic Gridscape Stag Hunt Protocol) *Each cell in the gridscape is initialized by placing on it either a hare hunter, H, or a stag hunter, S. H and S are the consideration set of policies of play for the experiment. Players using the H policy always hunt hare; and players using the S policy always hunt stag. Initialization is random in the sense that every cell has the same probability of being initialized with an H (or an S). After initialization, play proceeds by discrete generations. Initialization concludes generation 0. At the start of each subsequent generation each player (at a cell) plays each of its 8 neighbors using its policy in use from the consideration set. The player records the total return it gets from playing the 8 neighbors. After all players have played their neighbors, each player updates its policy in use. A player changes its policy in use if and only if one of its neighbors is using the counter-policy and has achieved a strictly higher total return in the current generation than has the player or any of its neighbors with the same policy in use achieved. (This is called the* IMITATE THE BEST NEIGHBOR *policy.) Policy updating completes the generation. Play continues until a maximum number of generations is completed.*

The table in figure 3.2(b) shows representative data when the gridscape is seeded randomly (50:50) with stag hunters (Ss) and hare hunters (Hs) and protocol 1 is executed. Here, after 5 generations the Hs go extinct. The stag hunters conquer the board. This is the usual case, but it is not inevitable. To see why, consider the examples in figure 3.3, which shows a 3×3 block of stag hungers (Ss). The remaining cells are blank and for the purposes of the discussion may be filled in as needed.

The first thing to notice is that in figure 3.3(a) at (3,3) we have an S that is completely surrounded by Ss. This cell obtains a total reward of $8 \times R = 8 \times 4 = 32$ in our reference example. It is impossible to do equally well or better, given the setup. In consequence, given the protocol, once a 3×3 block of Ss is created none of its members will ever change to H. This is true for all versions of the Stag Hunt game (under protocol 1). We say that a 3×3 block of Ss cannot be invaded. More generally, it is easy to see that no rectangular block of Ss larger than 3×3 can be invaded either. (Note further that blocks of hare hunters are not so advantaged. An internal hare hunter gets $8T < 8R$ in all Stag Hunt games.)

Can a block of stag hunters grow? Assume that in figure 3.3(a) the blank cells are all hare hunters. In general, the stag hunter at (2,3) will get a return of $5R+3S$ which is $5 \times 4 + 3 \times 1 = 23$ in our reference game. The hare hunter at (1,3) will get $5P + 3T$ in general and $5 \times 2 + 3 \times 3 = 19$ in the reference game. And in general, so long as $5R + 3S > 5P + 3T$ a hare hunter in this position will convert

	1	2	3	4	5	6	7	8
1								
2		S	S	S				
3		S	S	S				
4		S	S	S				
5								
6								
7								

(a) Generation x

	1	2	3	4	5	6	7	8
1			S					
2		S	S	S				
3	S	S	S	S	S			
4		S	S	S				
5			S					
6								
7								

(b) Generation x+1

	1	2	3	4	5	6	7	8
1		S	S	S				
2	S	S	S	S	S			
3	S	S	S	S	S			
4	S	S	S	S	S			
5		S	S	S				
6								
7								

(b) Generation x+2

Figure 3.3: Growth of a block of stag hunters when R=4, T=3, P=2, and S=1

	$1-\varepsilon$: TFT	ε: ALLD	Total
$1-\varepsilon$: TFT	$(1-\varepsilon)(l_eR)$ $(1-\varepsilon)40$	$\varepsilon((l_e-1)P+S)$ 9ε	$40-31\varepsilon$
ε: ALLD	$(1-\varepsilon)(T+(l_e-1)P)$ $12(1-\varepsilon)$	$\varepsilon(l_eP)$ 10ε	$12-2\varepsilon$

Table 3.1: State 1: Payoffs to Row in Stag Hunt example when the system is in state 1 and $l_e = 10$.

to stag hunting. Note that not all Stag Hunt games will support this conversion. For example, R=101, T=100, P=99, and S=0 will not. Figures 3.3(b) and (c) show the next two generations and the pattern is clear: the stag hunters will drive the hare hunters to extinction.

Is conquest by stag hunters inevitable if a 3×3 block is created and the game rewards are sufficient for it to grow in a field consisting entirely of hare hunters? Equivalently, if $5(R-P) > 3(T-S)$ and a 3×3 block (or larger) of stag hunters forms, is it inevitable that hare hunters are driven to extinction? No it is not. For example, the configuration in figure 3.2(c) is stable for the given payoff values.

There are many nice questions to ask and many interesting variations on the basic gridscape model for the Stag Hunt game. For present purposes, however, the following points are most on topic.

1. The gridscape Stag Hunt results described above are robust. What happens—whether Hs come to dominate or not, whether a stable mixture results and so on—depends on the game payoffs (What is it worth if both players hunt stag? etc.) and the initial configuration. For a broad range of cases, however, hunting stag will eventually dominate the society. Trust—in the form of hunting stag predominately—can arise spontaneously among strategically interacting individuals. In fact, this is far from an implausible outcome.

2. Trust in Stag Hunt on the gridscape is also robust in a second sense: once it is established in large part, it is not easily dislodged. Mutations occurring at a small rate in a field of stag hunters will create mostly isolated hare hunters who will convert to S in the next generation. If stag hunters do well without noise they will do reasonably well with it.

3. The gridscape model under protocol 1 evidences a clear social effect. What we may call the *shadow of society* appears and affects the outcome. The

policies played by one's neighbors may be, and usually are, influenced by policies played by players who are not one's neighbors. Recalling figure 3.3(a), what happens to a hare hunter at (1,3) depends very much on the fact that the neighboring stag hunters are themselves adjacent to other stag hunters. Thus, while the hare hunter at (1,3) beats the stag hunter at (2,3), in the sense that it gets more points in the one-on-one play, the stag hunter at (2,3) in aggregate does better and it is the hare hunter who is converted.

4. The Prisoners' Dilemma game arguably presents a trust dilemma in more extreme form than does the Stag Hunt. Can cooperators largely take over the gridscape? Yes, under certain, more restricted conditions. In Prisoners' Dilemma, we require $T > R > P > S$ and $2R > T + S$. Further, we relabel the policies. Hunt Stag becomes Cooperate and Hunt Hare becomes Defect. On the gridscape, the 3×3 (and larger) block is key in the analysis. If, for example, we set $T = R+1$, $P = 1$ and $S = 0$ in Prisoners' Dilemma, then so long as $R > 4$, a 3×3 (and larger) block of cooperators will be able to expand in a field of defectors. Defectors may not be eliminated, but they may become very much minority constituents of the gridscape. Note further that if Prisoners' Dilemma games are repeated (either infinitely with discounting or finitely), and the TIT FOR TAT policy replaces ALWAYS COOPERATE, then the payoff structure will, under broad conditions, become a Stag Hunt (cf. [8, chapter 7], [76, chapter 1]).

5. The agents on the gridscape have an update policy, IMITATE THE BEST NEIGHBOR, which they use to choose policies for play from their consideration sets. Under this update policy, from protocol 1, agents change their policies of play if, after a round of play, one of their neighbors has used the alternative policy and gotten more points than either the player or one its neighbors playing with the player's policy of play. This is a reasonable update policy, but there are reasonable alternatives. Is there a sense in which it is optimal? Is some other update policy optimal? Is there a sense in which it is an ESS (evolutionarily stable strategy) [53]?

These are all interesting questions, well worth investigation. Space is limited, however, and doing so would divert us from the main theme. A larger issue raised is this. Any reasonable update policy, including ours, may be interpreted as taking a stand on the uncertainty faced by the agent.[6] Agents in games may be interpreted

[6] I am using *uncertaintly* here in its technical sense, which contrasts with risk [50]. In a decision under risk we have an objectively supported probability distribution (or density) on the outcomes.

as seeking maximum return. That indeed is a presumption underlying the strategic framework. It is not a presumption that the agents are conscious or have intentions.

Regarding the stand on uncertainty, agents following our update policy are engaging in risky behavior whenever they opt for hunting stag. Yet when all agents so behave collective stag hunting robustly follows. Note the "Total Points" column in figure 3.2(b). In the first generation the agents collectively garnered 344960 points from the gridscape. If the agents had not updated their policies of play this is where it would stay. As we see, with the update policy used (IMITATE THE BEST NEIGHBOR) the agents collectively more than doubled their take from the gridscape. IMITATE THE BEST NEIGHBOR has this to commend itself: it does well when playing against itself. In Prisoners' Dilemma the same can be said for TIT FOR TAT. Notice as well that NEVER UPDATE, ALWAYS HUNT STAG does well against itself in Stag Hunt and ALWAYS COOPERATE does well against itself in Prisoners' Dilemma. Further, IMITATE THE BEST NEIGHBOR does well against NEVER UPDATE, ALWAYS HUNT STAG, as TIT FOR TAT does well against ALWAYS COOPERATE. Before pursuing these comments further, indeed as a means of doing so, let us turn to a more sophisticated model of learning by agents in games.

	ε: TFT	$1-\varepsilon$: ALLD	Total
$1-\varepsilon$: TFT	$\varepsilon(l_eR)$ $\varepsilon 40$	$(1-\varepsilon)((l_e-1)P+S)$ $9(1-\varepsilon)$	$9+31\varepsilon$
ε: ALLD	$\varepsilon(T+(l_e-1)P)$ 12ε	$(1-\varepsilon)(l_eP)$ $10(1-\varepsilon)$	$10+2\varepsilon$

Table 3.2: State 2: Payoffs to Row in Stag Hunt example when the system is in state 2 and $l_e = 10$.

	$1-\varepsilon$: TFT	ε: ALLD	Total
ε: TFT	$(1-\varepsilon)(l_eR)$ $(1-\varepsilon)40$	$\varepsilon((l_e-1)P+S)$ 9ε	$40-31\varepsilon$
$1-\varepsilon$: ALLD	$(1-\varepsilon)(T+(l_e-1)P)$ $12(1-\varepsilon)$	$\varepsilon(l_eP)$ 10ε	$12-2\varepsilon$

Table 3.3: State 3: Payoffs to Row in Stag Hunt example when the system is in state 3 and $l_e = 10$.

Not so in a decision under uncertainty.

	ε: TFT	$1-\varepsilon$: ALLD	Total
ε: TFT	$\varepsilon(l_e R)$ $\varepsilon 40$	$(1-\varepsilon)((l_e-1)P+S)$ $9(1-\varepsilon)$	$9+31\varepsilon$
$1-\varepsilon$: ALLD	$\varepsilon(T+(l_e-1)P)$ 12ε	$(1-\varepsilon)(l_e P)$ $10(1-\varepsilon)$	$10+2\varepsilon$

Table 3.4: State 4: Payoffs to Row in Stag Hunt example when the system is in state 4 and $l_e = 10$.

3.4 A Model for Exploring Rationality

We turn now to a more sophisticated model for learning by two agents engaged in repeated play of a stage game.[7] The model—MLPS: Markov Learning in Policy Space—is highly stylized and has quite unrealistic assumptions. It is, however, valid as an approximation of realistic conditions; I ask for the reader's indulgence.

The key idea is that agents have a *consideration set of policies for play, $\mathcal{S}$*. The *supergame* consists of an indefinitely long sequence of *games*, each of which is a finite sequence of *rounds of play* of a stage game (e.g., Stag Hunt). Agents draw elements from their $\mathcal{S}$s and use them as *focal policies* for a period of time, or number of rounds of play, called a *game*. Each game is divided into n_e epochs of length l_e. Thus, the number of rounds of play in a game is $n_e l_e$. During an epoch an agent plays its current focal policy with probability $(1-\varepsilon)$, and other policies from its consideration set the rest of the time, with probability ε.

At the end of each game, g_{t-1}, a player, p, picks a focal policy, f_p^t, from its consideration set, $\mathcal{S}$, for play in game g_t. The players use the *fitness-proportional* choice rule. Let $\widehat{V}(p,i,j,k)$ be the average value per round of play returned to player p for policy i, when p has focal policy j and $-p$ (the counter-player) has focal policy k. (Similarly, $V(p,i,j,k)$ is the value realized in a particular round of play.) Then

$$\Pr(f_p^{t+1} = i | f_p^t = j, f_{-p}^t = k) = \widehat{V}(p,i,j,k) / \sum_i \widehat{V}(p,i,j,k) \quad (3.1)$$

That is, the probability that a player chooses a policy for focus in the next game is the proportion of value it returned per round of play, compared to all the player's policies, during the previous game.

There is nothing egregiously unrealistic about these assumptions. The MLPS model strengthens them for the sake of mathematical tractability. Specifically,

[7] More extensive treatment of this model may be found in [40].

it is assumed that a mechanism is in place so that the two players are exactly coordinated. Each has its games begin and end at the same time (round of play in the sequence). Further, each game is neatly divided into epochs and the random choices are arranged so that each player's $\widehat{V}$ values exactly realize their expected values. The upshot of this is that the $\widehat{V}$ values seen by the players are constant, as are the underlying expected values. The resulting system is a stationary Markov process with states $\mathcal{S}_p \times \mathcal{S}_{-p}$ and the equilibrium distribution of states can be analytically determined.

To illustrate, assume the stage game is Stag Hunt with $R = 4, T = 3, P = 1$, and $S = 0$. Assume that each player has a consideration set of two policies of play: (1) TIT FOR TAT (TFT) in which the player begins (in the epoch) by hunting stag and subsequently mimics the behavior of the counter-player on the previous round of play, and (2) ALWAYS DEFECT (ALLD) in which the player always hunts hare. This system has four possible states: (1) both players in the game have TFT as their focal policy, (TFT, TFT), (2) player 1 (Row) has TFT as its focal policy and player 2 (Column) has ALLD as its focal policy, (TFT, ALLD), (3) (ALLD, TFT), and (4) (ALLD, ALLD). With $l_e = 10$ we get the payoffs for the various states as shown in tables 3.1–3.4.

Letting $\varepsilon = 0.1$, routine calculation leads to the transition matrix indicated in table 3.5.

	s(1)=(1,1)	s(2)=(1,2)	s(3)=(2,1)	s(4)=(2,2)
s(1)	$0.7577 \cdot 0.7577 = 0.5741$	$0.7577 \cdot 0.2423 = 0.1836$	$0.2423 \cdot 0.7577 = 0.1836$	$0.2423 \cdot 0.2423 = 0.0587$
s(2)	$0.5426 \cdot 0.7577 = 0.4111$	$0.5426 \cdot 0.2423 = 0.1315$	$0.4574 \cdot 0.7577 = 0.3466$	$0.4574 \cdot 0.2423 = 0.1108$
s(3)	$0.7577 \cdot 0.5426 = 0.4111$	$0.7577 \cdot 0.4574 = 0.3466$	$0.2423 \cdot 0.5426 = 0.1315$	$0.2423 \cdot 0.4574 = 0.1108$
s(4)	$0.5426 \cdot 0.5426 = 0.2944$	$0.5426 \cdot 0.4574 = 0.2482$	$0.4574 \cdot 0.5426 = 0.2482$	$0.4574 \cdot 0.4574 = 0.2092$

Table 3.5: Stag Hunt transition matrix data assuming fitness proportional policy selection by both players, based on previous Tables 3.1–3.4. Numeric example for $\varepsilon = 0.1 = \varepsilon_1 = \varepsilon_2$.

At convergence of the Markov process:

Pr(s(1))	Pr(s(2))	Pr(s(3))	Pr(s(4))
0.4779	0.2134	0.2134	0.0953

So 90%+ of the time at least one agent is playing TFT. Note the expected take for Row per epoch by state:

1. $(1 - \varepsilon)(40 - 31\varepsilon) + \varepsilon(12 - 2\varepsilon) = 34.39$
2. $(1 - \varepsilon)(9 + 31\varepsilon) + \varepsilon(10 + 2\varepsilon) = 11.91$
3. $\varepsilon(40 - 31\varepsilon) + (1 - \varepsilon)(12 - 2\varepsilon) = 14.31$
4. $\varepsilon(9 + 31\varepsilon) + (1 - \varepsilon)(10 + 2\varepsilon) = 10.39$

Further, in expectation, Row (and Column) gets $(0.4779\ 0.2134\ 0.2134\ 0.0953) \cdot (34.39\ 11.91\ 14.31\ 10.39)' = 23.02$ (per epoch of length $l_e = 10$, or 2.302 per round of play), much better than the 10.39 both would get if they played ALLD with ε-greedy exploration. Note that even the latter is larger than the return, 10 per epoch or 1 per round, of settling on the risk-dominant outcome of mutually hunting hare. There is a third, mixed, equilibrium of the one-shot Stag Hunt game. For this example it occurs at (($\frac{1}{2}$S, $\frac{1}{2}$H), ($\frac{1}{2}$S, $\frac{1}{2}$H)). At this equilibrium each player can expect a return of 2 from a round of play. Players playing under the MLPS regime learn that trust pays. A few points briefly before we turn to the larger lessons to be extracted from these examples.

1. Markov models converge rapidly and are quite robust. The results on display here hold up well across different parameter values (e.g., for ε). Further, relaxation of the mechanism of play so that agents get imperfect, but broadly accurate, estimates of the expected values of the V quantities will not produce grossly different results. We get a nonstationary Markov process, but in expectation it behaves as seen here.

2. The MLPS model also has attractive behavior for different kinds of games. Players in Prisoners' Dilemma games will learn a degree of cooperation and do much better than constant mutual defection. In games of pure conflict (constant sum games) the outcomes are close to those predicted by classical game theory. And in coordination games players go far by way of learning to coordinate. See [40] for details.

3. If we retain the core ideas of the MLPS model, but entirely relax the synchronization conditions imposed by the game mechanism, simulation studies produce results that qualitatively track the analytic results: the players learn to trust and more generally the players learn to approach Pareto optimal outcomes of the stage game [41].

3.5 Discussion

Neither the gridscape model nor the MLPS model with protocol 1 nor the two together are in any way definitive on the emergence of trust in repeated play of Stag Hunt games. They tell us something: that trust can arise spontaneously among strategically interacting agents, that this can happen under a broad range of conditions, that it can be stable, and so on. The models and their discussion here leave many questions to be investigated and they raise for consideration many new questions. Much remains to be done, which I think is a positive result of presenting these models. I want now to make some remarks in outline by way of abstracting the results so far, with the aim of usefully framing the subject for further investigation.

LPS models: learning in policy space. Both the gridscape model and the MLPS model with protocol 1 are instances of a more general type of model, which I call an LPS (learning in policy space) model. In an LPS model an agent has a consideration set of policies or actions it can take, $\mathcal{S}$, and a learning or update, L/U, policy it employs in selecting which policies to play, or actions to take, at a given time. In the gridscape model, $\mathcal{S} = \{H, S\}$ for every player. In the MLPS model with protocol 1, $\mathcal{S} = \{$TFT, ALLD$\}$ for both players. In the gridscape model the L/U policy employed by all players was IMITATE THE BEST NEIGHBOR. In the MLPS model, the players used the fitness-proportional update rule, in the context of the mechanism described in the previous section.

LPS models categorize strategies. In classical game theory the players are conceived as having *strategies,* complete instructions for play, which they can be thought of as choosing before the (super)game starts. The possible strategy choices constitute what we call the consideration set, $\mathcal{S}$. Because strategies are picked *ex ante* there is no learning, although the strategies can be conditioned on play and can mimic any learning process. The agents employ what we might call the *null learning/update rule,* $L/U_{\emptyset}$. In an LPS model with a non-null L/U policy, the consideration set of policies of play does not include all possible strategies in the game. Policies in $\mathcal{S}$ are tried sequentially and played for a limited amount of time, then evaluated and put into competition with other members of $\mathcal{S}$. The L/U policy constitutes the rules for comparison, competition and choice. The total number of possible strategies is not affected by imposition of the LPS framework, but the strategies are implicitly categorized and the agents choose among them during the course of play (instead of *ex ante*). The consideration set of *strategies* used by an agent is implicit in its consideration set of policies, its L/U policy, the structure of the game, and the play by the counter-players. Thus, LPS models subsume standard game-theoretic models. A *proper* LPS model, however, has a non-null L/U policy. Normally, when I speak of an LPS model I shall be referring to a proper LPS model.

Folk Theorem undercuts. According to the Folk Theorem,[8] nearly any set of outcomes in an indefinitely repeated game can be supported by some Nash equilibrium. In consequence, the Nash equilibrium becomes essentially worthless as a predictive or even explanatory tool, in these contexts. The problems of trust arise against this backdrop and against the following point.

Refinements unsatisfying. Refinements to the classical theory, aimed at selecting a subset of the Nash equilibria in predicting outcomes, have been less than fully satisfying. This is a large subject and it takes us well beyond the scope of the present chapter. However, the favored refinement for Stag Hunt would be universal hunting of hare, because it is the risk dominant equilibrium. (For a general discussion see [81, 80].) Agents playing this way might well be viewed as "rational fools" [72] by LPS agents.

LPS agents may be rational. At least naïvely, the L/U regimes employed by our gridscape and MLPS agents are sensible, and may be judged rational, or at least not irrational. Exploring the environment, as our LPS agents do, probing it with play of different policies, informed by recent experience, is on the face it entirely reasonable. Why not try learning by experience if it is not obvious what to do in the absence of experience? I shall now try to articulate a sense in which LPS agents may be judged rational, even though they violate the rationality assumptions of classical game theory and rational choice theory.

Contexts of maximum taking (MT). Given a set of outcomes whose values are known, perhaps under risk (i.e., up to a probability distribution), given a consistent, well-formed preference structure valuing the outcomes, and given a set of actions leading (either with certainty or with risk) to the outcomes, rational choice theory (or utility theory) instructs us to choose an action that results in our taking the maximum expected value on the outcomes. Presented with valued choices under certainty or risk, we are counseled to take the maximum value in expectation. Although the theory is foundational for classical game theory and economics, it has also been widely challenged both from a normative perspective and for its empirical adequacy.[9]

[8] A genuine theorem, described in standard texts, e.g., [3].

[9] Good, wide-ranging discussion can be found in [21, 29]. A classic paper [45] develops a model in which for the finitely repeated Prisoners' Dilemma game it is sometimes rational for a player to cooperate, *provided the player believes the counter-player is irrational.* Since both players would benefit by mutual cooperation it seems a stretch to call all attempts to find it irrational.

Contexts of maximum seeking (MS). In an MS context an agent can discriminate among outcomes based on their values to the agent, but the connection between the agent's possible actions and the resulting outcomes is uncertain in the technical sense: the agent does not have an objectively well grounded probability distribution for associating outcomes with actions. In seeking the maximum return for its actions, the agent has little alternative but to explore, to try different actions and to attempt to learn how best to take them.[10]

Exploring rationality is appropriate for MS contexts. The claim I wish to put on the table is that in MS as distinct from MT contexts, rationality is best thought of as an appropriate learning process. An agent is rational in an MS context to the extent that it engages effectively in learning to obtain a good return. In doing so, it will be inevitable that that agent engages in some form of trial and error process of exploring its environment. Rationality of this kind may be called an *exploring rationality* to distinguish it from what is often called *ideal rationality,* the kind described by rational choice theory and which is, I submit, typically not appropriate in MS contexts. See [39] for further discussion of the concept of an exploring rationality.

Evaluate exploring rationalities analytically by performance. LPS models with their articulated L/U regimes afford an excellent framework for evaluating forms of exploring rationality. Such evaluation will turn largely on performance under a given L/U regime. For starters and for now informally, an L/U regime may be assessed with regard to whether it is generally a strong performer. Rational admissibility is a useful concept in this regard.

General Definition 1 (Rational Admissibility) *A learning (update) regime for policies of play in an indefinitely repeated game is* <u>*rationally admissible*</u> *if*

1. *It performs well if played against itself (more generally: it performs well if universally adopted).*

2. *It performs well if played against other learning regimes that perform well when played against themselves (more generally: the other learning regimes perform well if universally adopted).*

3. *It is not vulnerable to catastrophic exploitation.*

[10]Classical game theory seeks to finesse this situation by assuming classical rationality and common knowledge. The present essay may be seen as an exploration of principled alternatives to making these very strong assumptions.

To illustrate, in the gridscape model IMITATE THE BEST NEIGHBOR performs well against itself in that when everyone uses it, as we have seen, trust breaks out and stag hunting prevails robustly. The null L/U policy of ALWAYS HUNT STAG also does well against itself, and both IMITATE THE BEST NEIGHBOR and ALWAYS HUNT STAG will do well against each other. ALWAYS HUNT STAG, however, is catastrophically vulnerable to ALWAYS HUNT HARE. IMITATE THE BEST NEIGHBOR on the other hand will do better, although how much better depends on the payoff structure of the stage game. Some stag hunters may open themselves up to exploitation because they have one neighbor who hunts stag and is surrounded by stag hunters. In sum, with reference to the set of these three L/U policies, IMITATE THE BEST NEIGHBOR is uniquely rationally admissible (robustly, across a wide range of stag game payoff structures). A similar point holds for the MLPS model discussed above.

Two additional comments. First, "not vulnerable to catastrophic exploitation" is admittedly vague. It is not to my purpose to provide a formal specification here. I believe that more than one may be possible and in any event the topic is a large one. The motivating intuition is that a learning regime is vulnerable to exploitation if it learns to forego improving moves for which the counter-players have no effective means of denial. Thus, an agent that has learned to hunt stag in the face of the counter-player hunting hare is being exploited because it is foregoing the option of hunting hare, the benefits of which cannot be denied by the counter-player. Similarly, agents cooperating in Prisoners' Dilemma are not being exploited. Even though each is foregoing the temptation to defect, the benefits of defecting can easily be denied by the counter-player following suit and also defecting. Second, the similarity between the definition, albeit informal, of rational admissibility and the concept of an ESS (evolutionarily stable strategy, [53]) is intended. In a nutshell, a main message of this chapter is that for repeated games it is learning regimes and consideration sets of policies, rather than strategies alone, that are key to explanation. (And dare one suggest that rational play in one-shot games may sometimes draw on experience in repeated games?)

Evaluate exploring rationalities empirically, for descriptive adequacy. As noted, it is well established that rational choice theory (ideal rationality) is not descriptively accurate at the individual level. In light of the results and observations given here, one has to ask to what degree subjects at variance from the received theory are perceiving and responding to contexts of maximum seeking (MS), rather than the postulated MT contexts. In any event, it is worth noting that foraging by animals—for food, for mates, for shelter or other resources—is a ubiquitous natural form of behavior in an MS context [27, 78], for which models under the LPS framework would seem a good fit. Experimental investigation is only beginning. I think it shows much promise.

* * *

In conclusion, the problems and paradoxes of trust are vitally important on their own. Trust is the "cement of society".[11] Understanding it is crucial to maintenance and design of any social order, including and especially the new social orders engendered by modern communications technologies, globalization, global warming, and all that comes with them. I have tried to contribute in a small way to understanding how and when trust can emerge or be destroyed. The gridscape and MLPS models are helpful, but they can be only a small part of the story and even so their depths have barely been plumbed. But it's a start; it's something. The more significant point, I think, is that the problems of trust lead us, via these very different models, to a common pattern that abstracts them: LPS, learning in policy space, and contexts of maximum seeking (MS), as distinguished from contexts in which maximum taking (MT) is appropriate. The fact, demonstrated here and elsewhere, that agents adopting this stance generate more trust and improve their take from the environment, is encouraging. So is the observation that such behavior is analogous to, if not related to or even a kind of, foraging behavior.

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`$Id: staghunt1.tex,v 2.3 2006/01/03 01:19:05 sok Exp $`

[11] Elster's term [13], after Hume who called causation the cement of the universe.

Chapter 4

Games in the Abstract

4.1 Interrogation

The scene is Sam Spade's apartment. Gutman and Cairo are partners in search of the Maltese Falcon. They have the guns. Spade has the information. Spade is being difficult.

> Cairo, his face and body twitching with excitement, exclaimed: "You seem to forget that you are not in a position to insist on anything."
>
> Spade laughed, a harsh derisive snort.
>
> Gutman said, in a voice that tried to make firmness ingratiating: "Come now, gentlemen, let's keep our discussion on a friendly basis; but there certainly is"—he was addressing Spade—"something in what Mr. Cairo says. You must take into consideration the—"
>
> "Like hell I must." Spade flung his words out with a brutal sort of carelessness that gave them more weight than they could have got from dramatic emphasis or from loudness. "If you kill me, how are you going to get the bird? If I know you can't afford to kill me till you have it, how are you going to scare me into giving it to you?"
>
> Gutman cocked his head to the left and considered these questions. His eyes twinkled between puckered lids. Presently he gave his genial answer: "Well, sir, there are other means of persuasion besides killing and threatening to kill."
>
> "Sure," Spade agreed, "but they're not much good unless the threat of death is behind them to hold the victim down. See what I mean? If

> you try anything I don't like I won't stand for it. I'll make it a matter of your having to call it off or kill me, knowing you can't afford to kill me."
>
> "I see what you mean." Gutman chuckled. "That is an attitude, sir, that calls for the most delicate judgment on both sides, because, as you know, sir, men are likely to forget in the heat of action where their best interest lies and let their emotions carry them away."
>
> Spade too was all smiling blandness. "That's the trick, from my side," he said, "to make my play strong enough that it ties you up, but yet not make you mad enough to bump me off against your better judgment."
>
> Gutman said fondly: "By Gad, sir, you are a character!"
>
> –From "The Fall-Guy" in *The Maltese Falcon* by Dashiell Hammett

The passage describes a context of strategic interaction. How might we model this game in the wild? A number of *game forms* are available. For the present we will use just one, the *strategic form*, which is well-suited to games with two players. See Figure 4.1 for the general, canonical strategic form for two players each having two strategies.

	C_1	C_2
R_1	r_1, c_1	r_2, c_2
R_2	r_3, c_3	r_4, c_4

Figure 4.1: Canonical game matrix for the 2×2 game in strategic form

The interpretation is straightforward. There are two players: Row, who choses R_1 or R_2, and Column, who choses C_1 or C_2. We say Row has available the *strategies* R_1 and R_2, and similarly Column has strategies C_1 and C_2. The form is called *strategic form* because the players' strategies are laid out so plainly. If players have more than two strategies, we add rows or columns to the *game matrix* as necessary. If there are more than two players we can use a game cube or hypercube (for more than 3 players) if needed.

Choosing simultaneously, or at least in ignorance of each other's choices, there are four possible outcomes and associated *rewards* for the players. In terms of the canonical game matrix, Figure 4.1:

Outcome	R's reward	C's reward
R_1C_1	r_1	c_1
R_1C_2	r_2	c_2
R_2C_1	r_3	c_3
R_2C_2	r_4	c_4

Returning now to our scene of interrogation in Sam Spade's apartment, the players are Spade (let us say Row) and Gutman (Column). Spade may either Blab (B) regarding the whereabouts of the bird, or Keep Silent (K). Gutman will interrogate. He will either Press (P) or Extreme Press (E), the latter killing Spade in the process, or at least severely disabling him. Spade's preference ordering is $KP > BP > KE > BE$. Gutman's is $BP > BE > KP > KE$. That is, Spade prefers most that Row play K and Column play P; Gutman prefers most that Row play B and Column play P. Converting these rankings to convenient numbers gives us Figure 4.2. The numbers range from 1 to 4, more being better. Here they simply record the ranking of the rewards or outcomes for the players in question.

	Press (P)	Extreme Press (E)
Blab (B)	4 3	3 1
Keep Silent (K)	2 4	1 2

Figure 4.2: Interrogation: Spade is Row, Gutman is Column

What will happen? Gutman is surely right that the situation "calls for the most delicate judgment on both sides." Putting that aside for the present and looking at the game as abstracted in Figure 4.2, we can see that no matter which strategy Gutman pursues, Press or Extreme Press, Spade will prefer—is rewarded more by—Keeping Silent. If Gutman Presses, Spade gets 4 for Keeping Silent and 3 for Blabbing. If Gutman Extreme Presses, Spade gets 2 from Keeping Silent and 1 from Blabbing. Either way, Spade does better by Keeping Silent. Notice that the actual values of the rewards to Spade matter little. We could replace 4 by A, 3 by B, 2 by C and 1 by D and this conclusion would follow for any numbers assigned to the letters, so long as higher numbers represent preferred rewards and $A > B > C > D$.

So we predict Spade will Keep Silent. What will Gutman do? Gutman is a smart fellow and he will presumably see the reasoning in the situation—as he appears to in the dialog—and conclude that Spade will choose to Keep Silent. Given that, Gutman's best strategy is to Press, since choosing Extreme Press won't get him the Maltese Falcon and burdens him with Spade's demise. Gutman's "By Gad, sir, you are a character!" is an admission of defeat.

Some terminology and associated concepts that we will need throughout: a dominant strategy, a Nash equilibrium, and a Pareto optimal outcome. We say Spade's (Row's) Keep Silent strategy *dominates* his Blab strategy, because on every alternative—here, either Press or Extreme Press—Spade's Keep Silent yields a higher reward to Spade than does his Blab. The typographic conventions we are using for game matrices were designed to make it easy to spot dominating or dominated strategies. See Figure 4.2. Gutman (Column) also has an absolutely dominate strategy: Press beats Extreme Press.

The *principle of dominance* enjoins us—or at least predicts of all rational players—never to choose a dominated strategy. If Spade and Gutman are rational in this rather minimal sense, KP (Gutman Presses and Spade Keeps Silent) will be the outcome. In this game the dominance principle is sufficient to predict a unique outcome. Is this in fact what will always happen? Gutman's remark that "men are likely to forget in the heat of action where their best interest lies and let their emotions carry them away" is surely apt. We should ask ourselves whether other rational causes might lead to violation of the principle. This is a matter for the sequel.

We say that KP is a *Nash equilibrium* (or NE) outcome because no player could change its strategy *unilaterally* and do better.[1] KP is an NE because for each player, given the play by the other player(s), its strategy is best. Specifically, KP is an NE because *given that Gutman chooses Press*, Spade can do no better with Blab, and *given that Spade chooses Keep Silent*, Gutman can do no better with Extreme Press. A Nash equilibrium outcome is a strategic standoff: no player can do better, given what the other players have played. One might, and classical

[1]Named in honor of John Nash, 1928–, who invented and developed the concept. His home page is http://www.math.princeton.edu/jfnj/. See http://www.nobel.se/economics/laureates/1994/nash-autobio.html for his biography for his The Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel, 1994, prize.

game theory does, predict that game outcomes among rational players will be Nash equilibria. Again, we should ask ourselves whether rational causes might lead to violation of this *Nash equilibrium principle.* Again, this is a matter for the sequel.

We say that an outcome is *Pareto optimal* if there is no other outcome for which every player does better.[2] If an outcome is not Pareto optimal, we say it is dominated. (Notice the difference between a dominated strategy and a dominated outcome.) In our Interrogation game, Figure 4.2, BE is a dominated outcome; both Spade and Gutman do better with BP. Similarly, KE is dominated, this time by KP. Both KP and BP are Pareto optimal. We say that the set of Pareto optimal outcomes constitutes the *Pareto frontier*. The *Pareto frontier principle* has it that among rational players game outcomes will always be in (or on) the Pareto frontier. Once again, we should ask ourselves whether rational causes might lead to violation of this, the Pareto frontier principle. And what if the principles fail to apply or even even conflict?

Summing up, we can label the outcomes in the game matrix, [N] for Nash equilibrium, [P] for Pareto optimal, [NP] for Nash and Pareto, and unlabeled for none of the above. Here is Interrogation with labeling.

[2]Named in honor of Vilfredo Pareto, 1848-1923, who invented and developed the concept. See http://cepa.newschool.edu/het/profiles/pareto.htm.

	Press (P)	Extreme Press (E)
Blab (B)	4 [P] 3	3 1
Keep Silent (K)	2 [NP] 4	1 2

Figure 4.3: Interrogation with Labeling

A last word on the Interrogation game. Dashiell Hammett, the author of *The Maltese Falcon,* actually worked for a time as a private detective. He knew whereof he wrote. *The Maltese Falcon* was published on February 14, 1930, well before the flowering of game theory.

4.2 Prisoner's Dilemma

The Prisoner's Dilemma is the most famous and the most studied of games in the abstract. Invented about 1950 (and attributed to A.W. Tucker), experimentation began on it immediately [17, 18] and continues to this day. Entire books have been devoted to it, including [1, 59, 61]. Luce and Raiffa give the standard interpretation [49, page 95]. There is no need to revise it:

> Two suspects are taken into custody and separated. The district attorney is certain that they are guilty of a specific crime, but he does not have adequate evidence to convict them at trial. He points out to each prisoner that each has two alternatives: to confess to the crime the police are sure they have done, or not to confess. If they both do not confess, then the district attorney states he will book them on some very minor trumped-up charge such as petty larceny and illegal possession of a weapon, and they will both receive minor punishment; if they both confess they will be prosecuted, but he will recommend less than the most severe sentence; but if one confesses and the other does not, then the confessor will receive lenient treatment for turning state's evidence whereas the latter will get "the book" slapped on him.

The game is standardly taken to hinge on cooperation. The prisoners (Row and Column) may behave cooperatively (C) by *refusing to confess* or they may behave uncooperatively and defect (D) on each other. If both prisoners cooperate (refuse to confess) each receives a reward, R, for cooperation. R=3 in much of the literature and in what I will call the *Default* Prisoner's Dilemma (PD) game. Think of 3 as the number of years the prisoner gets to live out of jail during the next 5 years. More is better. If both prisoners defect (confess), each receives a reward of P, the penalty for mutual defection. P=1 in the Default PD. If one player cooperates (C) and the other player defects (D), the cooperator gets S, the sucker's payoff and the defector gets T, the temptation to defect. In the Default PD, S=0 and T=5. These assumptions are recorded in the left-hand game matrix of Figure 4.4. Typically,

	D	C
D	1 [N] 1	0 [P] 5
C	5 [P] 0	3 [P] 3

	D	C
D	P [N] P	S [P] T
C	T [P] S	R [P] R

Figure 4.4: Default and Canonical Prisoner's Dilemma: $T > R > P > S$ and $2R > (T + S)$

and in what I call the *Canonical* Prisoner's Dilemma (right-hand game matrix in Figure 4.4), the payoffs to each player are symmetric in the sense that T for row chooser equals T for column chooser, and so on. This is not strictly required for PD and wasn't true in the original PD experiments [17, 18].[3] Prisoner's Dilemma strictly requires that $T_i > R_i > P_i > S_i$ for each player i. Further, it is usually, although not always, required that $2R_i > (T_i + S_i)$ for each player i. Both the Default and the Canonical PDs meet both of these conditions.

The PD game is a dilemma because our rationality principles are in conflict. The principle of dominance advises both players to defect, D, since it is a dominant strategy for each of them. The Nash equilibrium principle concurs. DD is the only Nash equilibrium outcome of this game. DD, however, is not Pareto optimal: *both* players would do better if CC were the outcome. The Pareto frontier in this game is: $\{CD, CC, DC\}$. On one side we have the dominance principle and the

[3]The game matrix used was:

(-1, 2)	($\frac{1}{2}$, 1)
(0, $\frac{1}{2}$)	(1, -1)

Nash equilibrium principle and on the other we have the Pareto frontier principle. The two sides are, in this game, directly in conflict.

4.3 Hawk-Dove

Two players confront each other over a resource whose full value is V to either of them. Each player may play one of two strategies: H (Hawk) or D (Dove). Doves signal that they wish to share the resource equally. Hawks signal they are willing to fight to get the resource. When two Doves meet, each gives the characteristic sharing signal and the resource is divided equally, or, perhaps, a fair coin is tossed and the winner gets all. In any case, the expected return to each of the two Doves is $V/2$. When a Hawk meets a Dove, the Hawk (as it always does) signals fight, the Dove (as it always does) signals share, then the Dove retreats and the Hawk takes the entire resource. Finally, when two Hawks meet, each signal fight, neither retreats, both fight at a cost of C. In the end, the resource is shared equally, minus the cost, or, perhaps, half the time one Hawk gets the entire resource and half the time the other Hawk gets it. In any case, the expected return to each of the Hawks is $\frac{1}{2}(V - C)$. We can summarize the Hawk-Dove game in the following strategic form representation. Let us assume, sensibly and without loss of generality, that

	H	D
H	$\frac{1}{2}(V - C)$ $\frac{1}{2}(V - C)$	0 [P] V
D	V [P] 0	$V/2$ [P] $V/2$

Figure 4.5: Hawk-Dove Game: $C > V > 0$

$V > 0$ and $C > 0$. Notice that the outcomes HD, DD, and DH are Pareto optimal: in each case it is impossible to find another outcome that will not make at least one of the players worse off. Note especially what happens when $C > V$ and when $V > C$. DD is not a Nash equilibrium in either case, since Row would prefer HD and Column would prefer DH. If $V > C > 0$, then HD is not Nash because Column would prefer H; similarly for DH, and HH is a Nash equilibrium. Conversely, if $C > H > 0$ fighting is very damaging. Row would refer DH to HH and Column would prefer HD to HH, so HH is not a Nash equilibrium, while HD and DH are.

	H y	D $(1-y)$
H x	$\frac{1}{2}(V-C)$ (Column); $\frac{1}{2}(V-C)$ (Row)	0 (Column); V (Row)
D $(1-x)$	V (Column); 0 (Row)	$V/2$ (Column); $V/2$ (Row)

Figure 4.6: Hawk-Dove Game with Mixed Strategies

What about *mixed equilibria*, in which strategies are played according to a probability distribution? Let Row play H with probably x and D with probability $(1-x)$. Similarly, let Column play H with probability y and D with probability $(1-y)$. See Figure 4.6, in which these probabilities are recorded in the margins. The expected return to Row, or the game value for Row, G_R in this regime is

$$G_R = xy(\frac{1}{2}(V-C)) + x(1-y)V + 0 + (1-x)(1-y)(\frac{V}{2}) \quad (4.1)$$

$$= x[y(\frac{1}{2}(V-C)) + (1-y)V - (1-y)(\frac{V}{2})] + (1-y)(\frac{V}{2}) \quad (4.2)$$

Taking a derivative with respect to x (see the Addendum to this chapter, §4.11.1):

$$\frac{dG_R}{dx} = [y(\frac{1}{2}(V-C)) + (1-y)V - (1-y)(\frac{V}{2})] \quad (4.3)$$

Setting this to 0, solving for y, and simplifying we get:

$$y = \frac{V}{C} \quad (4.4)$$

Making the same calculation for G_C, the value of the game to Column, we also get:

$$x = \frac{V}{C} \quad (4.5)$$

It is an equilibrium (indeed a stable one) for Row to play H with probability $x = \frac{V}{C}$ and for Column to play H with probability $y = \frac{V}{C}$. Note that this assumes $V < C$, which is what we'll assume in our subsequent studies of the Hawk-Dove game. Note further that when $V - C$ is positive, Hawk-Dove is a degenerate Prisoner's Dilemma: $T = V, R = V/2, P = (V - C)/2, S = 0$ violates the $2R > T + S$ condition.

The method we used immediately above to find the mixed equilibria of a game in strategic form is entirely general, and especially tractable in a 2×2 game. We will see it again. There is another, very fruitful perspective we can take on "solving" the game. Suppose now that the regime of play is repeated or iterated. We have an infinite population of players, some of whom play H and some of whom play D. We draw them at random from the population and have them play each other. We record the returns the two players get and we adjust their frequencies accordingly in the next generation. See Figure 4.7 for a representation relevant to our current regime of play, which is called a *replicator dynamic.* Figure 4.7 is a special version of Table 4.6, with $x = y$ and the perspective of the row player.

	H x	D $(1-x)$
H	$\frac{1}{2}(V-C)$ $\frac{1}{2}(V-C)$	0 V
D	V 0	$V/2$ $V/2$

Figure 4.7: Hawk-Dove Game with a Mixed Population

What about equilibrium? Think of it this way. Suppose you could pick which strategy to play, H or D, knowing the current value of x. At what value of x would you be indifferent between playing H and playing D? You would be indifferent when the expected return from playing H, $E(H)$, equaled the expected return from playing D, $E(D)$.

$$E(H) = \frac{1}{2}(V - C)x + V(1 - x) \tag{4.6}$$

$$E(D) = 0x + \frac{V}{2}(1 - x) \tag{4.7}$$

Setting them equal

$$\frac{1}{2}(V - C)x + V(1 - x) = \frac{V}{2}(1 - x) \tag{4.8}$$

and solving for x we again get

$$x = \frac{V}{C} \tag{4.9}$$

Thus, the equilibrium reached by the replicator dynamic is a Nash equilibrium. This *replicator dynamic equilibrium* is also stable.[4] A replicator dynamic equilibrium is a special case of a Nash equilibrium; it is an equilibrium reached by an infinite population of strategies under the regime of the replicator dynamic. Finite populations over finite times, driven by evolution and natural selection, may approximate it. The replicator dynamic equilibrium, like the Nash equilibrium, will be a useful benchmark.

4.4 Stag Hunt or Assurance

Two agents go hunting and take up their places in a blind, which hides them both from each other and from any stags that happen by. Together they can expect to bag a stag, which will feed them each for 3 days. If, however, one of the players reneges and goes hunting for hare, that player can expect to bag two hares, enough to feed him for two days. The other player will receive nothing, neither stag nor hare. If both players renege, each can expect to bag one hare, a day's worth of food. The game, presented in strategic form in Figure 4.8 is so named in honor of a passage in Rousseau's *A Discourse on Inequality*:

> If it was a matter of hunting a deer, everyone well realized that he must remain faithful to his post; but if a hare happened to pass within reach of one of them, we cannot doubt that he would have gone off in pursuit of it without scruple...

The Stag Hunt is also called the Assurance game. What assurance does a player have that the other player won't renege? The game has been used to model arms races. To see why, relabel. For the row player, change Hunt stag to Refrain from deploying missile defense and Hunt hare to Fully deploy missile defense. For the column player change Hunt stag to Refrain from deploying missile defense penetration system and Hunt hare to Fully deploy missile defense penetration system. Hunting stag (or its strategic equivalent) is a cooperative play, as chasing hare is uncooperative. The Stag Hunt game is thus another kind of strategic context in which issues of cooperation arise.

[4] And it qualifies as an ESS (Evolutionary Stable Strategy). See [54] for the seminal introduction of the Hawk-Dove game and the concept of an ESS.

	Hunt stag (S)	Chase hare (H)
Hunt stag (S)	3 [NP] 3	2 0
Chase hare (H)	0 2	1 [N] 1

Figure 4.8: Stag Hunt (aka: Assurance game)

Note that there are two equilibria: SS (both Hunt stag) and HH (both Chase hare), only one of which is Pareto optimal, SS. What about a mixed equilibrium? Symbolizing for the general case, with allusion to Prisoner's Dilemma we have the game matrix of Figure 4.9:

	Hunt stag (S)	Chase hare (H)
Hunt stag (S)	R [NP] R	T S
Chase hare (H)	S T	P [N] P

Figure 4.9: Generic Stag Hunt (aka: Assurance game): $R > T > P > S$

Calculating the replicator dynamic equilibrium we have

$$E(S) = Rx + (1-x)S \tag{4.10}$$
$$E(H) = Tx + (1-x)P \tag{4.11}$$

Equating and solving for x gives us

$$x = \frac{P+S}{R+P-T-S} \tag{4.12}$$

For the specific game in Figure 4.8

$$x = \frac{1+0}{3+1-2-0} = \frac{1}{2} \tag{4.13}$$

The expected return received by a player of the game at this equilibrium is

$$G_{x=0.5} = \frac{1}{4}(3 + 0 + 2 + 1) = 1.5$$

The single Pareto optimal outcome continues to dominate.

See “The Stag Hunt” by Brian Skyrms [75] for a thoughtful discussion of the significance of this game.

4.5 Chicken

Each player drives a car, racing it directly on a collision course with the other player’s car. If both players Drive Straight they will crash into one another with dire consequences for each. If one player swerves, leaving the road clear for the other, that player is “chicken” and the non-swerving player gets accolades for bravery. If both players swerve, both are “chicken” and both dishonored, but less so than being the sole “chicken.” Figure 4.10 abstracts this strategic context as a strategic form game. In our game, the Pareto frontier consists of SS (both swerve),

	Swerve	Drive Straight
Swerve	2 [P] 2	3 [NP] 1
Drive Straight	1 [NP] 3	0 0

Figure 4.10: Chicken

SD (row swerves, column drives straight), and DS. SD and DS are both Nash equilibria. Symbolizing for the general case, with allusion to Prisoner’s Dilemma gives us Figure 4.11. Calculating the replicator dynamic equilibrium we have:

$$E(S) = Rx + S(1-x) \tag{4.14}$$
$$E(D) = Tx + P(1-x) \tag{4.15}$$

Equating and solving for x gives us

$$x = \frac{P - S}{R - S - T + P} \tag{4.16}$$

	Swerve	Drive Straight
Swerve	R [P] R	T [NP] S
Drive Straight	S [NP] T	P P

Figure 4.11: Generic Chicken: $T > R > S > P$

In our specific case

$$x = \frac{0-1}{2-1-3+0} = \frac{1}{2} \tag{4.17}$$

This game apparently was originated in the movie, "Rebel without a Cause," starring James Dean and Natalie Wood. In the movie's "Chickie Run" scene, the two players race stolen cars towards a cliff overlooking the ocean. Whoever bails out first is "chicken." One of the players is unsuccessful in abandoning his car before it lurches over the cliff. James Dean survives and gets the girl.

4.6 Battle of the Sexes

We meet Della, loving wife of loving Jim in O. Henry's short story "The Gift of the Magi"[5] Della has a problem. They are a young couple, times are tough, and Jim's salary has been cut. They are poor.

> Della finished her cry and attended to her cheeks with the powder rag. She stood by the window and looked out dully at a gray cat walking a gray fence in a gray backyard. Tomorrow would be Christmas Day, and she had only $1.87 with which to buy Jim a present. She had been saving every penny she could for months, with this result. Twenty dollars a week doesn't go far. Expenses had been greater than she had calculated. They always are. Only $1.87 to buy a present for Jim. Her Jim. Many a happy hour she had spent planning for something nice

[5] Freely available at http://www.auburn.edu/~vestmon/Gift_of_the_Magi.html thanks to Project Gutenburg.

> for him. Something fine and rare and sterling—something just a little bit near to being worthy of the honor of being owned by Jim.

How is she to get money for his Christmas present?

> Now, there were two possessions of the James Dillingham Youngs in which they both took a mighty pride. One was Jim's gold watch that had been his father's and his grandfather's. The other was Della's hair. Had the queen of Sheba lived in the flat across the airshaft, Della would have let her hair hang out the window some day to dry just to depreciate Her Majesty's jewels and gifts. Had King Solomon been the janitor, with all his treasures piled up in the basement, Jim would have pulled out his watch every time he passed, just to see him pluck at his beard from envy.

So Della sells her hair and buys a gold chain for Jim's watch. Jim, of course, comes home with an expensive set of combs for Della's hair. He has sold his watch to buy them.

Abstracting this story we get the Battle of the Sexes as it is called in the game theory literature. Its story is slightly different. He (say Row) would like to go to the fight (F, boxing match). She (Column) would like to go to the opera (O). They both would prefer to attend the same event and they each have to commit to an event without communicating with the other (cell phones have been lost, time is short, etc.). Figure 4.12 gives a reasonable abstraction. (B=best return; S=second best return; 0=worst.) In pure strategies, there are two Nash equilibria and two

	F	O
F	1 [NP] 3	0 0
O	0 0	3 [NP] 1

	F y	O $\overline{y}$
F x	S [NP] B	0 0
O $\overline{x}$	0 0	B [NP] S

Figure 4.12: Battle of the sexes: Specific and Symbolic ($\overline{x} = (1 - x)$)

Pareto optimal outcomes: FF and OO. There is a mixed Nash equilibrium at

$$x = \frac{B}{B + S} \tag{4.18}$$

$$y = \frac{S}{B+S} \tag{4.19}$$

Because this is not a symmetric game (in a sense to be made clear in the sequel), the replicator dynamic equilibrium is not (yet) well defined.

4.7 Inspector versus Evader

A two-period game is played between the Inspector (Column) and the Evader (Row). Evader might be a drug smuggler, Inspector the Coast Guard, or Evader might be a country bent on developing nuclear weapons, Inspector the United Nations, and so on. Inspector can only inspect during one of the two periods. The evader has two strategies:

E: Evade the rules during the first period; evade during the second period if and only if Inspector inspects during the first period.

$\neg E$: Do not evade the rules during the first period; evade during the second period if and only if Inspector inspects during the first period.

Inspector has two strategies:

I: Inspect during the first period (and not during the second).

$\neg I$: Inspect during the second period (and not during the first).

In terms of outcomes, let us assume Evader's preferences are $E\neg I > \neg EI > \neg E\neg I > EI$ and Inspector's preferences are $\neg E\neg I > \neg EI > EI > E\neg I$. Let $0 < b, c < 1$, $0 < a, d$, and assign the outcome values as in Figure 4.13.

	I y	$\neg I$ $(1-y)$
E x	$-a$, 0	1, $-d$ [P]
$\neg E$ $(1-x)$	c, b	0, 1 [P]

Figure 4.13: Inspector versus evader game

The game has no Nash equilibrium in pure strategies. Using the technique of §4.11.1 we find that there is an equilibrium at

$$x = \frac{1-b}{1-b+d} \tag{4.20}$$

$$y = \frac{1}{1+a+c} \tag{4.21}$$

Checking further shows that this is a stable equilibrium. Notice that the larger a and c are the smaller y is, and the larger d is the smaller y is. Is the rationality of this plain?

See [60, chapter 11] for a thorough discussion of this game. I have honored the specifics of his example.

4.8 A Zero-Sum Game

The game in Figure 4.14 has the special property that for any outcome, Row's gain is Column's loss, and *vice versa.* Such *zero-sum* games are contexts of pure

	C_1 y	C_2 $(1-y)$
R_1 x	2 [P] -2	-4 [P] 4
R_2 $(1-x)$	-2 [P] 2	1 [P] -1

Figure 4.14: A zero-sum game

opposition. Every outcome is on the Pareto frontier simply because if one player is relatively better off, the other player is relatively worse off. It happens in this particular game that there is no Nash equilibrium outcome, i.e., no Nash equilibrium in pure strategies. There is a Nash equilibrium in mixed strategies. Using the general results from the Addendum to this chapter, §4.11.1:

$$x = \frac{(1+2)}{(2+1)-(-4-2)} = \frac{1}{3} \tag{4.22}$$

$$y = \frac{-1-4}{(-2-1)-(4+2)} = \frac{5}{9} \tag{4.23}$$

4.9 PD Property Games: ##12, 47, 48 & 57

Recall the canonical game matrix for the 2×2 game (Figure 4.1, page 78), reprinted below:

	C_1	C_2
R_1	c_1 r_1	c_2 r_2
R_2	c_3 r_3	c_4 r_4

Figure 4.15: Canonical game matrix for the 2×2 game in strategic form

If we accept limitations on the numerical values assigned to the rewards, the r_is and the c_js, then the number of 2×2 games can be restricted to a tractable size and the class studied systematically. If each reward value for an agent is unique and drawn from $\{1, 2, 3, 4\}$, then there are only 576 = $4! \times 4!$ distinct 2×2 games. Many of these are really equivalent, e.g., one can be transformed to another simply by exchanging rows, or one can be transformed to another by switching the rôles of the players: Row becomes Column, Column becomes Row. It turns out that there are exactly 78 unique 2×2 games under these assumptions. Anatol Rapoport and his co-workers have studied them all [63].[6]

Among these 78 unique 2×2 games, it turns out that exactly 4 have the *PD property*—first seen in the Prisoner's Dilemma—of having Pareto frontier outcomes distinct from Nash equilibrium outcomes. Figure 4.16 presents Rapoport's game #12.

[6]See [63, pages 14–7] for details on counting the number of 2×2 games.

	C_1	C_2
R_1	2, [N], 2	1, [P], 4
R_2	4, [P], 1	3, [P], 3

Figure 4.16: Game #12: Prisoner's Dilemma

Game #12 is a Prisoner's Dilemma: $T = 4, R = 3, P = 2, S = 1$. Figures 4.17, 4.18, and 4.19 present games #47, #48, and #57 respectively, with their [N] and [P] outcomes labeled.

	C_1	C_2
R_1	3, [N], 2	1, [P], 4
R_2	2, 1	4, [P], 3

Figure 4.17: Game #47

	C_1	C_2
R_1	2 [N] 2	1 [P] 4
R_2	3 1	4 [P] 3

Figure 4.18: Game #48

	C_1	C_2
R_1	3 [N] 2	2 [P] 4
R_2	1 1	4 [P] 3

Figure 4.19: Game #57

Points arising:

1. Prisoner's Dilemma, as in game #12, is typically specified in a *symmetric* fashion. In the special case in which we have a *game in symmetric form* it happens that $V_R(i,j) = V_C(j,i)$: the value that the Row player receives if Row plays strategy i and Column plays strategy j is equal to the value to the Column player of Row playing j and Column playing i. Here, e.g., $V_R(R_2, C_1) = 1 = V_C(R_1, C_2)$. Symmetric form games are more easily—hence more often—studied. In Part II we begin with symmetric form games, then move on to asymmetric games (aka: bimatrix games).

2. Games ##47, 48, and 57 are asymmetric games. They are further distinguished from game #12 and indeed all Prisoner's Dilemma games by having a Pareto dominated outcome that is *not* a Nash equilibrium. Thus is a nice setup arrived at to test our three principles (so far) of rationality. The principle of (strategy) dominance and the Nash equilibrium principle predict the

outcome R_1C_1 for all three games. The Pareto optimality principle predicts $\{R_1C_2, R_2C_2\}$. No principle predicts R_2C_1. Who is right and under what conditions?

3. None of these four games has a Nash equilibrium in properly mixed strategies.[7]

4. It is possible to have a Pareto outcome in mixed strategies, for example a mixture of the elements in $\{R_1C_2, R_2C_2\}$. Can it actually happen? If so, under what conditions?

4.10 Other Games, Other Forms

Games in the wild are natural phenomena and as such need to be represented or modeled for purposes of investigation. To do so, we abstract the phenomena into simpler, more tractable representations, which afford our inquiries. This is a general point, and it applies straightforwardly in contexts of strategic interaction.

My purpose in this chapter has been to provide an inventory of abstract games, linked insofar as possible to naturally-occurring games. We draw upon this stock in the sequel. The discussion has proceeded and will proceed by example, raising concepts and terminology as they are needed.

Each of the games discussed in this chapter has been a 2×2 game in strategic form. There are other important 2×2 games, there are plenty of important games that are not 2×2 games, and there are other game forms than the strategic. We shall see examples of each of these in the course of our discussion. We begin with the 2×2 game because it is an excellent place to begin our algorithmic, constructivist—"from the ground up"—study of contexts of strategic interaction.

> [I]f there is any hope of eventually constructing scientific theories of human behavior, we must first learn to perform controlled experiments with a view of drawing inferences from them that at least have *apparent* relevance to human motivations, learning, decisions—above all, to interactions. Gaming experiments include all these features, and experiments on 2×2 games are the simplest and most tractable that include the most important of them. The value of such experiments is that they can teach us not necessarily how people behave

[7] **Note JDL: Use Selten** *Models of Strategic Rationality,* **"A Note on ESS in Asymmetric Animal Conflicts". Agent normal form.**

in real life but how we can study certain aspects of characteristically human behavior systematically, from the ground up, as it were.

–*The* 2×2 *Game,* Rapoport & Guyer [63, page 13, underline added]

4.11 Addendum

4.11.1 Solving for Mixed Equilibria in 2×2 Games

Recall Figure 4.1, our canonical game matrix for the 2×2 game in strategic form, which is reprinted below with a bit of relabeling and additional information: Row

	C_1 y	C_2 $(1-y)$
R_1 x	a_r, a_c	b_r, b_c
R_2 $(1-x)$	c_r, c_c	d_r, d_c

Figure 4.20: Canonical game matrix for the 2×2 game in strategic form

plays R_1 with probability x and R_2 with probability $(1-x)$. Similarly, Column plays C_1 with probability y and C_2 with probability $(1-y)$. The expected return for Row, G_R, is

$$G_R = xya_r + x(1-y)b_r + (1-x)yc_r + (1-x)(1-y)d_r \tag{4.24}$$

The equilibrium values of x, y, $0 < x, y < 1$ if they exists, are found by taking the partial derivatives $\partial G_R/\partial x$ and $\partial G_C/\partial y$, setting the results to 0, and solving. We have

$$\frac{\partial G_R}{\partial x} = ya_r + (1-y)b_r - yc_r - d_r + yd_r = 0 \tag{4.25}$$

Solving for y gives us:

$$y = \frac{d_r - b_r}{(a_r + d_r) - (b_r + c_r)} \tag{4.26}$$

The analogous calculation for G_C yields

$$x = \frac{d_c - c_c}{(a_c + d_c) - (b_c + c_c)} \tag{4.27}$$

If (and only if) the resulting values for x and y are legitimate probabilities, we have found a mixed equilibrium for the 2×2 game.

4.11.2 Replicator Dynamic Equilibrium for 2×2 Games

Given our standard 2×2 game matrix: the replicator dynamic equilibrium may be

	S_1 x	S_2 $\overline{x}$
S_1 x	A (row), A (column)	B (row), C (column)
S_2 $\overline{x}$	C (row), B (column)	D (row), D (column)

Figure 4.21: Canonical game matrix for the symmetric 2×2 game in strategic form

found by solving

$$Ax + B(1-x) = Cx + D(1-x) \tag{4.28}$$

which gives

$$x = \frac{(D-B)}{(D-B)+(A-C)} \tag{4.29}$$

4.11.3 Pareto Optimal Mixed Equilibrium

Consider a canonical 2×2 *symmetric* game under the replicator dynamic, with x the proportion of C players and $\overline{x} = (1-x)$ the proportion of D players:
Although the labeling here resembles that often used for the general Prisoner's Dilemma, the case at hand should be taken generally. The expected value of the game, G, for a player is

	S_1 x	S_2 $\overline{x}$
S_1 x	A A	C B
S_2 $\overline{x}$	B C	D D

Figure 4.22: Canonical game matrix for the symmetric 2×2 game in strategic form

$$
\begin{aligned}
G &= Ax^2 + Bx(1-x) + C(1-x)x + D(1-x)(1-x) && (4.30) \\
&= Ax^2 + Bx - Bx^2 + Cx - Cx^2 + D - 2Dx + Dx^2 && (4.31)
\end{aligned}
$$

Differentiating

$$
\begin{aligned}
\frac{dG}{dx} &= 2Ax + B - 2Bx + C - 2Cx - 2D + 2Dx && (4.32) \\
&= 2(A - B - C + D)x + (B + C - 2D) && (4.33)
\end{aligned}
$$

Setting to 0 and rearranging:

$$
x = \frac{(B + C - 2D)}{2((B + C) - (A + D))} \qquad (4.34)
$$

which will be maximal when

$$
\frac{d^2G}{x^2} = (A + D - C - B) < 0 \qquad (4.35)
$$

Part II

Introducing Societies

Chapter 5

Players without Memory

Play Prisoners' Dilemma once, one on one, and the defector will always beat the cooperator. Suppose that a strategy converts to (imitates, is taken over by) its alternative if the alternative scores more points. Then a single play of cooperation against defection will produce two defectors. Does anything change if, instead of one pair of players, we have an entire society? Will the defectors conquer the (mirco)world? What happens in other games?

In this chapter we work with a simple and remarkably powerful style of modeling agents in *social* contexts of strategic interaction, and in doing so we begin to explore answers to these questions. We shall model a society as a collection of individuals arrayed on a two-dimensional ("territorial" [1] or "spatial" [31]) grid that I call the *gridscape.* Think of a checker board, with one agent on each square. Agents interact with—play a game with—each of their neighbors, count of up their points, and convert to (imitate, are conquered by) the strategy of the neighbor with the most points. Agents keep their existing strategies if none of the neighbors do better. (Ties are resolved randomly.)

To illustrate, this is a randomly-populated 12×12 gridscape, which we'll call generation 0:

101001000011
110010000111
100001010001
011011000100
100000110110
011110111110
101011000010

101111011101
101001010010
111010110011
110100100010
011111110010

The two strategies in play—the two types of players—are represented as 1 and 0. Letting the agents play Prisoners' Dilemma with its default reward structure (see Figure 4.4, page 83)—

	Defect (0)	Cooperate (1)
Defect (0)	1, 1	5, 0
Cooperate (1)	0, 5	3, 3

—this is how strategies are distributed in generation 1 after one round of play and updating:

000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000

Cooperation has been extinguished in one generation!

Before continuing, three words on the details of the update process. First, in this example "neighborhood" was defined as the so-called *Moore* neighborhood: the eight immediately adjacent cells to the cell whose points are being tabulated. Below, that is the cell labeled C; the eight neighbors are N, S, E, W, NW, NE, SE, and SW.

NW	N	NE
W	C	E
SW	S	SE

Also popular in this sort of model is the *von Neumann* neighborhood of four cells:

	N	
W	C	E
	S	

We shall mostly, and unless otherwise noted, use the Moore neighborhood of eight. Second of our two words of detail, is to note that we assume (mainly for convenience) that every cell has 4 or 8 neighbors as the case may be. This is achieved by "wrapping" the grid: the column on the far right has as its right-hand neighbors the column on the far left. Similarly, the row on top has the row on the bottom as its northerly neighbor. Think of the gridscape as really having the shape of a doughnut. The third word is to note that each player plays each of its neighbors twice: once as the center cell (C) and once as a neighbor.[1] So, if player i is C and plays j to i's north, then when j has the role of C, i is the player to j's south. In the special case in which we have a *game in symmetric form* it happens that $V_R(i,j) = V_C(j,i)$: the value that the row player receives if row plays strategy i and column plays strategy j is equal to the value to the column player of row playing j and column playing i. Our Prisoners' Dilemma game—above and as played here—is a game in symmetric form. Figure 5.1 shows the general or canonical form for symmetric 2×2 games.

	S_1 x	S_2 $\overline{x}$
S_1 x	A A	C B
S_2 $\overline{x}$	B C	D D

Figure 5.1: Canonical game matrix for the symmetric 2×2 game in strategic form

[1]It is not necessary to play the games twice, simply convenient to do so from the point of view of implementation. Conceptually, each agent/cell plays one game with each of its neighbors in succession.

With these remarks to hand, it should be clear how to calculate the points for a cell. For example,

1	0	1
1	C=1	0
1	0	0

gives a value for C (starting with N and going clockwise) of $(0 + 3 + 0 + 0 + 0 + 3 + 3 + 3) \cdot 2 = 24$. (The presence of symmetry allows use to multiply by 2 and dispense with the second batch of 8 neighborly calculations.) It should be apparent that the point totals for generation 0 are

30.0	64.0	24.0	56.0	56.0	24.0	48.0	32.0	32.0	48.0	30.0	36.0
36.0	24.0	32.0	32.0	12.0	40.0	40.0	24.0	32.0	12.0	30.0	42.0
30.0	56.0	40.0	40.0	48.0	18.0	40.0	0.0	40.0	40.0	56.0	24.0
48.0	18.0	6.0	32.0	12.0	18.0	56.0	40.0	48.0	12.0	48.0	48.0
12.0	56.0	56.0	56.0	48.0	56.0	24.0	24.0	64.0	30.0	24.0	40.0
40.0	24.0	18.0	24.0	18.0	56.0	24.0	24.0	24.0	30.0	24.0	56.0
18.0	64.0	30.0	80.0	36.0	30.0	56.0	56.0	64.0	64.0	24.0	56.0
18.0	64.0	18.0	30.0	30.0	24.0	56.0	12.0	18.0	18.0	48.0	30.0
30.0	72.0	24.0	64.0	56.0	24.0	64.0	24.0	56.0	48.0	24.0	72.0
30.0	36.0	24.0	48.0	12.0	48.0	24.0	18.0	32.0	40.0	18.0	36.0
30.0	36.0	72.0	30.0	56.0	64.0	30.0	56.0	32.0	40.0	18.0	64.0
56.0	30.0	30.0	24.0	24.0	24.0	24.0	12.0	24.0	40.0	18.0	64.0

and that these produce the generation 1 gridscape of all 0s.

5.1 Inevitable Conquest?

After initialization, our gridscapes operate deterministically (except for ties). Is it certain that cooperators will be eliminated in Prisoners' Dilemma? Here are summary statistics from a typical run on a 200×200 gridscape, with 99% 1s at initialization.

Generation	0s	1s	Total Points
0	418	39582	1913228.0
1	3608	36392	1828420.0
2	8983	31017	1667816.0
3	14390	25610	1504112.0
4	20501	19499	1307108.0
5	25794	14206	1133364.0
6	30150	9850	986952.0
7	33556	6444	869640.0
8	35954	4046	785384.0
9	37434	2566	733044.0
10	38409	1591	698488.0
11	39057	943	675044.0
12	39378	622	663176.0
13	39519	481	658040.0
14	39609	391	654840.0
15	39669	331	652624.0
16	39700	300	651436.0
17	39711	289	650980.0
18	39711	289	650980.0

After generation 17 the system is static; no further changes occur. A few cooperators have survived. Why?

If we look at the gridscape for generation 18 we see mostly 0s, of course. The 1s appear in distinct patterns. Here is an example with labeling:

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	1	1	1	1	0	0
3	0	0	1	1	1	1	0	0
4	0	0	1	1	1	1	0	0
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0

A little calculation should convince you that this is a stable pattern: a rectangle of 1s (at least 3 deep) cannot be invaded by 0s. Here is the table segment with the 1s replaced by their point totals.

	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	18	30	30	18	0	0
3	0	0	30	48	48	30	0	0
4	0	0	18	30	30	18	0	0
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0

In this (kind of) configuration, the largest number of 1s that a 0 can have as neighbors is 3, so the highest number of points a 0 can get is $(3 \cdot 5 + 5 \cdot 1) \cdot 2 = 40 = 2 \cdot (3T + 5P)$.[2] Each 1 is adjacent to a 1 surrounded by 1s, yielding $2 \cdot 8 \cdot 3 = 48 = 2 \cdot 8R$. So long as $8R > 3T + 5P$, cooperators can survive (assuming without loss of generality that $S = 0$). Satisfying this condition is not required for a Prisoners' Dilemma. Here are summary data from a typical run with $S = 0, P = 100, R = 101$, and $T = 103$ ($8R = 808 < 809 = 309 + 500$).

Generation	0s	1s	Total Points
0	394	39606	6.4020608E7
1	3430	36570	6.2332296E7
2	8836	31164	6.1245808E7
3	15374	24626	6.0852688E7
4	21922	18078	6.0959208E7
5	27755	12245	6.1415856E7
6	32406	7594	6.2057232E7
7	35670	4330	6.2734128E7
8	37732	2268	6.3273848E7
9	38895	1105	6.3601376E7
10	39522	478	6.3763824E7
11	39874	126	6.3904016E7
12	39981	19	6.398384E7
13	39998	2	6.3997288E7
14	40000	0	6.4E7
15	40000	0	6.4E7

[2] Recall that in Prisoners' Dilemma, P is the penalty for mutual defection, R is the reward for mutual cooperation, T is the temptation to defect, and S is the sucker's payoff.

5.2 Special Forms

Rectangular blocks of 1s, at least 3 wide and deep are indeed special forms. They are, in our present regime, safe from invasion by 0s. We can see this by experimenting with some non-random initializations.

Starting with this nonrandom generation 0 and the default Prisoners' Dilemma game values ($T = 5, R = 3, P = 1, S = 0$)

```
000000000000
000000000000
000000000000
000111100000
000111100000
000111100000
000111100000
000000000000
000000000000
000000000000
000000000000
000000000000
```

the system never changes. If we add a pair of 1s on the right the rectangle is filled out on the next round:

```
000000000000      000000000000
000000000000      000000000000
000000000000      000000000000
000111100000      000111110000
000111110000      000111110000
000111110000      000111110000
000111100000      000111110000
000000000000      000000000000
000000000000      000000000000
000000000000      000000000000
000000000000      000000000000
000000000000      000000000000
```

If we let a single 0 invade, disaster happens.

Generation 0:

```
000000000000
000000000000
000000000000
000111100000
000111110000
000011110000
000111100000
000000000000
000000000000
000000000000
000000000000
000000000000
```

Generation 1:

```
000000000000
000000000000
000000000000
000111110000
000001110000
000001110000
000001110000
000000000000
000000000000
000000000000
000000000000
000000000000
```

Generation 2:

```
000000000000
000000000000
000000000000
000000110000
000000110000
000000110000
000001110000
000000000000
000000000000
000000000000
000000000000
000000000000
```

And Generation 3:

```
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
000000000000
```

(But not all invasions work. Which do not?)

Finally, here's what happens to a 4×4 initial block when $S = 0, P = 100, R = 101$, and $T = 103$.

```
Generation 0:                 Generation 1:

000000000000                  000000000000
000000000000                  000000000000
000000000000                  000000000000
000111100000                  000000000000
000111100000                  000011000000
000111100000                  000011000000
000111100000                  000000000000
000000000000                  000000000000
000000000000                  000000000000
000000000000                  000000000000
000000000000                  000000000000
000000000000                  000000000000
```

And after that it's oblivion for the 1s.

One upshot of these remarks is that numerical values matter on the gridscape. Recalling §4.11.1, there are no mixed equilibria for the Prisoners' Dilemma, given the defining requirement that $T > R > P > S$. We have just seen, however, that whether cooperators survive or not on the gridscape depends on the particular values assigned to T, R, P & S. Gridscape, and more generally, structured interactions depart from equilibrium—notably replicator dynamic—interactions.

When default Prisoners' Dilemma is played with random initialization on a sufficiently large gridspace, indefinite survival of rectangular blocks of cooperators (1s) regularly occurs. In these blocks, each 1 has a 1 as a neighbor who is surrounded by 1s and gets $2 \cdot R \cdot 8$ points. When a line of 0s faces a line of 1s, the 0s (assuming they have 0s behind them) get $2 \cdot (3 \cdot T + 5 \cdot P)$ points each. The 1s will be protected from invasion so long as

$$2 \cdot R \cdot 8 > 2 \cdot (3 \cdot T + 5 \cdot P)$$

Without loss of generality, let $R = T - \epsilon$. Then the 1s are safe from invasion so long as

$$8(T - \epsilon) > 3T + 5P)$$

or

$$5T > 5P + \epsilon$$

At $T = 5, P = 1, \epsilon = 2 = T - R$ safety is assured. We saw, however, what happens when $S = 0, P = 100, R = 101$, and $T = 103$: the cooperators are wiped out.

Can it go the other way? For the 1s to expand at the expense of the 0s, it must happen that a 1 adjacent to a 0 gets itself more points in a round of play than are gotten by the 0 or any of its neighbors. If two homogeneous blocks of 1s and 0s meet on a line, the 1 will get $2(4 \cdot R + 4 \cdot S)$ and the 0 will get $2(4 \cdot T + 4 \cdot P)$. Since Prisoners' Dilemma requires that $T > R$ and $P > S$, the 1s cannot conquer territory in this case. Consider now the case in which a 0 with three 1s as neighbors abuts a 1 with five 1s as neighbors. The 0 gets $2(3T + 5P)$ points and the 1 gets $2(3S + 5R)$ points. Let $S = 0$, $P = 1$, and $T = R + 1$. The 1s can invade if

$$3T + 5 < 5R$$

or

$$R > 4$$

A case in point: $T = 7, R = 6, P = 1$, and $S = 0$. Here are the summary statistics from a typical run.

Generation	0s	1s	Total Points
0	5020	4980	558400.0
1	9535	465	197200.0
2	9487	513	201040.0
3	9108	892	231360.0
4	8563	1437	274960.0
5	7985	2015	321200.0
6	7412	2588	367040.0
7	6751	3249	419920.0
8	6041	3959	476720.0
9	5380	4620	529600.0
⋮	⋮	⋮	⋮

Generation	0s	1s	Total Points
⋮	⋮	⋮	⋮
27	963	9037	882960.0
28	965	9035	882800.0
29	1002	8998	879840.0
30	965	9035	882800.0
31	952	9048	883840.0
32	999	9001	880080.0
33	957	9043	883440.0
34	951	9049	883920.0
35	997	9003	880240.0
36	959	9041	883280.0
37	949	9051	884080.0
38	999	9001	880080.0
39	957	9043	883440.0
40	951	9049	883920.0
41	997	9003	880240.0
42	959	9041	883280.0
⋮	⋮	⋮	⋮

Generation	0s	1s	Total Points
⋮	⋮	⋮	⋮
86	999	9001	880080.0
87	957	9043	883440.0
88	951	9049	883920.0
89	997	9003	880240.0
90	959	9041	883280.0
91	949	9051	884080.0
92	999	9001	880080.0
93	957	9043	883440.0
94	951	9049	883920.0
95	997	9003	880240.0
96	959	9041	883280.0
97	949	9051	884080.0
98	999	9001	880080.0
99	957	9043	883440.0

Here is the gridscape for generation 99:

```
1111111110111111111111111111111111111111111110011111111111111111111111111101111111111111111111111111
1111111110011111111111111111111111111111111110111111111111111111111111111101111111111111111111111111
1111111110011111111111111111111111111111111110111111111111111111111111111101111111111111111111111111
1111111110011111111111111111111111111111111110111111111111111111111111111000001111111111111111111111
1111111110111111111111111111111111111111111110111111111111111111111111110111111000001111111111111111
1111111111110111111111111111111111111111111110111111111111111111111111110111111111110111111001111111110
0111111111110000111111111111111111111111111110111111111111111111111111110111111111110111110000000000
1111111111110110111111111111111111111111111110111111111111111111111111110111111111111101111111111111
1100111111101111000000000000000000000000000001111111111111111111111111110111111111110111111111111111
1110111111101111111111111111111111111111111110111111111111111111111111110111111111110111111111111111
1111111111110111111111111111111111111111111110111111111111111111111111100111111111110111111111111111
1111111111101111111111111111111111111111111110111111111111111111111111101111111111110111111111111111
1111111111101111111111111111111111111111111110111111111111111111111111101111111111110111111111111111
1111111111101111111111111111111111111111111110111111111111111111111111101111111111110111111111111111
1111111110000111111111111111111111111111111110111111111111111111111111101111111111110111111111111111
1111111101111111111111111111111111111111111000011111111111111111111111101111111111110111111111111111
1111111101111001111111111111111111111111110111001111111111111111111111101111110111110111111111111111
1111110111111011111111111111111111111111110111000111111111111111111111100000000011110111111111111111
1111100111111111111111111111111111111111011111101111111111111111111111111111111111110111111111111111
1111111111111111011111111111111111111110111111111111111111111111111101111111111111111100111111111111
1110111111111111001111111111111111111110111111111111111111111111111100111111111111111101111111111111
1100111111111111111111111111111111111011111111111110011111111111111001111111111111111111111111111111
1111111111111111101111111111111111011111111111111110111111111000000001111111111111111111111111111111
0111111111111111110111111111111111101111111111111110111111101111111011111111111111111111101111111111111
0111111111111111111111111111111111110111111111111110000000001111111011111111111111111110011111111110
1111111111111111110111111111110111111111111111111110111111101111111011111111011111111111111101111111111
1111111111111111111111111111111110111111111111111101111111011111111001111111111111111111111111110011
1111111111111111111101111111110111111111111111111101111111011111111110011111111111111011111111111011
1111111111111111111101111111001111111111111111111101111111101111111111111111111111100111111111111011
1111111111111111111101111110111111111111111111111101111111101111111111111111111111111111110000000011
1111111111111111111101111011111111111111111111101111111111101111111111111111111111111111111001110011
1111111111111111111000000011111111111111111111101111111111101111111111111111111100111111111101111110111
1111111111111111111101111101111111111111111111101111111111101111111111111111111111111111111111111111
0111111111111111110011110001111111111111111111101111111111101111111111111111111111100111011111111110
0111111111111111111111111100011111111111111111101111111111101111111111111111111111101110011111111111
1111111111111111111111111110000111111111111111011111111111110111111111111111111111110000001111111111
1100111111111111111111111110001111111111111111011111111111110111111111111111111111011110111111111111
1110111111111111111111111110001111111111111111011111111111110111111111111111111111011110111111111111
1111111111111111111111111110001111111111111111011111111111110111111111111111111111011111101111111111
1111110111111111101111111111100001111111111111110111111111111100111111111111111111111011111111110111111111
1111110011111110111111111111100011111111111111011111111111110001111111111111111111111011111111111011111111
1111111111111011111111111111100011111111111111011111111100000111111111111111111111011111111111101111111
1111111100000111111111111111111100010111100000000000000000001111111111111111111111011111111111110111
1111111110001111111111111111111000000110011111111111111110001111111111111111111111011111111111110001
1111111111011111111111111111111000100011111111111111111111111111111111111111111111011111111111110001
1111111111011111111111111111111111111110111111111111111111111110111111111111111111011111111111110001
1111111111011111111111111111111111111110111111111111111111111110011111111111111111011111111111111100
0111111110011111111111111111111111111110111111111111111111111111101111111111111110000111111111111111100
0111111110111111111111111011111111111110111111111111111111111111101111111111110011101111111111111110
1011111110111111111111110111111111111110111111111111111111111111101111111111110111111011111111111111
1111111110011111111111101111111111111110111111111111111111111111101111111111001111111111100111111111111
1111111110000000000000011111111111111110111111111111111111111111101111111111111111111111111001111111111
1111111111111111111111011111111111111110111111111111111111111111101111111111011111111111111111001111111111
1111111011111111111111011111111111111110111111111111111111111111100000000000011111111111111111100011111111
1111100111111111111111011111111111111100000000001110111111111111101111111111111111111111111111101111111
1110000111111111111111011111111111111111111111111110000001111111110111111111111111111111111111111111110011
0000111111111111111111011111111111111101111111111111110000000001111111111111111111111111111111111111000
1011111100111111111111011111111111111111111111111111110011111111111111111111111111111111111111111111
1011111110111111111111101111111111110011111111111111111101111111111111111111111111111111111111111111
1011111111111111111111110111111111110111111111111111111101111111111111111110111111111111111111111111
1011111111111111011111111011111111111111111111111111111001111111111111111110111111111111111111111111
0001111111111111000011111101111111111111111111111111111011111111111111111110111111111111111111111100
1111111111111111100001111101111111111111111111111111110011111111111111111110111111111111111111111001
1110111111111111111111111101111111111011111111111111110011111111111111111110111111111111111111111111
1111001111111111111110000011111111111000111111111111111011111111111111111111101111111111111111111001111
1111101111111111111111010011111111110011111111111111111111111111111111111110111111111111111111111011111
1111101111111111111111111111111111111011111111111111111111111111111111110111111111111111100011111111111
1111101111111111111111111100111100111111111111111101111111111111111111111101111111111111111000111111111111
1111101111111111111111111110111111111111111111111101111111111111111111111100000000000000000011111111111
1111101111111111111111111111100111111111111111111101111111111111111111111111110111111111111111111111111
1111101111111111111111111111101111111111111111111101111111111111111111110111110111111111111111111111111
1111101111111111111111111111101111111111111111111101111111111111111111100111110111111111111111111111111
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1110001111111111111111111111101111111111111111111101111111111111111110111111111101111111111111111111111
1100111100111111111111111111000000111111111111111110111111111111111100111111111110111111111111111111011
0011111110111111111111111110001111111110111111111111111111111111111111111111111111101111111111111111110000
1111111110000000000000011111111101111111111111111111111011111111111111111111111111111110111111111111111000111
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1111111110011111111111111111110111111111111111011111111111111100111111111111111101111111111111101111
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1110111111111111111111111000111111111111111111111111110000001111111111111111111111000011101111111111
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Here is the crucial generation 1 gridscape, one round removed from a random 50:50 start, and the low point of the 1s:

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1110010000000000000000000000000000000000000010100000000000000000000000000000000000000000011000000001
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Remarkably, the 1s immediately crash to a low point of 465 out of 10,000 cells, then recover to consistently just over 9,000. In fact, at generation 32 the system enters a stable cycle, lasting 6 rounds.

5.3 Quasi-Battle of the Sexes

The Battle of the Sexes game (§4.6 page 90) is not symmetric, but can be transformed into a symmetric version I'll call the Quasi-Battle of the Sexes Game.[3] The version we'll investigate is the following.

	C (0) x	D (1) $\overline{x}$
C (0) x	2 / 2	4 / 3 [NP]
D (1) $\overline{x}$	3 / 4 [NP]	1 / 1

	C (0) x	D (1) $\overline{x}$
C (0) x	R / R	T / S [NP]
D (1) $\overline{x}$	S / T [NP]	P / P

Figure 5.2: Quasi-Battle of the Sexes ($\overline{x} = (1 - x)$)

In addition to the two noted Nash equilibria in pure strategies, there is a third Nash equilibrium (also a replicator dynamic equilibrium) in mixed strategies, with C (coded as 0 in the simulations of this section, since it is the more cooperative strategy) played with probability $x = \frac{3}{4}$. In general this equilibrium will be at

$$x = \frac{T - P}{(T + S) - (R + P)} \tag{5.1}$$

Notice that the two Nash equilbria—and Pareto optimal outcomes—in pure strategies cannot be achieved in the sense that it is impossible for everyone to play them. If everyone plays C (0) then C (00) is achieved and if everyone plays D (1) then DD (11) results. Only a mixture of D and C will produce equilibrium or Pareto outcomes. Further, note that if both players play C at the equilibrium probability of $\frac{3}{4}$ the expected return to each player is 2.5. That is not Pareto optimal, however. If both players played C with probability $\frac{5}{8}$ their expected returns would be 2.5625 each. In general the mixed Pareto optimum is at

$$x = \frac{T + S - 2P}{2(T + S - R - P)} \tag{5.2}$$

(Recall §4.11.3 page 99.) Thus, the replicator dynamic finds an equilibrium which is not on the Pareto frontier. How will the gridscape perform?

[3] Colman [9, page 110] calls this specific game the Battle of the Sexes.

Here are summary data from a typical run:

Generation	0s	1s	Total Points
0	7161	7239	575856.0
1	14227	173	467056.0
2	13706	694	470864.0
3	14063	337	468592.0
4	13636	764	471440.0
5	14072	328	468832.0
6	13604	796	472096.0
7	14002	398	469856.0
8	13586	814	472448.0
9	13988	412	469328.0
⋮	⋮	⋮	⋮
295	13955	445	469472.0
296	13536	864	472032.0
297	13955	445	469472.0
298	13536	864	472032.0
299	13955	445	469472.0

Note that the point total declines from the original 1:1 distribution of 0s and 1s. This run uses as 120×120 grid, so there are 14,400 cells in all. For convenience, each agent/cell plays 16 games (8 as initiator with its 8 neighbors and 8 as responder to each of its 8 neighbors). In generation 0 the grid produces 575856.0 points for an average of $2.499375 = \frac{575856.0}{14400 \times 16}$ points per game. In generation 299 we have entered an oscillating stable condition and in that generation the cells receive an average of $2.037638889 = \frac{469472.0}{14400 \times 16}$ points per game. Further, in generation 299 the percentage of 0s (Cs) is $0.969097222 = 13955/14400$. Under random play, as in the replicator dynamic, if both players played 0 (C) with this probability, their expected return would be 2.088888407 each. The corresponding return for generation 298 is 2.1656, but the players on the gridscape realized on avaerage only 2.04875. Things have gotten worse since initialization and generation 0.

5.4 Leader

Leader is very similar to Quasi-Battle of the Sexes:[4]

	C (0) x	D (1) $\overline{x}$
C (0) x	2 2	4 [NP] 3
D (1) $\overline{x}$	3 [NP] 4	1 1

	C (0) x	D (1) $\overline{x}$
C (0) x	R R	T [NP] S
D (1) $\overline{x}$	S [NP] T	P P

Figure 5.3: Leader ($\overline{x} = (1 - x)$)

This version has a mixed Nash equilibrium playing C (0) with probability = $\frac{1}{2}$. The general formula is

$$x = \frac{S - P}{(T + S) - (R + P)} \tag{5.3}$$

It has a Pareto optimal mixture playing C with probability = $\frac{5}{8}$, the general formula being the same as that for Quasi-Battle of the Sexes, expression (5.2).

[4]The specific game and its name are Colman's in [9, page 109].

The following summary statistics are from a typical run. The results are robust for random initialization with the probability of 0 ranging between 0.25 and 0.75.

Generation	0s	1s	Total Points
0	7226	7174	575616.0
1	3051	11349	365296.0
2	5782	8618	458256.0
3	6352	8048	482352.0
4	5566	8834	470672.0
5	5936	8464	483344.0
6	5356	9044	471296.0
7	5834	8566	478464.0
8	5360	9040	470256.0
9	5894	8506	480256.0
⋮	⋮	⋮	⋮
490	5086	9314	455024.0
491	6972	7428	493232.0
492	5053	9347	455232.0
493	6944	7456	490512.0
494	5149	9251	457728.0
495	6814	7586	489232.0
496	5113	9287	457072.0
497	6881	7519	488560.0
498	5239	9161	458096.0
499	6898	7502	490656.0

Note that in generation 499 the average cell earns a return of $2.129583333 = \frac{490656.0}{14400 \times 16}$. Again, the expected return at equilibrium under the replicator dynamic is 2.5. Note particularly, the sustained bias towards the 1s, defying both the predictions of the replicator dynamic (Nash) equilibrium and the Pareto optimal value.

Interestingly, if we set T to 400, R to 200, S to 300 and leave P at 1, the results we get are not much different. Summary data from a typical run:

Generation	0s	1s	Total Points
0	7245	7155	5.2168624E7
1	7142	7258	3.6738936E7
2	8419	5981	4.5269956E7
3	7590	6810	4.3500676E7
4	7735	6665	4.470886E7
5	7050	7350	4.2635604E7
6	7612	6788	4.4425012E7
7	6839	7561	4.1875956E7
8	7733	6667	4.4443012E7
9	6712	7688	4.163526E7
⋮	⋮	⋮	⋮
490	8176	6224	4.523676E7
491	6306	8094	4.0025192E7
492	8190	6210	4.520948E7
493	6330	8070	4.011558E7
494	8177	6223	4.5229964E7
495	6301	8099	4.0013264E7
496	8187	6213	4.5217892E7
497	6336	8064	4.011672E7
498	8179	6221	4.5228348E7
499	6300	8100	4.0024052E7

Note throughout the rather violent shifts between generations, averaging near the 1:1 ratio predicted by the replicator dynamic equilibrium. When, however, the 0s are in the majority, their number approaches the $\frac{5}{8}$ value of the mixed Pareto optimum. Using generation 498 as an example, $8179/14400 = 0.567986111$ while $5/8 = 0.625$.

5.5 Stag Hunt

Recall the Stag Hunt game (Figure 4.8, page 88), reproduced below:

	Hunt stag (0)	Chase hare (1)
Hunt stag (0)	3 [NP] 3	2 0
Chase hare (1)	0 2	1 [N] 1

Figure 5.4: Stag Hunt (aka: Assurance game)

This game, and Stag Hunt generally, does have a mixed equilibrium, which is also a replicator dynamic equilibrium. For our particular game, at equilibrium Hunt Stag occurs with probability $\frac{1}{2}$ and players get a return of $1\frac{1}{2}$ on average. This is only half of what they would get at the Pareto outcome, further reinforcing Stag Hunt's connection with dilemma. There is, however, no Pareto optimum in mixed strategies. Generically we have for Stag Hunt:

	Hunt stag (0)	Chase hare (1)
Hunt stag (0)	A [NP] A	C B
Chase hare (1)	B C	D [N] D

Figure 5.5: Generic Stag Hunt (aka: Assurance): $A > C > D > B$

If a mixed Pareto optimum exists it is at

$$x = \frac{B + C - 2D}{2((B + C) - (A + D))} \tag{5.4}$$

requiring

$$B + C - 2D \leq 2((B + C) - (A + D)) \tag{5.5}$$

or

$$B + C - 2D \quad \leq \quad 2((B + C) - (A + D)) \tag{5.6}$$

or

$$2A \quad \leq \quad (B + C) \tag{5.7}$$

which is impossible for Stag Hunt, given our assumption that $A > C > D > B$.

If you simulate the replicator dynamics for this game, the results are highly sensitive to the initial distribution of strategies. Starting with a 1:1 ratio of 0s and 1s, the system is driven in about half the cases to all 0s. In the other half of the cases the 1s conquer the population. If the initial ratio differs from 1:1, the favored strategy reliably goes to fixation. The equilibrium is not stable under the replicator dynamic.

Running this particular Stag Hunt on the gridscape, starting with a uniform (1:1) random distribution of 0s and 1s, we get in a typical run:

Generation	0s	1s	Total Points
0	7167	7233	344960.0
1	8715	5685	451224.0
2	12288	2112	599424.0
3	13891	509	668312.0
4	14306	94	686720.0
5	14392	8	690712.0
6	14400	0	691200.0

Conquest by the Stag Hunters! Even when the initial frequency of 0s (Stag Hunters) is very low, conquest will often occur, since small patterns such as

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will expand to conquest.

What is an agonizing dilemma for an individual appears to be a no-brainer for this society. Stag hunting wins out decisively.

5.6 Chicken

Recall Chicken in its canonical form (page 89):

	Swerve (0)	Drive Straight (1)
Swerve (0)	2 [P] 2	3 [NP] 1
Drive Straight (1)	1 [NP] 3	0 0

Figure 5.6: Chicken

This instance of Chicken has a mixed Nash (and replicator dynamic) equilibrium at Swerving with probability $\frac{1}{2}$, giving each player in expectation $1\frac{1}{2}$. By alternating play with 01 and 10, the players could expect a return of 2. When simulated the replicator dynamic equilibrium is reliably approached and proves to be very robust to the initial distribution of strategies.

Here are summary statistics for a run with uniform (1:1) initialization on a 12×12 gridscape. What we see is a steady fluctuation about the 1:1 ratio of 0s and 1s (Swerve and Drive Straight).

Generation	0s	1s	Total Points
0	67	77	3288.0
1	11	133	624.0
2	29	115	1384.0
3	72	72	2968.0
4	91	53	3544.0
5	95	49	3680.0
6	94	50	3720.0
7	92	52	3568.0
8	87	57	3520.0
⋮	⋮	⋮	⋮

Generation	0s	1s	Total Points
⋮	⋮	⋮	⋮
189	80	64	3144.0
190	103	41	4008.0
191	73	71	2904.0
192	101	43	3976.0
193	74	70	2976.0
194	99	45	3840.0
195	79	65	3176.0
196	97	47	3816.0
197	70	74	2888.0
198	94	50	3864.0
199	61	83	2504.0

As we might expect from these numbers, the gridscape undergoes a great deal of change as its history unfolds.

Generation 3:	Generation 193:
111111000000	000000011000
111111000000	000000011100
100111000000	001110000000
110101000000	111110001111
110100000001	101110001111
011100000111	100011111111
011100011111	000011111110
000000011100	111111111111
000001111100	111111011111
000001111111	111100000011
111111111111	000000000000
111111000001	000000000000

Scaling up, we get these summary statistics for a run with uniform initialization on a 100×100 gridscape.

Generation	0s	1s	Total Points
0	4993	5007	239904.0
1	1024	8976	54320.0
2	3175	6825	141360.0
3	4532	5468	190752.0
4	5748	4252	230656.0
5	5690	4310	226128.0
6	6323	3677	247912.0
7	5854	4146	231000.0
8	6298	3702	245880.0
9	6089	3911	237648.0
⋮	⋮	⋮	⋮
990	6611	3389	253200.0
991	6601	3399	252600.0
992	6443	3557	248304.0
993	6385	3615	246792.0
994	6420	3580	247952.0
995	6402	3598	247200.0
996	6447	3553	248448.0
997	6469	3531	249288.0
998	6466	3534	248464.0
999	6458	3542	248112.0

Here is the gridscape for generation 999:

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1111110000001100000001100000001111111001000000111000000111100000100000000100000001111111111111111111
1111000000000111000000011100001101111100100000011110000000001001100000000000000001111111111111111111
0111000111011111000011111000000101111001100000111110000000001000010111110000000111000001000000000000
0111000111110000000011100000000000000000011000011100000000000000001111111100000111000001000000000000
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1111111100000000111101110011110000000001001000000111100001111100000111000000000000011101111111111000000
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0000000111101000000000100100000000000000001011111000111111000011111010000000111100111000000011000000110 10
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1110000000011100011110000111000000100000000000000001110000100000111111111100000000110011111000000000111000
0000000000001100011000011111000000100000000000000001111000010000110111111100000000000100001111000000000111000
1100000000000111001111111111111000000100000000110001110000100001111001110000000001100000111000000000111001
1100000000000111110000011110000000001000000000111001110000100000111111110000000000000000000111000000111111101
1100000000011111110000000100000000000100100000101000000110000000010011111100000000011000000110000001111111 11
1100000000011111000000001000000000000111110000000111000110000010100011111010000000010000011100000000000000111
1000000000000111100000001000000000011100000000011100010000000111000000010111000000100000011100000000100001 11
1000011000000001110000011001111111110000000000011100000000000100000000111010000000010000011100000000100001 11
1000010000000001110000110011111111111010011000000000000000000100001111110011100001100001110000000010000001
1100000000011111100001100111111111110111110000000000000111111100111111001111100100000000110000010000001
1000000001111111100000000000000000001110100110000000000000111111001111110011111001000000001100000000000001
1110000000011111111000000000000000001111110000000110000001100000010111011111010111000000000011111000001 1
0110000001110100111000000000000000001111100000001100000011100000111100011111111111000000000111111000001 1
0000000011111011100000000000000000001110000011101110000011000000111000111111111100100001111111100000000
0000001011101111100000000000001000011100000111110000000011100000100000000001110000011000011100000000000
0000011100000000111000111111110011100000001111100000111110000000000000000001110000011100011100000000000 0
```

```
0111001000000000000000111000001100111000001000000000011111111010000000000001110000011100011100000000000
1111110000000000000001110000000001110000010000011100101111111110000000000111000000111111111111111111
```

Here is generation 999 from a different run of a 100×100 gridscape:

```
0000001110011100000000000000000110000000000000000001111000000000000000000000000000100000110000000000011111
1111111110011100000001000000011100011110111111001100110111100000000000000000000010000001000001100011111
1111111100011100000001000000011100011111011101111000000000100000000000000000000010000011000001100000111
0011111100011100000010000000000000101100000000100000000110000000000000111110110000100000000000001110
0000000111011100111000000111000000000110000000000000000011000000011110011110111000000000110000000001110
0000000111000011111110000111000000000110000000000000000111000000110001100011000000001000011111100001110
0000000111000011111001010111100000000000000110000000000001000000000000000000000000000110001100111000111 0
0000111111000011110000111101100000000000011110000000000001100000000000000000000000000000000100111111110
0000111111000000110000111111000000000000011111000000000011111000000001110000000000000000000101000100 0
0000111111011111100000001111100000000110001111100000000001111100000000111110000110111000000100100001000
0011000000011111110000000000111000000011011111111111000000000111000000011111001111111111000011011000010 00
0001000000011111110000110001110000000011111000011110111000010000000001110000000111111000011011000010 00
0001110000001111111111111100011111001001111100000100011100000000000000111000000000000111000010000000 1000
0001111111111111111111100000000011010000011000000000001000010000000000000000000000001110010110000001000
0001000111111111001100000000001110100000110000100000001000010000000000000000000000000111111011000000 1000
1101000111111111100000000000001111110000000000001000001110011101000000100000000100000011001000000000001001
1111100001110000100000000000001111110001100011111000000100001111100011000000001000000111111000000100011 1
1001100001110000110000000000001110000001110010111000000000001010000011100000100000110010000000100000101
0001100001110000100000000000001110000000110000111111000000000000000000000100000001000011110100000000000 100
0000000000000000111000000111011100000000110000000111000000000000000000011000001000011110000000010000 0100
0000000000000000111000000111011100000000110011100111000000000000000000011011111110011110000000001110000 100
0000011111000000111000000111011111101111111101110011100000000000000011000100000001101110010000001100001100
0001011111000000000110000001101111111111110000111001110000000110000011001000000000011100100000100000 00100
0001000111111110000110000001111111001000000001100111110001111111111110000000000000000011110001100000000 00
1101000011111110000110000001111000000100000000110000010000001000111001100000000000011000000001110000000 001
1111100011111111000000110000111000011000000000000000100010000000000110000000000001100000000111011100000 1
1011100000000111100000001000000010000001110000000110000011001000000001000010000110000001110000011011110 00000
1011100000000111100000001100001110000111000000010000001100000000001000010000110000011000000010001110001 00
1100000000000001100000001000000111000000000000000100000000000000000000000011000000110001111110000000000 01100
0100000000000001110000111000000111011001111100100000000000000010111101110000000011000111111000000000001 1100
0100000000000001110000111000000111111111111111100000000000000111100000001000000000000000111100000000000 11100
0110001111000011000001110000011001111111111100000000000000111100000011110000111000100000000000011100
0110001111000010000000001100001000011100011000000000100000111000000001110000110000111000000000011100
0010001111001100011111111100111000001100001110111001111100001000000000000000000111111111100000000001000
0010000111000000000000001000111000001100001111111111100100001000000000010000011111111111111110000100
1100000000000011100000000000111000000100001110010011100100001000000000010000010000000111111110000011
0111111000000011000000000000000011110010000111000011110010000100000000001000011100000011111111100000
1100000100000011100000000000000011110100001110111010000100001000000000100000110000001000011111111110
1000000100000011100000000000000000100001100000101111111111100001000000000010000111000000000001111101
1000000100000011100000000000000000000011100000001111100110000100000000001100111000000000000000011100
0000000100000000000000000000000000000001000000000011100001111111100000000011111100110000000000000111
1000000100000000000000000000000011111000000000111000011000001000111000000111110011000000000000000111
1000000101010000111111100000000000111110000000000100000000100100011100000001111000010000000000000000111
0000000101110111111111110001111110001110000000001000001101001000111000000001100001000000000000000110
1111111111001111111111111111111110000000000000000001000001101111000100001000011000010000000000000001111
1100000100000011100000000111101000000000000000000001000011111111000000000100000100001000000000100001111
0000000000000011100000000011100000000000000000000000111111111110010000000000100000011111000011111100001110
0000110000000011100000000011100000001100000000000110000111110010000000010000010000111100000000111111100
0000010000000001111110000011100000011100000000111000000110000100011000111110000000000100000001111111000
0000010000011111111111111011100000001000000000011100000000000001001110001111001110011100000001111111000
0000001001111111000001111111000000011111100111110000000001110100000000000000001110011110000001111100000
0000011000111110000100110000000000011111110100000000000001111100000000000000000111001000111100000110000
0000011000000000000001111000000000111000010000000000011001000011111000011000001111000000111000000110000
1100000000000000000111111111001111111000000000000000111111001000100001110000011110000001100000110000
1100000000000000111110100000000111111000000111000000000010001011110001100000000011000000010011110000
0000111000000000011100010000000100111110000111000001000000001011100000001100000000000000000111110000
0001111000000000001110011100000001000011100001110000010000000011100000000011100000000010000000000111000
1001111000000000001110011000000001000011100001000000010000000111000001111111000000000011111111100011111
1001110000000001111100000100000010000000000110000000000001000000000011111110000000001111101100111000 0
1000001100000100110000000000001101110011110001100000100110000001110111100000000000010111000001110000
1000001111111000000000000100000110111001110001110000011111100001111110000000000000001000000000111000 0
0100000110000000000000000010000000111001000000011000000111100001111011000000010000000000000000011100 00
1110000111000010000000000010011000000011111000000000000011100111110000000000001000000000000000011100 00
1110000011000011000000111111110000000111110000000000000011001110000000111100011000000101100000011100 01
1100000001000000010000001111101100000000111010000000001000000110011100000001110000001000000111000000 1111111
1110000001000000010000001111001100000000000001100000000100000101100000000000010000000000011110000000 1111101
1110000110000000100000000001100000000000000100000000100000111000000000000000000000000001101000000001100 01
1110001111000000110000000000110000000000000100000000100000101110000000000100000000000000001000000011111 0
1111100000000001111110000000110000001000000000100000001100001011100000000011000000001111000010000000111100
0111101000011111111110000111000011000000000100000000110000101110000000001101111111010000001000000000000 00
1111111100011111111110000111000000100000000010000001110000001000000000000011000000000000000001000000000000 0
```

0111101100011111100001111111000001000000000100011111000001111110001000111000000110000001100000000000
000000000110000110000011110000000001111110111111111000001111100011001111000001100000011000111111000
0000000001100000110000111000000000111111101111111110000001111111111001000000000000000000000111111000
0000000000011111100000110000000000000011110111001111010000001111111010000000000001101100000000111111000
00000000000000001000000110000001110100000000000101100111010000000111111111100000001111101100000000001110000
0000000001000000100000000000111111100000000000101100000010000000000000000000000000111000011000000001110000
00001110010000000100000000011111111100000000110110000000100000000000000000000000111111000011000000110110000
1111111111100000010000000111111110000000000011000011000100000000000000000000001111111000000100000110110000
111111101110000000000000000100000111000110001101111110000000000000000000000001111011100000001000000111000000
0011100011100000011111111100000111000110001011101000000000000000000000000001110000000000100000011000000
000100000000000000000000000100000111000111101011000000000100001100001100001111011000000010000000000000000
00000000000000111100000000100000100000111111110000000000000000110001101111011111000000001000000000000000
00111000000000001111000000001100111000001111110100000110000000000001110100000000011100001111100000111000
0001000000111111110000001111000000000001111100000000110000000000000111000000000111100001111111110111100
1101000000111100000000111111100000000000010010000000000000000100000011100000000111111001111101110100011
11111000000000011101011100001000000000000100100000000000000010000000100000000000000111000000001110000011
00011001100000111111010000011100000000001101100000000100000100000000010001000000000111000000000000000011
00001011000000111110000000001110001111111110000000001100000111111000111111100000011100000000000000000000
00000000000000111000000000011000011111111000001100010111001111100110000100000001110000000000000000000000
0000000000000001000000000000000000111111010011111000001111001111001100011100000011100000000000000000000
00000000000000110000000111000001111000000000000111000001010001110000000001110011111111100000001111100000
000000100000000110000000111000001111000000000000000000000100000111000000000111000111000111111111111110000
0011101000000000000000001110000000011100000000000000000011000001110000000111000001100000000000011011111000
0011100000000000000000001110000000011100011000000000000011000001110000000110000010000000000000110000100000
111110001001111100011111000000000100011000000000000011100000000000000000000010000000000000100000100000
1100000110001100111111111100000000011110000000000000000111000000000000000111001000000000000000111111100000
000000111000110000011111100000111111100000000000110011111011000000000100001111110000011111100000001110000
0100001110011110000111110000011100000000010000001111110000000000000100000110011000001111111100000111111
00000011100111100000111000000111000000000100000011001000000011111000000000000000110000111111110000011111

And here are the summary statistics from this run:

Generation	0s	1s	Total Points
0	1001	8999	60896.0
1	5228	4772	220096.0
2	4783	5217	202640.0
3	5403	4597	225216.0
4	5273	4727	216456.0
5	5562	4438	225632.0
6	5713	4287	229312.0
7	6014	3986	238584.0
8	5942	4058	234768.0
9	6074	3926	238792.0
⋮	⋮	⋮	⋮
990	6353	3647	243680.0
991	6678	3322	255784.0
992	6385	3615	245800.0
993	6673	3327	255712.0
994	6187	3813	240272.0
995	6480	3520	250464.0
996	6204	3796	240736.0
997	6699	3301	256384.0
998	6298	3702	243312.0
999	6663	3337	255960.0

The runs give indistinguishable results, yet the first began with a 1:1 ratio of 1s and 0s, while the second began with on 10% 0s (see table above, generation 0). Suppose instead we start with 10% 1s and 90% 0s uniformly (randomly) distributed? Here are the summary statistics from a typical run.

Generation	0s	1s	Total Points
0	8995	1005	316768.0
1	3866	6134	165128.0
2	5773	4227	238064.0
3	5119	4881	207528.0
4	6032	3968	239632.0
5	5780	4220	229704.0
6	6243	3757	244272.0
7	6051	3949	236880.0
8	6310	3690	246984.0
9	6069	3931	238232.0
⋮	⋮	⋮	⋮

Generation	0s	1s	Total Points
⋮	⋮	⋮	⋮
990	6456	3544	248864.0
991	6499	3501	249888.0
992	6449	3551	248016.0
993	6501	3499	249456.0
994	6741	3259	257048.0
995	6377	3623	245624.0
996	6724	3276	256344.0
997	6464	3536	248072.0
998	6583	3417	253272.0
999	6422	3578	247664.0

And here is the gridscape for generation 999:

```
0000000000000000000000000000000100000001110000000000000111111110011111100000111000111111001000001111 0
00011000000000000000000000000000000000000000011000111111101001110000000011000010000111000111111 0
0001110000000000000000110011000000000000000000000010111111111001000000000000011000010000111000011 00000
1000111111000000000000011111111111000000000000000000000001000000000100000000000000000000010000111000111 00111
1000111111100000001110000111101100000000000000000000000000000010000000000000100001110001110001110 0111
0000111001111000001110000110000000000000000000000000000000000010000100000001011111110001110001110 0111
0000000011111101111110000110000001101110001111000000000000000000101110000000101111111000000011111 00011
0000000011000000001100000110000000111110000000111001000000000000001000001100001111111100000011000000 111
0000011000000000000110000000101111111100000000000000101110000000000000011111111000000011111111000100000 0010
1111111000000000010110000000111101011000000000000000010000000000000000101111000001101111110000000000 0110111
0000000001110001101100011111111000010000000000000001001100000000001101111100001000000000000000000110000
0000000011111111100001111101100000011011000000000110011100000000001100011100000000000000000000000 100000
0000000111111111100000111111000000111111000000001100001111000000110000111000000000111000000000000000 000
0000000100001110011111111111100101001000000110000000110011110001101100001110000000001110000000000000 0000
0000000100000000011011100001111110011000000010000000111011000000110100000111000000000110000000000000 00000
0000000100110000001100000000011101000111000000000000000111000000001111100000111111001111000000000000 10000
0000000011110000111000000000111100000011100000000111100111000000000111000011100000001110111011111111 11000
0000000000000000111000000000111100000011100000000111110011000000000011101110000000000011111000010000 011000
0110001111111000000000000001111111000000111000000000111111110000000000001000011110000000011111000010000 011000
```

011000111011000000011111110011100000111000000000111111110000000010000001110000000001110000000001111110
001100111001000000000000011000000111111111110011001111001001011110000001110000000001111000100000011110
011110000000100000000000011110000111111111111111000000000111000111000000000100000000011100110000111110
011111111011000000000000011100000111111111110000000000000100001110000000001100000000011111110011111000
001111111011101100000000000000000000001111000000000111000000000011000000000000000000011111110001111000
001111111001111100000000000000001000001111100000000111000000000001000000000000000000011110100111111100
001110000100000000000000000001100000010000000000011100000000110000000000010000000000111000001000011100
001110001100000000000110000001100000110000001100011111000001100110000000010000000001110000000000011100
001110000100000000000110111110000000100000001110001111001100000010000000100000000001000000000000001000
001110000000000000000011111111111111000001111100011110010000000111011111100000000000000000000000001100
111110000000000000011110100000000000000110001111111111100011100000110011111000100000000101110000000001111
111111000000000000111110000000000000000000011001111111100011100000000011000011111000000101110000001111111
111001000000000000001111100000000000000001110100111100000000111000000000110000100100001110111000111111111
000000000111111101001111011000000000001110100011100000000111000010000110000100000000111000100000111000
000000011111111111001110011000000000001110100011100000000011000110000110000100000000111001110000000000
000000011111111111000000110000000000000011000111101000000110000110011100001000000001100011100000000000
000000011100000001100000011000000001111110010001111111111110000010111111100100000011100011100000011100
000000011100000001100000011100000000111111101111111110011110000010010111100100000000000000110000001100
000000011100000001111101001100000000111110000111110000111111111110110011100100110000000110001111011100
011001111111111001011111110000000011111100000011111000011111000000001100000011111100000011000111111110000
111001111111111011100011100000000111110000000000000000011111110000111000000111111111100111000111101111
001111111111111101000011100000001111111010000100000000000111000011100001111111111110111000111000010000
001110000000000100000000000000001111111110100100000000000111000000001111111110000110001100000000001000
001110000000000110000000000000000111111111100100111000000111000000001111111110000000000000000000000001000
001111100000000100000000000000000111000000000000111111000000111000000000111111110000000000000000000000001000
111111100000000100000000000000111110000000001110011111000000000111100000111111000000000000000000000000001111
001111101000000110000000000111111000000000000111000110000111001111110000110000000111000000000000000000000000
001011111111111101010111100000010000000000111000110000111100001110000010000000000100111111111110010000000
001000011111111001111110000000000000000000100000111000000000000111100000100000000000001111111111001000000
001000000111111000110000100000001000000000110000111000000000000000111000001000000000000001111111110001000000
011100000111000000001000010000000100000000111000011100000000000000111000011100000000000000101110000111000000
011100000111000000001111010000000100000000000000011101100000000000111100011100011100000001001000000111000000
011111110110000000001111000000000100000100000000001110010000000000011111111110011110001011111110000000000000
001111110100111000000111001000000100001111110000001110000000000000011111100000111100010111111100000000000000
011111111111111110000000000100000111111101100000000000000001100110000110000000111100000101110000000000000
011100001110111100000000000100000001111111000000001000000000111111000111100000000000000111000000001000000000
011100000000111100000011010110011111111100000011100000000000000100011100100000000000001110000000001000000000
100100000000000011100011000000000000110000000000111000000000000000000111000000111101111111110000000000111011
111110000000000011100011000000000000111000000000111000000000000000110011100000111101111111100000000000000010011
100100001101111111000011100000000011100000000000000000110000000111111000000111101111111000000000001110000011
000000001111111111000011100000000011100010000000111111110000011111100000000011111100001000011001100000000
000000000111110111000011111100000000000011111100000000000000000000111111111111001111100001011100000000000000
000000000100000111000001111100000000000000110000000000000011000000111111111111100111111110100000000000000000
000000000100000111000000001110000000111101100000000000000010000001111011111100100011111100000000000000000000
100000000101111111111000001110000011111011000000000000000001100100000000000001100111110000000000111000000011
100000000111111111111000000111000001111111000000000111000011111000000000000000101111000000000000101110000011
100000001111111110111000000111000001111111111000000111000000100100000000000000100011000000000111011100000011
100000000100011101110011111111110111111111111100000111001000011000000000000001000000000111000000111000000011
000000001111000000111001111111111110011111000000011111111001111011000000100000111110000100111000000111
000000001111000000111001111111111110000110100000000111001001010111111000000100000110000000100000000000011
111000000111100000011100000011100000100001111000000000000000000100100000010000001001011000000011100000000000011
111000000111000000011100000011100000100001101100000000000000001101000000100000000111111000000000100000000001111
011000000000000000011100000011100000000000000100000000000000000100100000001000000010000000111000000100000001110
011000000001000000011000000110000110000000010111000000000000000100000001000001110000011110000001000000001110
011000000001100000001110000011100011110000111110000000001000000000001000001100000001111000000100000001110
011001000000110000000000000000111000011100000011101111100111011000000010000011000000000100000000100000001110
011010000000000000000000000111000011111000000011111110000000010000000100111111000000011000000010000011110
000000000000000000000001000000111000000011000000011111110000000010000000110111111000000000100000000000000011100
000000000000000000000001000000111000000011000000010000001100000001000000000000000100111000000000000000000011100
000110000111100000000001110001110000011000000111000000111000011000000000000000011111111000000000100000011100
100100000111101000000001011000000000000110000001110000000010000110000000010000001111111000000000000000111001
111110001111000111111001111100000000000000000000000000000011011110000000111000000000001110000000010000111111
100110001100000000000000001110000000000000100000000000000000111111111100011100000000000111000000000000001110011
101100000000000000000000001100000000000000111111000000000011111111110001110000000000011111100110001100000000
111100000000000000000000001100000010111111100001100000000011100001110001110000000000011111100100001110000000
001100000100000000000000001100000001111111100001110000000000000000000011111100110000000111110000000011100000000
000000000100000000000000011011000000111111111000011100000011000000000011111111001100011111100011110011011011
000000000101100000000001101111000111001111100011100000010000000000111111111000011111111100001111000000111110
000000000011100000000000000111100111001111100011100000001000110011000000000000001011111110001111000000111100
000110000000011110001000011110001100111111111100000001000111111000000000000000110001110000111000000001000
000111100000000111001110000000000000000001111111110000000111010000100000000000000100001110000111000000001000
000011100000000111001110000001110000000001111111110000000111110000010000000000000100000111011111110000100000
011110110000000111001110000001110000000110000000001111100011100111110001110000100000001110111111111111000
011110110000000111101100000001111000000111000000001111110011100010000000111000010000000111011101000000000
111100000000000111100111111101110000000100000011111110000000001100000000011100000000011100000110000000001
100000000000000000000011111111110001111110000000111000000000000010000000001110101000001110000001100001111
000000000000000000000011111111110001111111100011100000000000001110000000111110011000010000000011000000100

```
1000000000000000010000000111111011111111110001110000000000001110000000111100001000001000001111000000001
100000000000000111110000000000000011111111110111111000000000000011000000111110000100000100000111100000001
0000001101111101111100000000000000111110011111111000000000000001000111111111001100000001000000110000001110
00000011000000111000000000000000001000000111111100000000000000001011111111100001100001111000011000011110
```

Again, the results are indistinguishable from the earlier runs.

Consider now another version of Chicken: As always, there is a Pareto optimal

	x Swerve (0)	$\overline{x}$ Drive Straight (1)
x Swerve (0)	A=2 [P] A=2	C=4 [NP] B=1
$\overline{x}$ Drive Straight (1)	B=1 [NP] C=4	D=0 D=0

Figure 5.7: Chicken Version 2: $C > A > B > D$

mixed strategy if $2A < B + C$. Version 2 of Chicken, above, is like version 1, except that C has been increased from 3 to 4. At these values there is a Pareto mixed optimum at

$$x = \frac{B + C - 2D}{2((B + C) - (A + D))} = \frac{5}{6} \tag{5.8}$$

The replicator dynamic (and mixed Nash) equilibrium is at

$$x = \frac{D - B}{A - B - C + D} = \frac{-1}{2 - 1 - 4} = \frac{1}{3} \tag{5.9}$$

Here are summary statistics from a typical run:

Generation	0s	1s	Total Points
0	7212	7188	403680.0
1	79	14321	5828.0
2	452	13948	24628.0
3	1098	13302	51312.0
4	2008	12392	89144.0
5	2959	11441	126104.0
6	3958	10442	165500.0
7	4800	9600	198528.0
8	5407	8993	224600.0
9	5720	8680	242596.0
⋮	⋮	⋮	⋮
490	3255	11145	192348.0
491	3281	11119	193408.0
492	3271	11129	191252.0
493	3345	11055	194664.0
494	3156	11244	188940.0
495	3322	11078	195704.0
496	3453	10947	202020.0
497	3050	11350	183100.0
498	3438	10962	201504.0
499	3236	11164	192088.0

Note the decline in total points after the initial random start. The system reliably settles down, as we see. The percentage of 0s in generation 499, $3236/14400 = 0.224722222$ is not near Nash and very far from Pareto.

5.7 Hawk-Dove

The generic Hawk-Dove game (Figure 4.5, page 84), is copied below. There is

	H (0)	D (1)
H (0)	$\frac{1}{2}(V-C)$ $\frac{1}{2}(V-C)$	0 [NP] V
D (1)	V [NP] 0	$V/2$ [P] $V/2$

Figure 5.8: Hawk-Dove Game: $C > V > 0$

a replicator dynamic mixed equilibrium at playing H with probability $\frac{V}{C}$. This is not changed if we add a constant, K, to each return.

	H (0)	D (1)
H (0)	$\frac{1}{2}(V-C)+K$ $\frac{1}{2}(V-C)+K$	K [P] $V+K$
D (1)	$V+K$ [P] K	$(V/2)+K$ [P] $(V/2)+K$

Figure 5.9: Hawk-Dove Game with arbitrary constant K: $C > V > 0$

Letting: $V = 10, C = 12, K = 1$ we get the concrete payoff matrix:

	Hawk (0)	Dove (1)
Hawk (0)	0, 0	11, 1
Dove (1)	1, 11	6, 6

At the replicator dynamic equilibrium the Hawk strategy is played with probability $\frac{V}{C} = \frac{5}{6}$. Simulation of the replicator dynamic robustly approximates this probability. Similarly, letting $V = 100, C = 200, K = 50$ reliably and robustly results in Hawk being played with approximately probability of $\frac{V}{C} = \frac{1}{2}$.

Matters are different on the gridscape. At $V = 100, C = 200, K = 50$, here are typical summary data from a run. The results are robust and the initial random proportion of 0s and 1s does not, within a broad range, much influence the outcome.

Generation	0s	1s	Total Points
0	7559	2441	6848400.0
1	5512	4488	1.00748E7
2	6838	3162	7569200.0
3	5451	4549	1.00144E7
4	5599	4401	9590400.0
5	4996	5004	1.0508E7
6	4751	5249	1.08724E7
7	4531	5469	1.10944E7
8	4516	5484	1.10756E7
9	4106	5894	1.1706E7
10	4199	5801	1.15308E7
11	3738	6262	1.22172E7
⋮	⋮	⋮	⋮
990	3689	6311	1.21924E7
991	3403	6597	1.26508E7
992	3611	6389	1.23052E7
993	3601	6399	1.23488E7
994	3416	6584	1.25616E7
995	3501	6499	1.24712E7
996	3518	6482	1.24476E7
997	3371	6629	1.26896E7
998	3579	6421	1.2368E7
999	3456	6544	1.2614E7

At the replicator dynamic equilibrium the probability of playing Hawk is $\frac{1}{2}$; on the gridscape it stabilizes near $\frac{1}{3}$. A player in a replicator dynamic regime can expect a return per play of $\frac{1}{2}(\frac{1}{2}(0) + \frac{1}{2}150) + \frac{1}{2}(\frac{1}{2}50 + \frac{1}{2}100) = \frac{1}{4}(300) = 75$. A player in this gridscape society can expect at stability roughly $\frac{1}{3}(\frac{2}{3}(150)) + \frac{2}{3}(\frac{1}{3}(50) + \frac{2}{3}(100)) = \frac{2}{9}200 + \frac{4}{9}100 = \frac{800}{9} \approx 88.9$. The imposition of the gridscape regime has resulted in higher average returns to the players.

Here is generation 999 from this run. Notice, once again, the emergence of organization highly distinct from a random distribution of the strategies.

111111001111111111111111111000111110000110001110000111110000000000000001011111110011111110111110 11

0000110000111111111111111111110111111000011000111000011011100000100001011100111111001000000000111110 11
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Things go differently with $V = 10, C = 12, K = 1$ (and $\frac{V}{C} = \frac{5}{6}$). Here are the summary data from a typical run.

Generation	0s	1s	Total Points
0	2472	7528	901296.0
1	9236	764	121488.0
2	9575	425	63768.0
3	9754	246	37752.0
4	9840	160	24624.0
5	9861	139	20520.0
6	9858	142	20736.0
7	9877	123	18288.0
8	9860	140	20208.0
9	9890	110	16464.0
10	9894	106	16032.0
11	9906	94	14352.0
12	9917	83	12768.0
13	9920	80	12480.0
14	9938	62	9792.0
15	9946	54	8424.0
16	9946	54	8424.0

The Doves are all but wiped out. Only 54 Doves out of 10,000 players survive indefinitely. A look at generation 15, the first "stable" generation, is revealing. Notice the 9 cases of 6 Doves (1s) in various rotations.

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These *6-pointer* emergent patterns of Doves have their own dynamic, which can be followed over generations.

Generation 0	Generation 1	Generation 2
0000000000	0000000000	0000000000
0000000000	0000000000	0000000000
0000000000	0000000000	0000000000
0000000000	0000000100	0000000000
0001111000	0000000100	0000000000
0000011000	0000001100	0000000000
0000000000	0000001100	0000011000
0000000000	0000000000	0000011110
0000000000	0000000000	0000000000
0000000000	0000000000	0000000000

Generation 3	Generation 4
0000000000	0000000000
0000000000	0000000000
0000000000	0000000000
0000000000	0000000000
0000000000	0001111000
0000110000	0000011000
0000110000	0000000000
0000100000	0000000000
0000100000	0000000000
0000000000	0000000000

The surviving 6-pointer patterns of Doves (1s) rotate clockwise $90°$ each generation. Notice that in generation 2 *all* of the locations occupied by Doves in generation 0 have been taken over by Hawks, albeit temporarily. Once established in a field of Hawks, this emergent pattern of Doves cycles forever.

The interest of these observations is not diminished by adding labels to the Hawk-Dove game in strategic form (C_l is a *label,* not to be confused with the outcome parameter C):

	Hawk (0)	Dove (1)
Hawk (0)	$\frac{1}{2}(V - C) = D$ $\frac{1}{2}(V - C) = D$	$0 = B$ [P] $V = C_l$
Dove (1)	$V = C_l$ [P] $0 = B$	$\frac{V}{2} = A$ [P] $\frac{V}{2} = A$

Figure 5.10: Hawk-Dove Game: $C > V > 0$

and noting that $C_l > A > B > D$. Hawk-Dove is a game of Chicken.

5.8 Observations

These simple models demonstrate and suggest much. It is time now to pause for summary and interpretation.[5]

5.8.1 Brief Points

1. We have examined symmetric 2×2 games in some detail, comparing outcomes of play under the replicator dynamic with outcomes of play on the gridscape.

   We used a very simple regime on the gridscape—we might call it the *basic gridscape regime*. Under this regime every cell on the gridscape plays each of its (4 or, normally, 8) neighbors twice, once as initiator and once as responder, without memory of previous play. Points are totaled up and each cell keeps its strategy if it equals or exceeds all of its neighbors, and failing that converts to the strategy of its most successful neighbor, ties being resolved by chance. This completes one generation of play. Subsequent generations ensue until a stopping condition is met, usually a specified maximum number of generations.

   Under the replicator dynamic regime each strategy is represented as a portion of the population. Strategies are drawn at random according to their proportion in the population during a given generation. Pairs of strategies play against each other and the return each gets is accumulated in association with the strategy producing it. Each generation consists of a number of such plays, 5000 in the studies for this chapter. At the end of a generation, a new population is constituted with the proportions of strategies in accordance with the total returns obtained in the previous generation. The replicator dynamic thus models in a basic way an evolutionary dynamic: strategies obtaining comparatively higher returns from their play increase in frequency.

2. The strategies we investigated are maximally simple. They have no memory of previous play, they are not reactive or conditional in any way, they do

[5] All of the gridscape results reported in this chapter were generated by the program Symmetric2x2M0GridScape.java, v 1.2. Replicator dynamics simulation results were generated by ReplicatorDynamic2x2.java, v 1.2. Both programs available from the author at http://opim-sun.wharton.upenn.edu/~sok/agebookcode/.

not benefit from learning. They adapt by imitation, but do not learn in any genuine sense of the term. Yet complex patterns are produced deterministically from random starts. Will this change, and if so how, as we investigate more realistic models, with more intelligent agents, with post-initialization elements of chance, and so on?

3. The results are often surprising. Even granting—rather a stretch—that careful reflection on these games would yield the results seen here, I think it has to be said that reflection would have produced surprising results. This should–and will—occasion some reflection on the concept and nature of rationality. If Prisoners' Dilemma is to be played once and outside of a society, I suppose we would all choose to play 0, Defect. What if you had to bet on the 1s or the 0s on balance in an evolving society on the gridscape?

4. The gridscape systems often display optimum-seeking behavior. In the Prisoners' Dilemma run described beginning on page 112 at initialization the grid produces 558,400 points for the players. The Cooperators immediately undergo a near-catastrophic population decline. At their low point the gridscape yields only 197,200 points. As the Cooperators recover productivity of the gridscape increases. By generation 99, 90% of the players are Cooperators and the gridscape yields 883,440 points. In Stag Hunt, we saw for the standard game that the gridscape was entirely conquered by those who would Hunt Stag. This extracts the maximum possible value from the system. The gridscape regime did not do especially well in maximizing value in the game of Chicken we investigated. It was able to recover from an initial excess of those who would Drive Straight, but it failed to exploit an initial excess of those who would Swerve. Finally, in one version of Hawk-Dove the Hawks nearly conquer the gridscape, yielding a net value well *below* that obtainable at the replicator dynamics equilibrium (RDE). Yet in another version, the gridscape extracted significantly more "wealth" than is available at the RDE.

5. Reward quantities matter. Two specific games may satisfy the requirements for Prisoners' Dilemma, Hawk-Dove, and so on. Their Nash equilibrium, Pareto, and replicator dynamics properties will be identical or very similar. Yet they may produce very different results on the gridscape.

6. Our method of simulating—of executing play on the gridscape—is useful for making discoveries. For example, it might be difficult to predict that

rectangular blocks of 1s—and only rectangular blocks of 1s—would survive in the canonical Prisoners' Dilemma. Similarly, we found dynamically stable 6-pointers in the Hawk-Dove game. Once we see something empirically, analysis can more easily be undertaken to explain it.

7. Patterns in the gridscape often arise from random starts. We have here a clear and unmysterious case of emergence: Play of the game converts a random, high entropy grid to a highly non-random, low entropy grid. Structure has been created from an underlying process. Further, the process is distributed: there is no central control; what happens happens as a result of local actions taken in parallel.

8. Geometry matters. Suppose we run the Default Prisoners' Dilemma ($T = 5, R = 3, P = 1, S = 0$) on our 4×4 block of 1s, but use the von Neumann neighborhood instead of the Moore neighborhood.

Generation 0:	Generation 1:	Generation 2:
000000000000	000000000000	000000000000
000000000000	000000000000	000000000000
000000000000	000011000000	000010000000
000111100000	000111100000	000011000000
000111100000	001111110000	000111100000
000111100000	001111110000	000111110000
000111100000	000111100000	000011100000
000000000000	000011000000	000001000000
000000000000	000000000000	000000000000
000000000000	000000000000	000000000000
000000000000	000000000000	000000000000
000000000000	000000000000	000000000000

Generation 3:	Generation 4:
000000000000	000000000000
000000000000	000000000000
000000000000	000000000000
000010000000	000000000000
000011100000	000000100000
000111100000	000001000000
000001000000	000001000000
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9. The search spaces presented by the gridscapes are more than astronomical. 10^{80} is an astronomical number; it is the estimated total number of atomic particles in the universe. Our 200×200 gridscape has $2^{200\times 200} = 2^{40000} = 2^{20^{2000}} > 10^{6^{2000}} = 10^{12000}$ possible states.

10. More intelligent play may well yield very different dynamics. What would happen, for example, if agents were introduced that cooperated on the first game played with a neighbor but defected on the second, if the neighbor defected on the first? In the next chapter we take up *reactive strategies.* Agents that meaningfully learn are not far beyond.

In conclusion, a few slightly longer comments.

5.8.2 Computational Explanation

Gridscape models, such as we've seen in this chapter, afford us means to provide *explanations,* indeed computational explanations (see [38]). The essential pattern is:

- Observe phenomena to be explained; then
- Ask how the phenomena could have (or did) come about; then

- Construct a gridscape model whose behavior mimics (perhaps very roughly) the observed phenomena; and
- Appeal to the model in explanation of the phenomena.

There is, of course, a big difference between *an* explanation (which the above pattern offers) and *the* (true, correct, best) explanation and even *a good* explanation. These are interesting issues, and need to be kept in mind, but they need not divert us. Instead, we can judge each case on its merits. Sometimes calls for explanation can be resolved rather straightforwardly with these gridscape (and similar) models. We observe games in the wild and find cooperation occurring. We ask Hobbes's question: Can—and if so, how can—cooperation emerge and be sustained in a society of individuals not controlled by a central authority? The experiments on Prisoners' Dilemma and on Stag Hunt described in this chapter provide existence proofs that there are indeed circumstances in which cooperation not only can be sustained but can triumph. The models have also provided some insight into how this can come about. In virtue of being *computational* models we have full access to their machinations and, in principle at least, to understanding of why they behave as they do. None of this guarantees that the computational mechanisms mirror in any accurate way the ”real world” mechanisms that produced the original phenomena. But it's a start.

5.8.3 Societies

How is it that gridscapes (and related structures, such as networks) can be said to model societies? A *game,* recall, is a context of strategic interaction in which 2 or more players engage in interdependent decision making. The return that a given player receives from a play of the game depends on the reward structure of the game, what strategy the player plays, and what strategies the other players play. A *society,* at least as I shall use the term technically, is also a context of strategic interaction: agents (players) interact with one another and receive returns as a function of their decisions as well as the decisions (the strategies played) by their counter-players. Societies, however, have an additional, special feature: the games played by an agent in a society are affected—via their payoff structures or via the strategies employed by the agent's counter-players—by *games the agent does not play*, i.e., by games played by other agents in the society. Games require at least 2 players, societies at least 3.[6] The pattern of who plays whom—the

[6] I am reminded of a recorded interview I once heard with Bertrand Russell. The interviewer began by discussing Russell's family background (privileged). "It would not be accurate to describe

social network—constitutes (or at least is a major feature of) the *social structure* of a society.

Our gridscape models fit this characterization of society exactly. The rewards received by an agent, as well as the strategies of the agent's counter-players, are dependent in part upon the strategies and rewards received of the agent's counter-players' counter-players. The agent itself never plays against these important counter-players' counter-players. Put differently, if I am a player on the gridscape, it matters much to me what strategy will be played by the guy to the east. And it matters to that guy what the guy to his east plays. Hence, over the generations, it matters (at least potentially) much to me what the entire distribution of strategies on the gridscape is. In time, social effects can reach everywhere.

The *social principle* holds that social structure matters. *Social explanation* appeals to social situations, situations in which what agents see (directly interact with) depends on things (games played by other agents) the agents do not see. The extent to which social explanation matters is a matter of investigation for us.[7]

5.8.4 Negotiation and Contracting

If players/agents are able to negotiate enforceable contracts, many if not all of the games of this chapter would have a very different flavor. The dilemma in Prisoners' Dilemma would disappear if the players could negotiate an iron-clad contract; presumably or at least often both would agree to cooperate. In Stag Hunt, both players would better off if they could bind themselves to Hunt Stag. Players of Chicken could arrange never to collide. And so on. The point is not that negotiation is trivial or without its puzzles, for it is not. Rather, the point is that negotiation brings new factors into play.

Negotiation, at least the non-trivial sort, serves the purpose of restricting the players in their actions. Such restrictions may be in the interest of the players, or not. Similarly, the gridscape is an imposed structure that restricts actions in the society it defines. This happens without courts, lawyers, or even agents with

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the Russells as well-connected. Instead, the Russells were the people to whom the well-connected were connected."

[7]The term *ecology* is perhaps more apt than *society* because the latter suggests a grouping homogenous in some sense, while the former does not. We speak of human society and the ecological balance among humans and the bacteria living within them. Which term we use does not much matter. I have chosen *society* mainly because it is the term used by John Dewey (see *Human Nature and Conduct*) and other pragmatist philosophers when discussing phenomena of the ilk addressed here and because I am broadly sympathetic with those discussions.

intentions or purposes. The gridscape, notably with its capturing of an important aspect of a society, similarly gives us conceptual tools for understanding—for explaining and intervening in—broadly social phenomena in plants, animals, and humans. That, at least, is the promise.

5.8.5 Social Structure

We have explored a two-dimensional gridscape regime for symmetric 2×2 games. Other games will follow. Why just a two-dimensional gridscape? Why indeed.

Consider the Prisoners' Dilemma played on a one-dimensional (1-d) gridscape. As usual, 0s are Defectors and 1s are Cooperators. Suppose a field of 0s abuts a triple of cooperators:

$$\ldots 0000011100000 \ldots$$

This triple is safe from invasion if

$$P + T < S + R \tag{5.10}$$

which is never true for the Prisoners' Dilemma. The triple is also safe if

$$P + T < 2R \tag{5.11}$$

which can happen, e.g., $T = 101, R = 100, P = 1, S = 0$. Notice, however, that two triples of 1s separated by a single 0 and surrounded by 0s are not safe from invasion. What about invasion of the 0s by the 1s? This will happen if

$$R + S > \max\{2P, T + P\} \tag{5.12}$$

In Prisoners' Dilemma, it is certainly possible that $R + S > 2P$, but definition of the game disallows $R + S > T + P$. In 1-d, a clustered triple of 1s may or may not be safe from an invasion by a clustered triple of 0s, and the latter are always safe from invasion by the former.

In general, we are interested in what happens at the interface between 3-cubes of strategies. In one dimension the 3-cubes are three of one strategy in a row. In two dimensions, as we have seen, they are a 3×3 square of one strategy. In three dimensions we have a $3\times3\times3 = 3^3$ cube, and in general we have a 3^d hypercube.

Recall now our canonical symmetric 2×2 game in strategic form:

	S_1 x	S_2 $\overline{x}$
S_1 x	A / A	C / B
S_2 $\overline{x}$	B / C	D / D

Figure 5.11: Canonical game matrix for the symmetric 2×2 game in strategic form

In d dimensions, a hypercube of S_1s can invade an abutting hypercube of S_2s if

$$A[2 \cdot 3^{d-1} - 1] + B3^{d-1} > \max\{D[2 \cdot 3^{d-1} - 1] + C3^{d-1}, D[3^d - 1]\} \quad (5.13)$$

Similarly, a hypercube of S_2s can invade an abutting hypercube of S_1s if

$$D[2 \cdot 3^{d-1} - 1] + C3^{d-1} > \max\{A[2 \cdot 3^{d-1} - 1] + B3^{d-1}, A[3^d - 1]\} \quad (5.14)$$

Mapping these to Prisoners' Dilemma (0s for Defectors, 1s for Cooperators): $A \Rightarrow P, B \Rightarrow T, C \Rightarrow S$, and $D \Rightarrow R$. Then a cube of the Cooperators will invade an abutting cube of the Defectors if

$$R[2 \cdot 3^{d-1} - 1] + S3^{d-1} > \max\{P[2 \cdot 3^{d-1} - 1] + T3^{d-1}, P[3^d - 1]\} \quad (5.15)$$

Setting $S = 0$ and noting that $P[2 \cdot 3^{d-1} - 1] + T3^{d-1} > P[3^d - 1]$, this simplifies to

$$R[2 \cdot 3^{d-1} - 1] > P[2 \cdot 3^{d-1} - 1] + T3^{d-1} \quad (5.16)$$

or

$$R > P + \frac{T3^{d-1}}{[2 \cdot 3^{d-1} - 1]} \quad (5.17)$$

In 2-d:

$$5R > 5P + 3T \quad (5.18)$$

In 3-d:

$$17R > 17P + 9T \quad (5.19)$$

In the limit as $d \to \infty$:

$$2R > 2P + T \tag{5.20}$$

And unneglecting S gives us

$$2R > 2P + T - S \tag{5.21}$$

in the limit. So, the Cooperators *can* expand in 2-d and higher (depending on the actual values of T, R, P and S), and things don't change much after 3-d.

Here are the summary statistics from a run with $T = 150, R = 100, P = 5$ and $S = 0$. We begin with a single 3-cube (3×3 square) of Cooperators (1s) in a 20×20 gridscape of Defectors (0s).

Generation	0s	1s	Total Points
0	391	9	48560.0
1	379	21	68240.0
2	363	37	94000.0
3	339	61	132640.0
4	327	73	151600.0
5	319	81	164480.0
6	283	117	220640.0
7	243	157	282880.0
8	195	205	358000.0
9	159	241	413440.0
10	119	281	476400.0
11	115	285	485360.0
12	123	277	472480.0
13	107	293	495360.0
14	75	325	546160.0
15	95	305	517200.0
16	79	321	540800.0
17	79	321	541520.0
18	95	305	516480.0
19	63	337	565840.0
20	107	293	497520.0
21	43	357	594080.0
22	107	293	497520.0
23	43	357	594080.0
⋮	⋮	⋮	⋮

At generation 20 the system settles into a two-state oscillation, dominated by Cooperators.

Generation 0:	Generation 1:	Generation 2:
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00011100000000000000
00000000000000000000	00011100000000000000	00111110000000000000
00011100000000000000	00111110000000000000	01111111000000000000
00011100000000000000	00111110000000000000	01111111000000000000
00011100000000000000	00111110000000000000	01111111000000000000
00000000000000000000	00011100000000000000	00111110000000000000
00000000000000000000	00000000000000000000	00011100000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000
00000000000000000000	00000000000000000000	00000000000000000000

Generation 19:	Generation 20:	Generation 21:
11111111111111011111	11111111111110001111	11111111111111011111
11111111111110001111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111110001111	11111111111110001111	11111111111111011111
11111111111110001111	11111111111110001111	11111111111111011111
11111111111110001111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111110001111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
10100010111110001111	00000000000001010000	11111111111110001111
00000000000000000000	00000000000000000000	00000000000000000000
10100010111110001111	00000000000001010000	11111111111110001111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111
11111111111111011111	11111111111110001111	11111111111111011111

The behavior is simplified on an infinite gradiscape. Assume:

1. An infinite gridscape in d dimensions.
2. Prisoners' Dilemma is being played.
3. Expansion of a Cooperating 3-cube obtains, as in Expression 5.16, page 151.

4. The gridscape is filled with Defectors, except for a single 3-cube, V, (in d dimensions) composed entirely of Cooperators.

   Remark: Results will hold if the 3-cube is extended to a larger hyper-rectangle (in d dimensions)

Then:

Proposition 1 *The 3-cube, V, cannot be invaded at any of its cells.*

Remark: Expression 5.16 governs. Again:

$$R[2 \cdot 3^{d-1} - 1] > P[2 \cdot 3^{d-1} - 1] + T3^{d-1} \tag{5.22}$$

Invasion occurs on the surface of V. Every surface cell is adjacent to an interior cell. Every interior cell receives a return of $R[3^d - 1]$ each generation and $R[3^d - 1] > R[2 \cdot 3^{d-1} - 1]$. No cell on the surface of V is exposed to more than 3^{d-1} Defectors. Since $T > P$, V cannot be invaded.

Proposition 2 *Every expansion is permanent; conversion of a cell from Defect to Cooperate cannot be reversed.*

Remark: This is not true, as we have seen, in a finite gridscape, since a defector may abut more than one side of V. In support of the proposition, note that every converted cell belongs to some 3-cube in d dimensions and no 3-cube can be invaded.

Proposition 3 *Every cell adjacent to V will be converted to Cooperate within two generations.*

Remark: Corners and edges are even more favorable for expansion than other surface points, since they abut nearly internal cells; they have more than $[2 \cdot 3^{d-1} - 1]$ Cooperating neighbors.

Proposition 4 *V—a cluster of Cooperators playing Prisoners' Dilemma in a field of Defectors—will expand forever on the gridscape; every Defecting cell within a finite distance from V will be converted to Cooperate in a finite number of generations.*

There are other ways to generalize our gridscape models. So far, we have focused on the Moore neighborhood of depth 1. In two dimensions, this is the 8 immediately adjacent cells to a given cell. The depth 2 Moore neighborhood (in two dimensions) adds to this the 16 immediately surrounding cells. Whereas in the depth 1 Moore neighborhood we focused on 3-cubes in d dimensions, we attend to 5-cubes for the depth 2 Moore neighborhood. For the sake of simplicity (and without substantive consequence), let us assume that in each generation each cell plays itself. Thus, for the depth 1 neighborhood in d dimensions, each cell has 3^d neighbors (including itself). The revision of Expression 5.13 (for depth 1 Moore models) is:

In d dimensions, a 3-hypercube of S_1s can invade an abutting 3-hypercube of S_2s if

$$A[2 \cdot 3^{d-1}] + B3^{d-1} > \max\{D[2 \cdot 3^{d-1}] + C3^{d-1}, D[3^d]\} \tag{5.23}$$

Setting $A \Rightarrow R, B \Rightarrow S, C \Rightarrow T$, and $D \Rightarrow P$ for Prisoners' Dilemma and rearranging gives us what was earlier the limiting formula:

$$2R + S > T + 2P \tag{5.24}$$

Under this regime even 3-cubes in 1 dimension may expand in Prisoners' Dilemma, depending on actual values of the rewards.

Generalizing to a wider Moore neighborhood, a 5-hypercube of S_1s can invade an abutting 5-hypercube of S_2s if

$$A[3 \cdot 5^{d-1}] + B[2 \cdot 5^{d-1}] > \max\{D[3 \cdot 5^{d-1}] + C[2 \cdot 5^{d-1}], D[5^d]\} \tag{5.25}$$

With the usual translation to Prisoners' Dilemma this simplifies to

$$3R + 2S > 2T + 3P \tag{5.26}$$

which is *less* favorable to expansion by the Cooperators. Generalizing further, let δ be the depth of the Moore neighborhood in use, then Expression 5.25 goes to

$$A[(\delta+1)\cdot(2\delta+1)^{d-1}]+B[\delta\cdot(2\delta+1)^{d-1}] > \max \begin{cases} D[(\delta+1)\cdot(2\delta+1)^{d-1}]+ \\ \quad C[\delta\cdot(2\delta+1)^{d-1}] \\ D[(2\delta+1)^d] \end{cases} \tag{5.27}$$

With the usual translation to Prisoners' Dilemma this simplifies to

$$(\delta+1)R + \delta S > \delta T + (\delta+1)P \tag{5.28}$$

and in the limit as $\delta \rightarrow \infty$

$$R + S > T + P \tag{5.29}$$

As neighborhoods expand Cooperators have an increasingly difficult time expanding at the expense of Defectors.

There is more to say about the mathematics of play on the gridscape and we shall return to this topic in due course. Larger issues lurk, however. Although the two-dimensional gridscape is a good place to start, we should think more generally, in terms of social networks. The gridscape is such a network, with certain properties, such as regularity (e.g., everyone has the same number of neighbors). But societies of agents will often be able to find, design, or impose social networks with other forms, and these forms will matter. The creation, exercising, and destruction of social networks pervades social systems. It can be seen as among the primary drivers of human history. (The delightful and provocative essay by J.R. and William H. McNeill, *The Human Web: A Bird's-Eye View of World History,* argues just this [57].) It has also become part of common folklore, via the notion of "six degrees of separation" between typical individuals in our society. (Note that on the gridscape there are very many pairs of cells more than 6 cells apart.)

The gridscape is among the simplest of social networks. One should expect *greater* social effects to attend other less simple social network structures. That we have found as much of this as we have strikes me as remarkable.

5.8.6 The Shadow of Society

When agents play games repeatedly it is well known that the "shadow of the future" (discount rate) may greatly affect play. Similarly, the gridscape society has greatly affected the play we have examined in this chapter. It is important to see that these are two distinct factors. The agents in this chapter have no memory and no capacity to respond to experience. They do or die and that's all. The fact that neighboring agents play each other twice each generation is merely a computational convenience; the agents are unable to exploit any information so gained.

The system behavior we have seen is due to the structure of the games played, the strategies played by the agents, *and* the imposed gridscape structure with its attendant rules. The latter, as I have argued, brings to the table inherently social factors. What we might call the *shadow of society* affects outcomes just as does the shadow of the future. But the two are distinct factors. We turn next to agents with the barest modicum of intelligence: they can respond to experience and alter their behavior. The shadow of the future may interact with the shadow of society.

Chapter 6

Memory in 2×2 Games

The behavior of agents without memory measures the effects of social structure. What will happen when agents become more intelligent, are given a modicum of rationality? Will their society be more efficient in the sense of extracting more wealth? More or less heterogeneous? More or less stable? How will the preponderance of strategies played change? These are some of the questions we investigate in this chapter.

6.1 Prisoners' Dilemma with Memory

A strategy is said to have a *memory-n* if it conditions its play on what happened in the previous n rounds of play. Allowing that the strategy needs to specify what to do in the first round, there are eight memory-1 strategies for the repeated 2×2 game, taking into account only the counter-player's play (and not jointly the play of the two players). With the Prisoners' Dilemma in mind, coding C, or cooperate, as a 1 and D, or defect, as a 0 we have:

0. 000. (ALLDEFECT) Defect on the first round; continue to defect no matter what.
1. 001. Defect on the first round; defect if the counter-player cooperated (played 1) on the previous round; cooperate if the counter-player defected (played 0) on the previous round.
2. 010. (SUSPICIOUS TIT FOR TAT)
3. 011.
4. 100.
5. 101.

6. 110. (TIT FOR TAT)

7. 111. (DOORMAT)

We'll begin as we did in the previous chapter, with the canonical Prisoners' Dilemma on a 12×12 gridscape. Every encounter results in 100 plays. We use the von Neumann neighborhood. Here are the summary statistics for a typical run:

Generation	0s	1s	2s	3s	4s	5s	6s	7s	Total Points
0	18	22	20	19	16	11	23	15	518912.0
1	69	33	0	0	33	0	9	0	341372.0
2	96	3	0	0	35	0	10	0	259882.0
3	96	0	0	0	20	0	28	0	291706.0
4	77	0	0	0	8	0	59	0	382568.0
5	40	0	0	0	7	0	97	0	494666.0
6	6	0	0	0	0	0	138	0	661756.0
7	0	0	0	0	0	0	144	0	691200.0

Initialization is random with probability spread equally across each of the 8 strategies. By generation 7, strategy 6, TIT FOR TAT has conquered the population. It is instructive to watch how play unfolds.

Generation 0:	Generation 1:	Generation 2:
333255223016	600011140006	660000000066
776331243062	000611140044	660011444066
702366732234	000611144044	006444444000
635733266101	444441161444	006444000000
144471743342	444441100444	444444000044
717071600101	444441100004	000444000040
564011241016	010011100000	000444000000
116422602150	011000040000	000000000000
176410457673	000000011100	000000000000
670575351626	100000011161	000000001000
414306040324	400000011164	440000000044
620266212655	600000000006	440000000044

Generation 3:	Generation 4:	Generation 5:
666000000666	666600006666	666660066666
666000000666	666600006666	666664066666
666444440666	666644006666	666666066666
064444440000	666660006666	666666066666
066640000000	666660000000	666666066666
000044000000	666660000000	666666000006
000044000000	000004000000	666666000006
000000000000	000000000000	000000000000
000000000000	000000000000	444440000004
000000000000	444400000004	666660066666
444000000004	666600006666	666660066666
666000000666	666600006666	666660066666

Now, unlike the the 1s in the memory-0 case (last chapter) with the canonical numbers, a line of backed up 6s confronting a line of backed up 0s will invade. But not every run ends with takeover by the 6s. The following run is perhaps more usual:

Generation	0s	1s	2s	3s	4s	5s	6s	7s	Total Points
0	14	22	23	20	10	21	16	18	543546.0
1	70	32	0	0	32	9	1	0	347686.0
2	103	8	0	0	33	0	0	0	258564.0
3	118	0	0	0	26	0	0	0	231408.0
4	133	0	0	0	11	0	0	0	230864.0
5	144	0	0	0	0	0	0	0	230400.0

Let's watch it unfold.

Generation 0:	Generation 1:	Generation 2:
327163113775	051100444000	011100004000
532130344020	044400444000	004000044000
704161117257	044400444400	044440044000
271135564116	044400044411	000000444440
500640775527	000400044411	000000444440
665375727122	000400041115	000000004440
525630441632	060000001110	000000000000
155051062260	040000011100	004000000000
244627121461	040001111100	004440000000
612322335755	555411000141	014440000000
353731203652	551111000000	014440000000
336127507203	551111000000	011100000000

Generation 3:	Generation 4:	Generation 5:
004400000000	000400000000	000000000000
044400000000	000400000000	000000000000
044400000000	044400000000	000000000000
000000000000	000000000000	000000000000
000000000440	000000000000	000000000000
000000000440	000000000000	000000000000
000000000000	000000000000	000000000000
000400000000	000000000000	000000000000
004400000000	044000000000	000000000000
044444000000	044000000000	000000000000
044400000000	044000000000	000000000000
044400000000	000000000000	000000000000

Unless a block of 6s can form, occasioned by the vagaries of initialization, it appears that the 0s will take over. If so, then larger gridscapes, all things being equal, should go to fixation with 6s more often, since they stand a better chance of creating a block of 6s. Indeed, in 10 runs of a 100×100 gridscape, with the agents playing 12 iterations, conquest was total by the 6s except in one case in which a few 7s remained. A similar experiment on a 12×12 grid resulted in 6 conquests by the 0s and 4 by the 6s.

Does it matter how may iterations are played? It might because some of the encounters evidence cycles in their returns. For example, if 1 plays 2 (001 vs. 010) n times, then the return to 1 is $(T+R+P+S)n$ if n is evenly divisible by 4. If, instead, n has a remainder of 1 when divided by 4, the the return is $(T + R + P + S)(n - 1) + P$. So it can matter, but in large gridspaces the effects are not seen. (At least I haven't seen them.) It would be possible to decide this analytically, but unjustifiably tiresome.

Comparison with our memory-0 results is instructive. Starting with a block of 6s

```
000000000000
000000000000
000000000000
000666600000
000666600000
000666600000
000666600000
000000000000
000000000000
000000000000
000000000000
000000000000
```

the 6s expand to conquest under the canonical Prisoners' Dilemma rewards (12 iterations of play). Recall the reward structure that led to conquest by 0s in the last chapter: $T = 103, R = 101, P = 100, S = 0$. Using those rewards here, there is an immediate standoff between the 0s and the 6s. A bit of intelligence staves off elimination.

6.2 Stag Hunt

Recall the Stag Hunt game (page 88):

	Hunt stag (0)	Chase hare (1)
Hunt stag (0)	3 3	2 0
Chase hare (1)	0 2	1 1

Here is a typical run with 100 game iterations and $V_R(0,0) = 3.0, V_R(0,1) = 0, V_R(1,0) = 2.0, V_R(1,1) = 1.0$.

Generation 0:	Generation 1:	Generation 2:
05534012446145671626	66222024444600000222	22222004444000000022
32020514042603602623	22222464444600000422	42222204444000000022
33744651756666004230	27222460444000000424	02222244444000000020
70700357132677406342	07444467775560000420	02222442220000020000
10407571362541131560	67000072225522222666	04444422222222222000
03351352255152422756	63377722223322222266	66222222222222222226
73550567415356301110	63322222223322222366	22222222222222222223
71352072721660545345	72222222253336663337	22222222222220002233
57223502676650161317	62222222222222055336	22222222202222222236
17320451766720150406	66222335000222222077	44222222002222222204
06114355601324020157	44422232444222222077	44222224002222222204
54472133247414320411	44422232422222222077	44222222222020222224
42247051002407170376	44422222222202222222	22222222222202222222
33712272321074212322	22222222222222222222	22222222222222222222
42433130270521022236	22222222222222222222	22222222222222222222
50416476072527635511	22277762225222222225	22222222222222222222
17473750771723673655	22200666605222376555	22220002222222222222
52500166173563357143	22666076666677776555	22220000000222235552
74260700776117156435	22666006666633333333	22260004444463333332
56630057145003035756	62666024444633333333	22220004444400002222

Generation 3:	Generation 4:
22222000440000000022	22222000000000000022
02222000440000000022	02222000000000000022
02222220200000000020	02222220222200000000
02222222222220000000	02222222222220000000
00222222222222220000	00222222222222220000
00222222222222222200	00222222222222222200
22222222222222222222	22222222222222222222
22222222222220002222	22222222222220002222
22222222202222222224	22222222202222222222
22222222000222222220	22222222000222222222
42222220002222222222	22222220002222222222
22222222220022222222	22222222220022222222
22222222222202222222	22222222222202222222
22222222222222222222	22222222222222222222
22222222222222222222	22222222222222222222
22222222222222222222	22222222222222222222
22220002222222222222	22220002222222222222
22220000222222222222	22220020222222222222
22200000000040022222	22200002222200022222
22222000440000000222	22222000000000000222

Stasis is reached in generation 4, summarizing:

Gen.	0s	1s	2s	3s	4s	5s	6s	7s	Total points
0	56	52	51	49	40	54	46	52	460870.0
1	43	0	184	38	38	17	53	27	703404.0
2	73	0	266	11	41	3	6	0	916800.0
3	95	0	296	0	9	0	0	0	959680.0
4	93	0	307	0	0	0	0	0	960000.0
5	93	0	307	0	0	0	0	0	960000.0
6	93	0	307	0	0	0	0	0	960000.0

Change the game a little, so that $V_R(1,1) = 2.0$ (formerly it was 1.0) and set the number of iterations to 12. Here is a typical result.

Gen.	0s	1s	2s	3s	4s	5s	6s	7s	Total points
0	42	38	55	50	44	59	51	61	132468.0
1	6	0	194	12	0	7	119	62	168496.0
2	26	0	314	4	0	6	30	20	207556.0
3	33	0	364	0	0	0	2	1	228792.0
4	33	0	367	0	0	0	0	0	230400.0
5	33	0	367	0	0	0	0	0	230400.0

Now boost $V_R(1,1)$ to 2.9. Here are results from a typical run:

Gen.	0s	1s	2s	3s	4s	5s	6s	7s	Total points
0	53	48	44	50	50	54	52	49	148808.39999999994
1	0	0	82	64	4	2	127	121	209299.5999999999
2	0	0	68	23	0	0	145	164	217378.79999999952
3	0	0	77	7	0	0	154	162	216576.79999999976
4	0	0	81	2	0	0	160	157	216251.9999999998
5	0	0	84	0	0	0	159	157	215810.39999999976
6	0	0	88	0	0	0	154	158	216870.39999999982
7	0	0	92	0	0	0	157	151	216258.39999999985
8	0	0	94	0	0	0	157	149	215209.9999999998
9	0	0	96	0	0	0	162	142	215813.5999999999
10	0	0	98	0	0	0	165	137	215350.39999999988
11	0	0	98	0	0	0	170	132	214514.3999999999

Here is generation 299 from another typical run with the same parameter settings:

```
77776666777777667677
77766676777777767777
77776666677777777767
77777666677777767767
77777767777777767667
77777777777767767666
77776777777766776666
77666777777766676667
77666777667776677667
22266777666767622222
22276666766672222222
22277666676662222222
22266666666662222222
22266666666662222222
22266766666672222222
```

66666667666777766666
66666677766677776666
66666676766677776666
67666667677777776666
77766667777776677766

Scaling up to a 100×100 gradscape, here are some summary statistics from a run:

Gen.	0s	1s	2s	3s	4s	5s	6s	7s	Total points
0	1247	1248	1242	1261	1210	1217	1265	1310	3780676.800000019
1	132	18	2325	1414	93	99	2992	2927	5208942.799999901
2	146	0	2335	698	62	11	2964	3784	5368719.199999687
3	164	0	2644	454	64	9	3056	3609	5380001.999999627
4	189	0	2891	380	69	9	3085	3377	5378320.799999609
5	223	0	3097	324	77	9	3102	3168	5386079.199999636
6	264	0	3277	301	87	9	3129	2933	5385326.799999635
7	309	0	3431	287	93	9	3090	2781	5384908.799999655
8	350	0	3546	270	96	9	3046	2683	5387367.599999659
9	412	0	3631	260	96	9	3018	2574	5384621.999999667
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
990	3572	0	6255	0	164	9	0	0	5754160.000000003
991	3576	0	6251	0	164	9	0	0	5753926.000000003
992	3580	0	6247	0	164	9	0	0	5754043.600000002
993	3587	0	6240	0	164	9	0	0	5754170.400000003
994	3587	0	6240	0	164	9	0	0	5753982.000000003
995	3579	0	6248	0	164	9	0	0	5754150.800000003
996	3583	0	6244	0	164	9	0	0	5754072.000000003
997	3580	0	6247	0	164	9	0	0	5754458.000000004
998	3576	0	6251	0	164	9	0	0	5754246.800000003
999	3568	0	6259	0	164	9	0	0	5754181.600000002

Here is generation 999:

4444444402002222222222220220222222222222222222222222222222222222222222222222222222222000200200000024
4444444422200222222222022222222222222222022202222222222222222222222222222202222222222002020022220244
4444444422220000222222222222222222222220002222222222222222222222222222222200222222222022222002022244
4444444400000000022222222222222222200220002220222222222222022222222222222222222222222222222222022244
4444444400000022022222222222220222202220002220222222202222220222222222222222222222222222222220022244
4444444400000000002022222222222220220200002222222222202022222222222222222222222222222222222222000044
4444444400000000200202222222222200220000000222222222220222222222222222222222222222222222222200000004
0000000000000000002002222222222222220222222222222202222222222222222222222222222222222222222220000000
0000000000000000020000222222222222222222222222222000222222222222220222222222222222222222220022000000

```
000000000000000000000002222222222222222222222222222222222222222222222222022222002002222202222222202022200000
0000000000000000000020022222222222222222222202022222222222222222220020000220022222020222222020020000
0000000000000000002022222222222222222222222220222222222222222220222000000200000002000222022220000000
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I repeated this experiment five times and got very similar results on every run.

Points arising:

1. As in the Prisoners' Dilemma case, the agents collectively are extracting a great deal of value from the environment. Stag Hunt as a one-shot game has exactly one outcome on the Pareto frontier: (0, 0) and our agents very often find it, even in the extreme case of $V_R(1,1) = 2.9$.

2. In this game, 0s (000) and 2s (010) are indistinguishable *in the presence of only 0s and 2s.* On another run, e.g.,

Gen.	0s	1s	2s	3s	4s	5s	6s	7s	Total points
0	52	44	48	51	55	42	53	55	478563.0
1	59	2	134	28	51	11	78	37	669884.0
2	124	0	176	3	79	6	9	3	914214.0
3	180	0	190	0	30	0	0	0	958912.0
4	204	0	196	0	0	0	0	0	960000.0
5	204	0	196	0	0	0	0	0	960000.0

   the proportion of 0s and 2s could be very different. In 10 runs of the 20×20 gridspace for the standard reward configuration, however, all resulted in a mixture of 0s and 2s by the fourth or fifth generation.

3. In the one-shot Stag Hunt game, playing 1 is said to be the *risk dominant strategy* because the player is guaranteed 2.0 (in the standard reward configuration), while playing 0 risks receiving 0.0. The effect, the riskiness, is *increased* as $V_R(1,1)$ increases. At $V_R(1,1) = 2.9$, for example, a player risks 2.9-0.0>2.0-0.0 by playing 0. Thus, it is perhaps not surprising to find that these societies will often retain 1-playing strategies as the relative risk of playing 0 increases. It, again perhaps, remarkable that this happens in a system undergoing deterministic updating. Setting $V_R(1,0) = 2.9$ (formerly 2.0) and $V_R(1,1) = 1.0$ (as usual), this is what typically happens:

Gen.	0s	1s	2s	3s	4s	5s	6s	7s	Total points
0	1250	1250	1285	1294	1230	1209	1182	1300	3303998.8000000045
1	496	181	1911	2177	435	1417	381	3002	3032253.399999962
2	682	47	4164	1495	573	753	147	2139	3871702.5999999526
3	952	11	6371	879	531	306	56	894	4810292.999999974
4	1121	0	7794	373	294	118	22	278	5389870.59999999
5	1198	0	8469	128	104	44	5	52	5648753.799999985
6	1239	0	8700	34	12	14	0	1	5735450.199999997
7	1252	0	8746	0	2	0	0	0	5759818.399999997
8	1254	0	8746	0	0	0	0	0	5760000.0
9	1254	0	8746	0	0	0	0	0	5760000.0

A some point considerations of risk take over. Here are summary data from a typical run with $V_R(1,0) = 2.9 = V_R(1,1)$.

0	1212	1224	1285	1192	1297	1257	1286	1247	4239927.999999721
1	67	44	223	1881	61	393	416	6915	5361849.19999936
2	36	0	30	585	39	134	180	8996	5514045.399999085
3	36	0	27	460	36	136	178	9127	5518250.999999059
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
995	36	0	27	460	27	142	375	8933	5524561.599999073
996	36	0	27	460	27	142	381	8927	5524544.199999073
997	36	0	27	460	27	142	377	8931	5524544.199999073
998	36	0	27	460	27	142	376	8932	5524549.999999073
999	36	0	27	460	27	142	375	8933	5524561.599999073

6.3 Chicken

Recall Chicken in its canonical form (page 89):

	Swerve (0)	Drive straight (1)
Swerve (0)	2 2	3 1
Drive straight (1)	1 3	0 0

Here are summary statistics from a typical run with a 100×100 gridscape, play iterated 12 times, and the standard (above) reward schedule.

0	1231	1242	1235	1264	1251	1216	1265	1296	2871232.0
1	428	1142	413	1605	359	4051	54	1948	2315122.0
2	124	272	547	2481	103	3100	20	3353	1635508.0
3	119	148	1129	2803	74	1969	17	3741	1503084.0
4	135	91	1997	2602	76	1769	10	3320	1731990.0
5	167	91	2979	2297	78	1466	4	2918	1975464.0
6	199	66	4026	1849	81	1372	2	2405	2282214.0
7	198	61	5044	1589	89	1017	0	2002	2529784.0
8	225	22	6077	1141	83	1007	0	1445	2854512.0
9	259	7	7055	891	55	591	0	1142	3067234.0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
28	333	0	9654	2	2	0	0	9	3837136.0
29	326	0	9658	9	2	0	0	5	3836720.0
30	319	0	9669	7	2	0	0	3	3837200.0
31	322	0	9672	4	2	0	0	0	3839072.0
32	324	0	9672	2	2	0	0	0	3839752.0
33	316	0	9673	9	2	0	0	0	3837200.0
34	320	0	9674	4	2	0	0	0	3839232.0
35	319	0	9676	3	2	0	0	0	3839016.0
36	323	0	9675	0	2	0	0	0	3840000.0
37	323	0	9675	0	2	0	0	0	3840000.0

A typical result when $V_R(1,0) = 5.0$:

0	1231	1242	1235	1264	1251	1216	1265	1296	2871232.0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
38	0	1	9834	29	0	16	83	37	3806316.0
39	0	9	9931	20	0	7	29	4	3831728.0
40	0	1	9879	2	0	3	83	32	3819552.0
41	0	9	9955	0	0	3	29	4	3840816.0
42	0	1	9881	0	0	3	83	32	3820704.0
43	0	9	9955	0	0	3	29	4	3840816.0

The agents would be better off taking turns swerving. Why don't they learn to do this? Even with $V_R(1,0) = 500$ and $V_R(0,0) = 0.5$, this is what we get:

0	1231	1242	1235	1264	1251	1216	1265	1296	2871232.0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
90	0	0	7018	0	2760	0	222	0	3836052.0
91	0	0	7806	0	2019	0	175	0	3856440.0
92	0	0	6935	0	2843	0	222	0	3836780.0
93	0	0	7670	0	2155	0	175	0	3857368.0
94	0	0	6786	0	2992	0	222	0	3837580.0
95	0	0	7548	0	2277	0	175	0	3857592.0
96	0	0	6645	0	3133	0	222	0	3837436.0
97	0	0	7453	0	2372	0	175	0	3858176.0
98	0	0	6535	0	3243	0	222	0	3837156.0
99	0	0	7384	0	2441	0	175	0	3858848.0

6.4 Battle of the Sexes

Recall the canonical version, page 91:

	F (0)	O (1)
O (0)	0 0	3 1
F (1)	1 3	0 0

The summary table from a typical run:

Gen.	0s	1s	2s	3s	4s	5s	6s	7s	Total points
0	1250	1226	1250	1283	1271	1256	1257	1207	1927478.0
1	0	929	50	2516	3	3119	59	3324	1304852.0
2	0	118	17	3356	7	1171	5	5326	658488.0
3	0	175	6	2297	0	1935	7	5580	831088.0
4	0	134	8	2247	0	1277	10	6324	678992.0
5	0	162	21	1616	0	2249	5	5947	881456.0
6	0	84	8	1636	0	1278	17	6977	636048.0
7	0	154	12	1161	0	2464	12	6197	891952.0
8	0	65	11	1303	0	1341	15	7265	616376.0
9	0	113	20	837	0	2627	12	6391	882016.0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
990	0	1	1663	0	0	0	8336	0	1081728.0
991	0	3	8313	0	0	0	1684	0	1089408.0
992	0	1	1663	0	0	0	8336	0	1081728.0
993	0	3	8313	0	0	0	1684	0	1089408.0
994	0	1	1663	0	0	0	8336	0	1081728.0
995	0	3	8313	0	0	0	1684	0	1089408.0
996	0	1	1663	0	0	0	8336	0	1081728.0
997	0	3	8313	0	0	0	1684	0	1089408.0
998	0	1	1663	0	0	0	8336	0	1081728.0
999	0	3	8313	0	0	0	1684	0	1089408.0

Here is the generation 999 gridscape:

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26222622262222222222222222662222222222666222622262222222222622262226222622222262226222622222222
22222622222222262222262226222662226222622262222262226222622222622262222222222622262226222622262222
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2226222622266222222222222222622262222222222222222222222222222226222622262222222222222222222222222622262222226
222222222222222262222262226222222222222622262226222662226222222262226222622262226222226222222222222222
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22622222222222622266222666222622222622222622222222222226222222222622262226222622262222222222222222222
22662222262222622222222222622262222222222226222622262226666222622266222622262226222622222666222666622
22222222222222226622262226222622262226222622262226222622222222222222222222222262226222666222222222222222
22226666222662226622222226222622222222622222226222622262222622266222622262222222222222222262226222226
22222622222222222262226662226662226222622222222222222222222222222222222222222222262226222222222222222222
26222622262222262226222222222222222222262226222622262222622262226222622262222222222222222662226222226622
22222222222222222222222226222222622222262226222222222222222222222222226222622266222622222222222222222
22262226222226222622262222222222222222666222662222622226622262226662226222622222222262222622266622226
22222222222622222222222222226222666222622222222222222222222222222226222622222222222666222622222222222222222
66222622222222262226222222222622222222226622226662226222222622262226222622262222226222622226222226222
22222222222222222222222622266622262222222222222222222222222262222222222222226222666222622222222222222
22262226662226222622262222222222222222262222266222226222666666222226222622222222222222266662226666622262
22222222262222222222226222622222622262222222222222222226222622222222222222226222622262222222222222222622222
26222622266622262222222222222222222222262222266222622262222266622262222222222222222262222262222662222
22222222222622262226222622266622262222222222222222226222222222222222222222226222622262222222222222222222
22262226222622222222222222222262226222662226662222666222666622262222222222222222226222662222662222266
22262226222666222622262226222622262222222222226222222222222222222622262222622262226222222222222222222266
22222222222222222222222222622222222222622222226222226622222622222222222222222222222222222666222262262
22222622262222622266222622262226222222222222222222222222222222262226622262226222622262226222222222222
22222222262222222222222222226222622266662226222622222622266222222222222222222222622222222222262222622222
66222222262226222222226222622222222226666222222262222222222262222662226622262226222622262222222222222222
22262222222222222222222222226222222222266222226222622262222222222222222222222226222222262226222262222
22262226622262226666222622222222262222222222222266222222222222266222622222222222666222222222222222222222226
2222222222222222222222222222622262222222222226222662226222622222222222222262222222222222622262226222222622
22222662226222622222262222222222262226662222222222222222222262222222622262226622222622222222222222222222622
26222222222222222222222262226222222222226222622262226662222222222226222222222222266222662226222622262222622
22222222622262222662226222222222622222222222222222222222666662222222222262226222222222222622222262222
62226222622222222222222222226222222222222262226222622222222222222222262226222622262226222622222222262222
22222222622222266222262222222222262222666222222222222222222222222266622262226222222222222622262222222222
22622266622222222222222262226222262222262226222666222262222222222622262222222222622266662222622662
22222262226622222262222622222222266622222222222222226222222222266222622222222222622262226222622222222222
62226662226222222222222262226222262222222226622262226666222222222222262222262222266662222222622222662222
62226222226222666222662222222222226222262222222222222222622262222222222222226662222222222222222222222222
62226222622222622222222222622262222222222222226622262222222222222222622262226222222222226222662226662222266
22226222622222622262222222222222222266222622222222222222222262226662222222222262226222222222222222222222222
22622222622262222222222622226222622222222622222622266222222222622262222222222222222622222622262222266222
22622222222226222622222222222222222662226662222222222222226222622262226222622262222222222222222222222222222
22222222222662222222622262226222622222222222222226222262226222222222222222222222222222222222266222622262222262
622222622222222222226222222222262222226222666222622222262226222622262226222622262222222222222222262222266
22222222262226622222222622222266222222222222222662226662222222222222222222222222222222622266622226222226622
22666622262226622222222222222226666222622222622222222222222262226222622262226222222222222222222222222222
2222222222222226222662226666222222622262222222222222622222622222222222222262222222222222262226622266622262222
66222266222622262226222222226222222622262226222222222222222262226222226622262226222222222262226222222222266
22222222222222222222622262226662222222222222222262226662226222622222222222222262226222622262226622226222
22666622262226222222222222222222666222622262226222222222222262226222662226222622222222222262222222222222226222
22262222222222622262226222662222222222222222222222226222222622262226222622262226222622222222622262226666222
22262226222666222622222222222222226222622262222226222222222222662222222262226222222222622222222222222222222666
22222222222226222666222622266222222222622222222222222622262226222222222222222222222222222622222222222622222222
```

What is the maximum number of points attainable on this gridscape? It will occur when, say, half the cells play 0 (or 2) , half play 7 (or 6), and all are arranged checker board style, alternating. Clearly, the gridscape for generation 999 fails to achieve anything like this. With 12 iterations, the total number of points available on the gridscape is $(3 \cdot 4 + 1 \cdot 4) \cdot 12 \cdot 2 \cdot 100 \cdot 100 = 3,840,000$. The summary table reports a maximum of $1,089,408$, confirming what we can see visually on the gridscape. The agents do better using the Moore neighborhood. Here is generation 999 from a typical run:

```
2626262226262622262226222626222622222222222262222262626262626262226222222222222226222222222222622226
6222222262222222622262622262226222626262626222626222222262222222626222626262626222626262626262222222
2226262626222626262622262222622222222222222222222626262226222622222222222222262626222262226262622
6262222222626262226222222262622262622262626262626262222262226262626262626262222222226222622222222
2226262226222222262226222622222622222222222222222222226222222222262622222222262626222622262626262626
2262226262226262622262626222626222226262626262626262622262626262622262626222222262626262626222626222
2222222222262622222622222622222626262222222222222262226222622222262622262226262626222222222222262222
6262626262622262626222626262222222626262626262626222626262226222226222226262622262626262626222222262
2626262222262622222222622222626222626222222222222226262622262226262226262666266622262226222222262626
6262622262262222262626222626222226222226262626262622222262626222226222226622666266626666622262626262
2626666226262626222222262222262626262222222222222226262626222626222626262622262666662666662622222666
6666626662222622222626222226222222222626262226262622262226262226222222222626266626266666266666226266
2626666662626222226262226262222262626262222262222222222262222222622262622262626662626666626666626662626
6266666222222262622222622222626222626262626262222262226262622262626222626662666662626666626666626666
2666262666262626222626222626222262222222222262622262222626262226262222666262666662626666626266626
6262626662622222222222226222226262226222226222622262226222222662226262666662666262666662626662666262
6626666662626262626262226262226262222626262626222622262222626262626662626666666666626266666262666666
6666626266626222222262622262222626222622222622262222666226266626266666262666266626666626666626262
2626666666626662666222222626222626222226222622262226266626266626666626662666266626662626666262666666666
6262666262666266626262222222262222262622222622262226262626266666262626262626662626662626262626666266
2666262626662666262222222262622262622222626262222626266666262626266626666666626662666666666662626626
6666666662626666262226262226262622222626222226222666666666666666226262626262626662626262626266626666
2626262666662626262626266222222626262226262226266226662262666666266266666666666666266666626266626666
6262666262626266666262666262222222222222626222622666662266266666626262622266262626262626266626666666626266
6666266666666662626666662666222262626222222222266266626662626266626262666266666666666666666262626666626
6266626662626262626262666666222222262626262666662626662666666626266666666262626262626662626262626662666
2626666666666666626662626662622226222222222626266666262626262666626266626666662666662626266666666626626
6662626266666266666662666666626262622262226266666626262666666626666626666666666626262666662626266666
2666666626262666266666662626262226266226666662666266626262626266626262666626266666666626662626262626
6262626262666666626266666666666666266666226266626662666666666666626666666666626666262626666666666666666
2666662666266626266626262626262626266626222626666662666262626262666662626266626266262666262626262666262626
6262626266666266666266626666666666666666662626666666666666666262666666626666666666626666626666662626666
2666266626666666626266666262626262626262662666662626262666266666262626666666266262626666666626266626
6262626666626662666662666662626666666666626262626666626666666626666666262626266666266626262666666666262
6666662626662626662666262666666626666266666266626266626266666262626666666666666666626666626262626666
6262666662626666626266666662666666262666266626266626666666266666666666662626222626666626662626662626
2666262666262666666626262666262626626666262666266626626262626626666262622666262626666666626266626266666
6666626662666662626266666266666626626266666666626266666266666666266666666626266626662626666626262666666
66262666666626666666626666662666666666662626266666666626662626266666662626662666666666662626666662626
6262666266626626626262626626626262666666626666662626266626266666626266626666666266626666666662626666666626666
26666666666266666666666666666666666666666262626262626626666666666666666626262662226666666626666266666626
66626262626662626666626662626626262666666662666666666662626262626662666262662666262626262666266626262
6626666666662666262666666626626666626262626262626262666266662626662626662666666666626666662626666
6266626262626666666666626266666626266666666666666266666666262626266666626662626262666262666262626262
626266666666262626262626666262626626662626626666262626266626266666666662626626666666666666666662666666666662
666626262666666666666666666626666666626666262666666666666626666262666266626262626666666662626666
626666626262626626666626262626262626266666626666662626666626626666666626262666666666666626626266666662
2666266666662666626262626666666666626262626626262626662626266666666262626662666666626266666666666666666
6262626262666666666666666662622266666666662666666666662626662626666662626666662626266666626666662626262
6666666626262626262626266666666626666662626662666262666626266626262666262626262666626662626666626662666
6262626266626666666666666662226666626266662626266666266666626662666666666626262666666266626626666
26666666666666626626262626666666662666266626266662666262666666626662666262626666666666662626666666666626
626262626262626666666666266266666666626666662626662666262666666262626666666662626666666662666262626666
2666666626662666662666666666266626266666666262626666666666262626266626666266626662626666662626
6662626666662666262626222626266626262666262666266666666666266666262626666626626662666666662666662626666666
26666662626662666666666666626626666662626662666266626262626662626626262626666666666626626266666666666
626262666662626262626626626666666662666666666262666662666666262626626662666666666666666662626266
6666662626662666662666662666662626266626266626666626262626626666666666666666666262626626626262666666626
6662666266666666266666626666666666666626662666266626262626626262626666666666666626266666666
26662666666262626626626626266662626666626666666666662666666666666626666666666662666262626666666262626
66626262666666666262666666626666662626666626666662626662626266666626262626266666666626666666666626626
66666662626266666626666666662626266662626266666626666666666666662666626666666666666666666266666666626
62626266666262666262626626666626666666666666666262626666666662626262666262626666666666262666262
662666666666666266666266666626666662626666626626666266662662626266666666666662666666662666
6662626262626266666666626262626666266666666662626662626266666662626662666662626262666666662666666262
26266666666266626262626666662666266626626666626666262626262626266666666262666666666266626626626
6662626662666666626666666666626666262626666666626262666626666266666626262666626262666666666262666626266
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2626662626662666266666262626662666266666666626662666662666666666622666266666266666666662666262666626
6666626266626666666626266666266626666626266626262626266626662626266666666666626666626266626666626262
6666266666266626662666662666266666262626262666666666626266666626262226262626262626666662666666666666
6262626262626662626262666666626666666666662666262666266666626266666262626666666666626662666262666626262
2626662626666666266666262626662626666626662666266666266262666662666666666626262626262666662626666666
6666666666626262626666666666662626662626666626662626666666262626266626662626666666666626262666662666262
2626262626266666662626262666266626662626266626666666266266666266666666266666626262626666626266626262626
6266666666666626266666666662626666626266666262666262666626662626662666266626266666666662666662666666222
2626266626662666662666262666662666666626666626666626666262666262666666666266666626262626666666626262626
6266626666666266666262666262626662626262626666626666266626266666662626262666262626666666662626666662222
2226262626666662626262626666666666666666666262626266662666662626666666666666266666666262626262626262626
2222226666662222666262226262626262626262666666666626666266626662626662666262626662626266666666666222222
2622262666262222622262262622666666666666662626262626626262666666666626662626666626266666666626266666626226
2222262662262666262222266266626262626262626266666666666666666666262626262626666666666262666262626262626262622
2262226262222262262626622222666662666262666262626626266666626666666666262626266626666666666666662626262
2626222222262226262622222262262626662622666262666666266666266626262666266666666266626262626262626666262626
2222622262626262626622226222222222626262262666262666662626262666266666662626266626666666666666262222222
2626262226222622262626222626222626266662666666662666666666626262626266626662666662626262626262626262626
2222222222222622222622262222222222662666622666662626266666266626666662666662626666666666662626262622222
2626262626262226262626262226262626262622266626266666662626266626262666262666662626262626262622222262626
2262226222226222222222222222222222222222262626266626266666262666266666666262666266666666262622222626262
26222222262626262626262626262622262266222666662626666626666666666662626262626262626262626662222262622222226
2222622222622222622262226262622222622666222226666666666626262626666666666662666662666222622222226222
2626222622262222262622262222222626222666222626262626262666662666262626262626262626222622262626262626
2222226262622262626222622262622222226262226222626666666666266626666666666662666662222262622262222222222
2626222222222222222622262622262626262222262226262626262626666626266626262626262622222226262222222622
6222226262626262626262226262626222222262622262226262626266626262626666666226262626222626262222226262626262
2622262622262226262226262226222226262622222622262226262626666662222262222262222222622222622262622222222
2262622262622222226262222262626262222222626222622262626262622222626262622222626222626262626222226262
2622222622222626262222226262222262222226262222222222622262626262626262622262222222222222226222222226222226
2262626222626262222262622222622262622222226262226222626222222222222262226262226262622262626262626262
2622222626222222222626222626262222222222222622222622262222626262626262622222222222222622222222222222226
2262622222222222626262222262222262626262626226262222222222222222222222222262626262626222626262626262626222

Now the maximum possible return for the society is $(3 \cdot 2 + 1 \cdot 2) \cdot 12 \cdot 2 \cdot 100 \cdot 100 = 1,920,000$, i.e., half of the von Neumann case. The summary table confirms that percentage-wise, the Moore agents do better. This is, however, a very difficult situation for these agents.

Gen.	0s	1s	2s	3s	4s	5s	6s	7s	Total points
990	0	0	5937	0	0	0	4063	0	952320.0
991	0	0	4059	0	0	0	5941	0	984384.0
992	0	0	5950	0	0	0	4050	0	942144.0
993	0	0	4032	0	0	0	5968	0	968256.0
994	0	0	5949	0	0	0	4051	0	951360.0
995	0	0	4024	0	0	0	5976	0	977280.0
996	0	0	5968	0	0	0	4032	0	947712.0
997	0	0	4022	0	0	0	5978	0	972480.0
998	0	0	5936	0	0	0	4064	0	948288.0
999	0	0	4035	0	0	0	5965	0	979584.0

6.5 Observations

The results of this chapter buttress the remarks at the end of the previous chapter. Hence the concluding remarks here can truly be brief.

Worth noting is that cooperation has robustly emerged in gridscape societies for two games: Prisoners' Dilemma and Stag Hunt. Both games capture an intuitive element of trust or cooperation, yet from a game-theoretic view they are quite different. In Stag Hunt, there are two Nash equilibria—both hunt stag and both hunt hare—(when the game is played once), one of which (hunting stag) is the sole Pareto optimal outcome. In Prisoners' Dilemma (played once), there is a single Nash equilibrium (both defect). It is not, but the other three outcomes are, Pareto optimal. In both games, the socially most attractive outcome (the one maximizing total points on the gridscape; no comment on whether this is genuinely good or not) is riskier than its alternative. Yet, typically and robustly (but not under all conditions), the gridscapes lead to mutual, maximal cooperation.

This has not be demonstrated for Chicken or for Battle of the Sexes. In Chicken we saw a strong tendency to evolve towards mutual swerving: both agents pick their least risky (pure) strategy. In Battle of the Sexes, each pure strategy (or particular play) is equally risky; players can end up with 0 either by playing a 1 or by playing a 0. A measure of how poorly these agents play is the fact that they extract less than one-third of the total possible value from the (von Neumann) gridscape. Similarly, in Chicken, the agents failed to exploit increases in the value of $V_R(1,0)$.

This raises questions we shall return to: What do the agents need to do to do better in games such as Chicken and Battle of the Sexes? How will this affect play in games such as Prisoners' Dilemma and Stag Hunt? Are there general mechanisms that broadly do well in attaining successful play? Contexts of strategic interaction are, in part, about power. Those favored by the structure of the situation will tend to do better than those not so favored. How can we understand—asses and use in prediction—the underlying power configurations on the gridscape? There are other questions, too. These will do for now.[1]

`$Id: symmetric2x2.tex,v 2.7 2006/01/03 01:19:05 sok Exp $`

[1]The gridscapes of this chapter were all generated using Symmetric2x2M1GridScape.java, v 1.7, which is available from the author at http://opim-sun.wharton.upenn.edu/˜sok/agebookcode/.

Chapter 7

Experiments with Simple Societies

Having introduced the gridscape as a simple model for social interaction (that is, of strategic decision making in a social context) and having discussed the basics of this model, we shall now conduct experiments and explore particular gridscape models. Two applications, written in NetLogo (`http://ccl.northwestern.edu/netlogo/`), serve as the platforms for all the experiments described in this chapter. Their file names are `symmetric-2x2.nlogo` and `m1-symmetric-2x2.nlogo`, and they are available for downloading at the Web site for this book: `http://opim-sun.wharton.upenn.edu/~sok/agebook/`. Specifically, they are stored at `applications/nlogo/` under the `agebook/` directory. In addition, a number of QuickTime movies are available. These will be mentioned in the discussion and may be found at `movies/` under the `agebook/` directory.

Recall (cf., page 151) our canonical symmetric 2×2 game in strategic form:

	S_0	S_1
S_0	A A	C B
S_1	B C	D D

Figure 7.1: Canonical game matrix for the symmetric 2×2 game in strategic form

Also recall that in d dimensions, a hypercube of S_0s can invade an abutting hypercube of S_1s if

$$A[2 \cdot 3^{d-1} - 1] + B3^{d-1} > \max\{D[2 \cdot 3^{d-1} - 1] + C3^{d-1}, D[3^d - 1]\} \quad (7.1)$$

Similarly, a hypercube of S_1s can invade an abutting hypercube of S_0s if

$$D[2 \cdot 3^{d-1} - 1] + C3^{d-1} > \max\{A[2 \cdot 3^{d-1} - 1] + B3^{d-1}, A[3^d - 1]\} \quad (7.2)$$

In two dimensions—on the gridscape in particular—expression (7.1) reduces to

$$5A + 3B > \max\{5D + 3C, 8D\} \quad (7.3)$$

That is, a 3-cube of S_0s can invade an adjacent 3-cube of S_1s only if expression (7.1) holds. Similarly, a 3-cute of S_1s can invade an adjacent 3-cube of S_0s only if expression (7.4) holds.

$$5D + 3C > \max\{5A + 3B, 8A\} \quad (7.4)$$

Other formulas will be useful to us, but expressions (7.3) and (7.4) will be the first we apply and usually the most important for understanding the unfoldings of computational processes on the gridscape.

7.1 Experiments with symmetric-2x2.nlogo

The `symmetric-2x2.nlogo` application is a NetLogo program for playing 2×2 games on the gridscape when each player may have one of two strategies for play in the stage game. The program assumes a Moore neighborhood of 8 neighbors, so expressions (7.3) and (7.4) apply. We shall now use the program to observe behavior on a large gridscape. The default configuration which we shall use has a gridscape that is 329 by 157 cells, for a total of 51653 cells. Since each cell can be in one of two states (playing strategy S_0 (aka: 0) or playing strategy S_1 (aka: 1)), there are a total of 2^{51653} possible states in the system, any of which might be realized in a random initialization. Nevertheless, we shall find recognizable patterns stably arising.

7.1.1 Standard Prisoners' Dilemma

In the Standard Prisoners' Dilemma, the Reward for mutual cooperation is 3 ($= A$ in figure 7.1), the Temptation to defect is 5 ($= C$ in figure 7.1), the Sucker's payoff is 0 ($= B$ in figure 7.1), the Penalty for mutual defection is 1 ($= D$ in figure 7.1). Defection (playing S_1 or 1) is dominant to Cooperation (playing S_0 or 0) in this and all versions of Prisoners' Dilemma. What will happen on the gridscape, however?

In general for Prisoners' Dilemma we may map the canonical 2×2 symmetric game, figure 7.1, to Prisoners' Dilemma (0s for Cooperators, 1s for Defectors) as follows: $A \Rightarrow$

$R, B \Rightarrow S, C \Rightarrow T$, and $D \Rightarrow P$. Then a cube of the Cooperators will invade an abutting cube of Defectors if

$$R[2 \cdot 3^{d-1} - 1] + S3^{d-1} > \max\{P[2 \cdot 3^{d-1} - 1] + T3^{d-1}, P[3^d - 1]\} \tag{7.5}$$

Setting $S = 0$ and noting that $P[2 \cdot 3^{d-1} - 1] + T3^{d-1} > P[3^d - 1]$, this simplifies to

$$R[2 \cdot 3^{d-1} - 1] > P[2 \cdot 3^{d-1} - 1] + T3^{d-1} \tag{7.6}$$

or

$$R > P + \frac{T3^{d-1}}{[2 \cdot 3^{d-1} - 1]} \tag{7.7}$$

In 2-d:

$$5R > 5P + 3T \tag{7.8}$$

Similarly, a cube of Defectors will invade an abutting cube of Cooperators if

$$5P + 3T > 8R \tag{7.9}$$

(In Prisoners' Dilemma $8R > 5R + 3S$.) Note that Prisoners' Dilemma permits expression (7.8) to be true, e.g., $T = 101$, $R = 100$, $P = 1$, and $S = 0$. Prisoner's Dilemma also permits expression (7.9) to be true, e.g., $T = 100$, $R = 51$, $P = 50$, and $S = 0$. In both cases the essential requirements for Prisoners' Dilemma are met: $T > R > P > S$ and $2R > T + S$.

In the Standard Prisoner's Dilemma:

1. $5R = 15$
2. $5P + 3T = 20$
3. $8P = 8$
4. $8R = 24$

By expressions (7.3) and (7.4)—and specifically by expressions (7.8) and (7.9)—this implies that 3-cubes (rectangles in the 2-dimensional case) of cooperators cannot invade 3-cubes of defectors and vice versa. In addition, we know that in every one-on-one encounter between a cooperator (0) and a defector (1), the defector gets more points, more points in fact than in any other type of one-on-one encounter. What will happen on the gridscape?

Figures 7.2–7.5 show generations 0 (initialization), 4, 12, and 19 of a run of Standard Prisoners' Dilemma on the gridscape. At initialization, 99% of the agents are cooperators (colored black; the defectors are colored white). By generation 4 (figure 7.3), more than half of the agents are defecting. Notice that there has been a decline in social welfare. In generation 0, the society extracted (had a total take of) 1,235,412 points from the environment. Generation 4 yields them only 839,186 points. The plot called "Trace of Total Take" at the bottom of the figure shows a consistent decline in points extracted over time. Decline continues and by generation 12 (figure 7.4) only 773 cooperators (0s) remain and the society's take is down to 427,986. The system reaches static equilibrium (stasis) at generation 19 (figure 7.5). Only 314 cooperators remain and the total system take is down to 419224. Running the program with different random number seeds will produce results that are different in detail, but these results are typical. The cooperators are at a huge disadvantage in the Standard Prisoners' Dilemma game. Whether any survive or not entirely depends on whether 3-cubes appear during the unfolding of events. This will depend upon the initial configuration. With the same random number seed (2), if the program is run with 50% cooperators at initialization, by generation 2 they have been eliminated.

In the stage game, played once, defectors in Prisoners' Dilemma always beat cooperators. This is true in the Standard Prisoners' Dilemma game and in all others, regardless of the payoff structure. On the gridscape, even in the Standard case, it is possible for cooperators to survive. To the extent they do, they will individually prosper more than the defectors. We see, for example, in figure 7.5 that at stasis the cooperators are averaging 14.675 points per generation, while the defectors get on average only 8.076. Changing the payoff structure will not affect the analysis of the stage game. Let us see how it matters on the gridscape.

Note: Besides running `symmetric-2x2.nlogo` to duplicate these results, they may also be see in the QuickTime movie `standard-PD-start-99C.mov`, which is available for downloading at the AGEbook site.

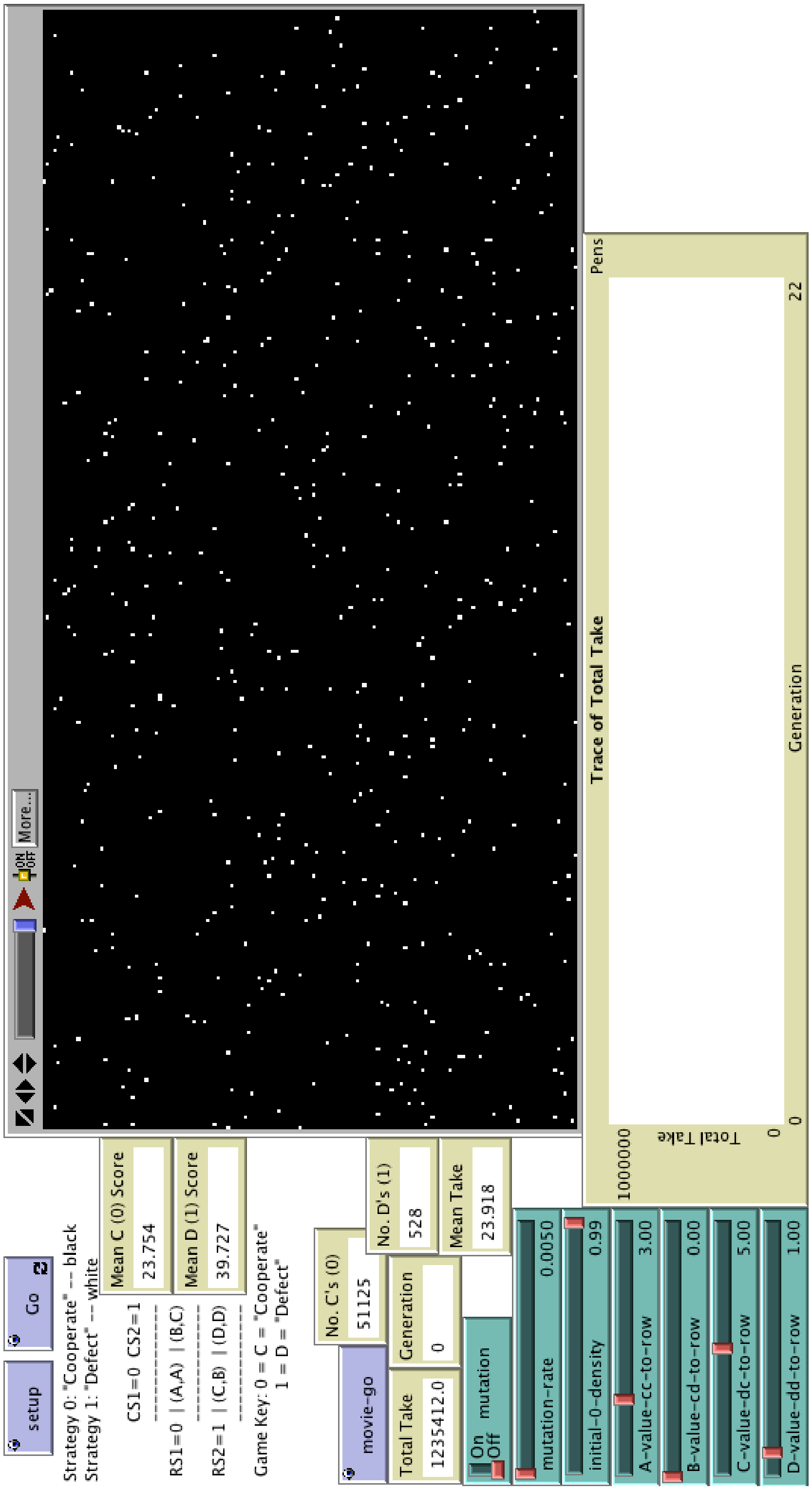

Figure 7.2: Standard Prisoners' Dilemma Stage Game, Initialized on the Gridscape. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2

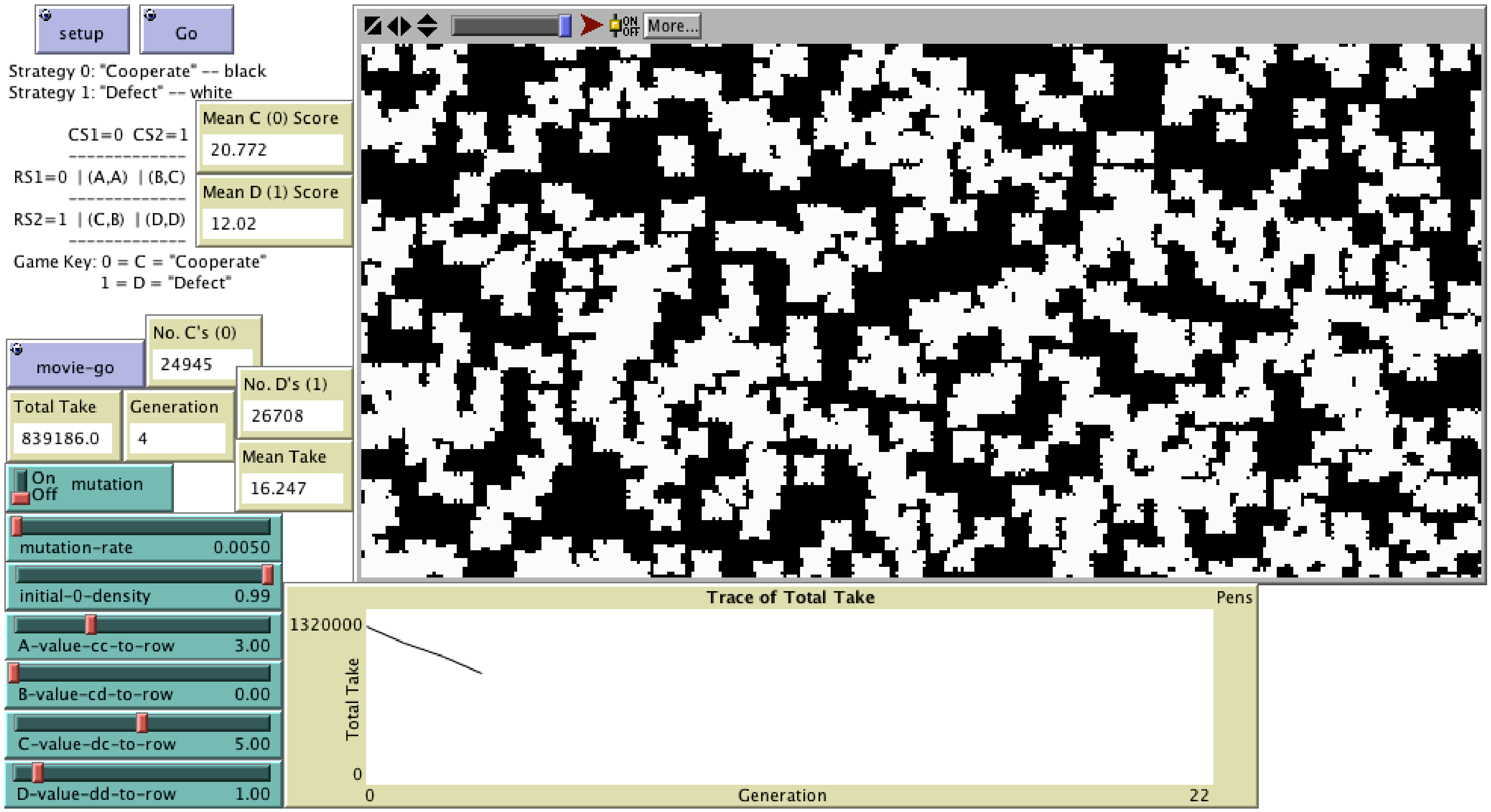

Figure 7.3: Standard Prisoners' Dilemma Stage Game, at Generation 4. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2

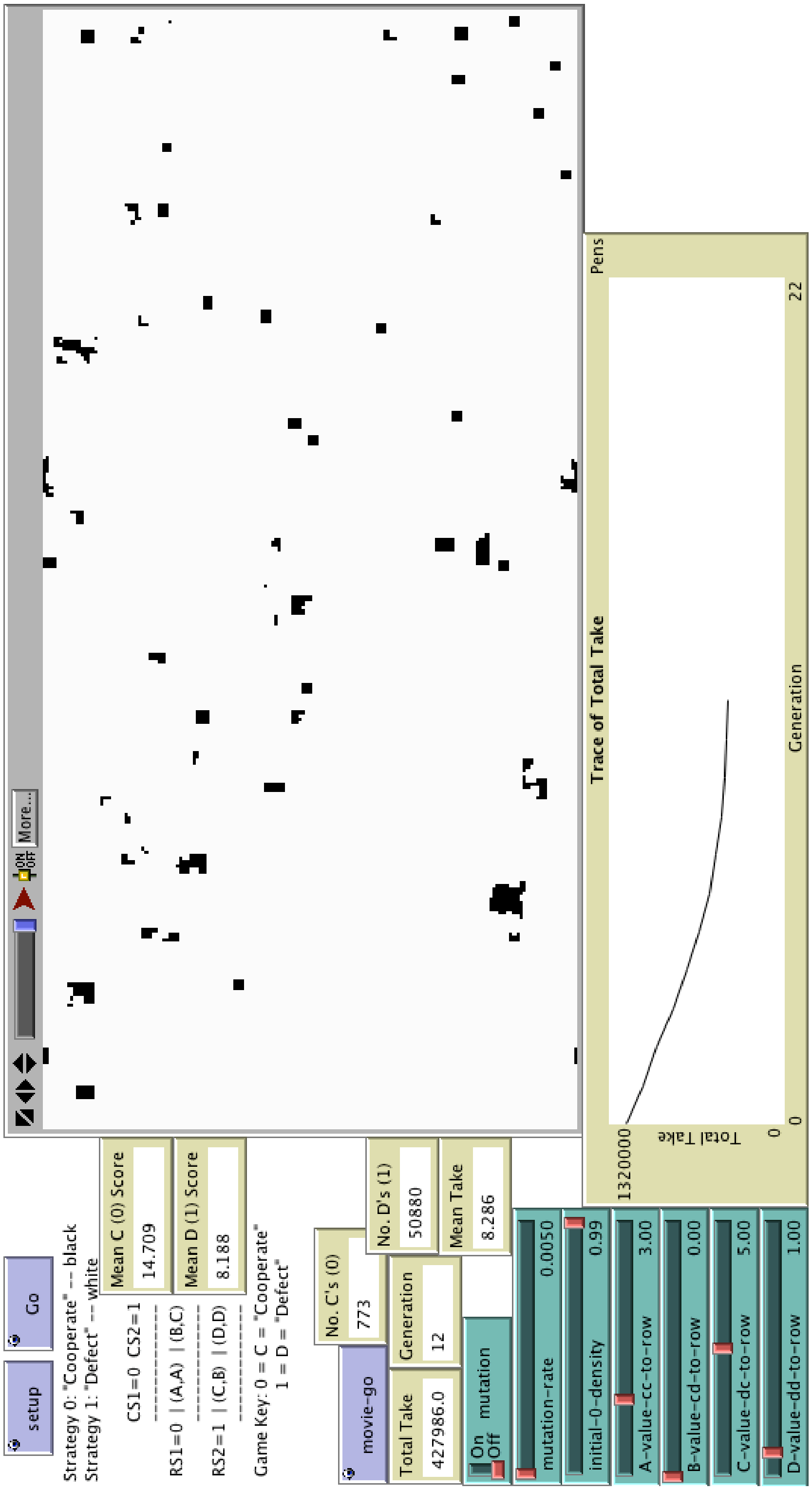

Figure 7.4: Standard Prisoners' Dilemma Stage Game, at Generation 12. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2

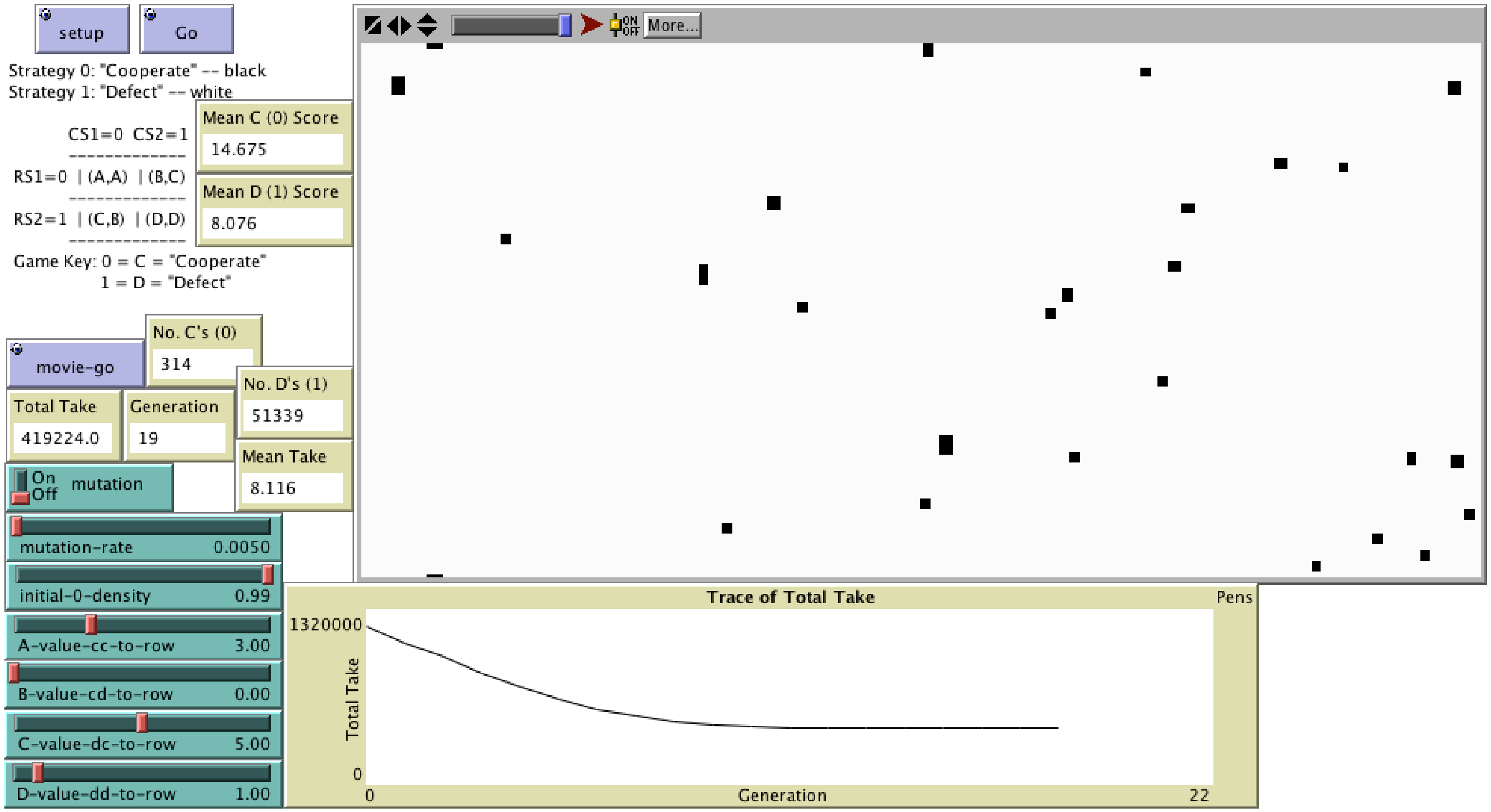

Figure 7.5: Standard Prisoners' Dilemma Stage Game, at Generation 19. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2. The system is static after generation 19.

7.1.2 Prisoners' Dilemma with a Payoff Structure Favoring Cooperation

We make one change to the Standard payoff structure: C (the Temptation to defect) goes from 5 to 3.2. Notice that we still have a genuine Prisoners' Dilemma stage game. What changes is that $5D + 3D$ goes from 20 to 14.6 ($= 5 \cdot 1 + 3 \cdot 3.2$). Because $14.6 < 15$, 3-cubes of cooperators can expand in a field of defectors. Even knowing this, the behavior of the gridscape is interesting.

Figure 7.6 shows the gridscape 30 generations after random initialization with 99% cooperators (initially only 528 defectors). The number of defectors varies by generation, but is consistently higher than the initial 528. In generation 29 it is 2937 and in generation 30 it is 1972. In fact, in generation 17 the system enters into a cycle condition, repeating itself every 12 generations. Because the same random seed was used (2), at generation 0 the system is as is shown in figure 7.2. With this new payoff structure, the defectors cannot expand very well against the cooperators. They cannot, however, really be eliminated, since a lone defector surrounded by a field of cooperators will temporarily expand at their expense. Notice that the total take actually declines from generation 0. The system is unable to meliorate after initialization. See also the QuickTime movie `PD-T3.2-99Cinit.mov`, in the AGEbook Web site.

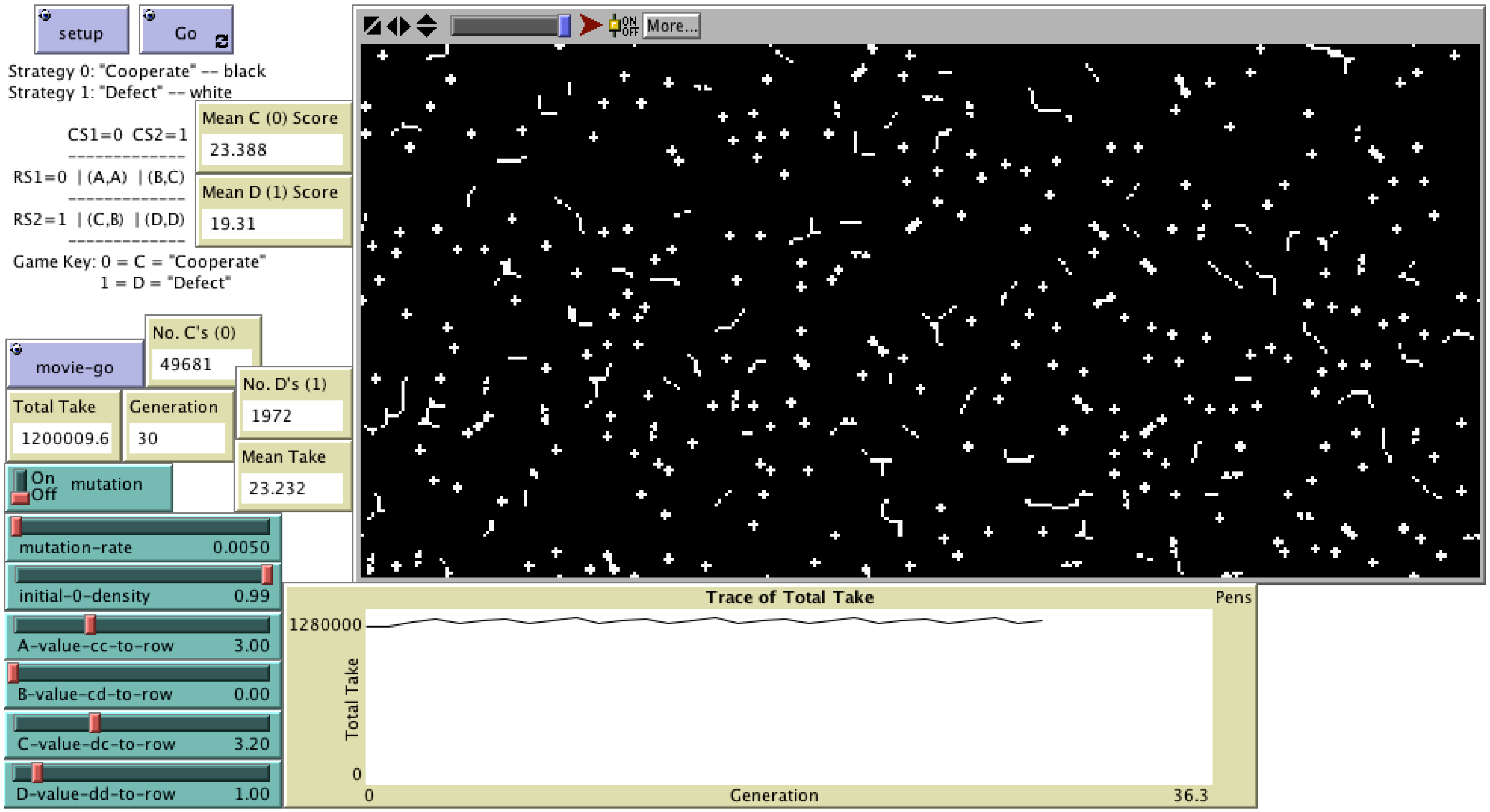

Figure 7.6: Prisoners' Dilemma Stage Game, Favorable to Cooperation, at Generation 30. Initialization was random with 99% cooperators and 1% defectors. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2. The system is in dynamic equilibrium, cycling every 12 generations, starting in generation 17.

A different pattern emerges if we initialize with 80% cooperators. See figure 7.7. Notice the *irregular checkered pattern* that prevails. The white areas, occupied by defectors, are able to withstand conversion by the cooperators. Several points are worth noting:

1. There is a decline in the number of defectors (agents playing 1) over the course of the run. At initialization there are 10,431 defectors, while at the end of generation 30 there are 8,987. The decline is regular throughout the run.

2. The total take in the society drops to a low of 679,366 after generation 2. From there it steadily increases and then levels off. This is, we shall see, a characteristic pattern.

3. At convergence, e.g., in generation 30, the mean takes of the two strategies are very close, 20.7 for cooperators and 19.4 for defectors. The mean take overall is 20.5. The maximum mean take, realized in a society of cooperators only, is 24. Thus we might say that the efficiency of the system is $20.5/24 = 85\%$. In figure 7.6 we found an efficiency of $23.2/24 = 96\%$.

See also the QuickTime movie `PD-T3dot2-80Cinit.mov`, in the AGEbook Web site.

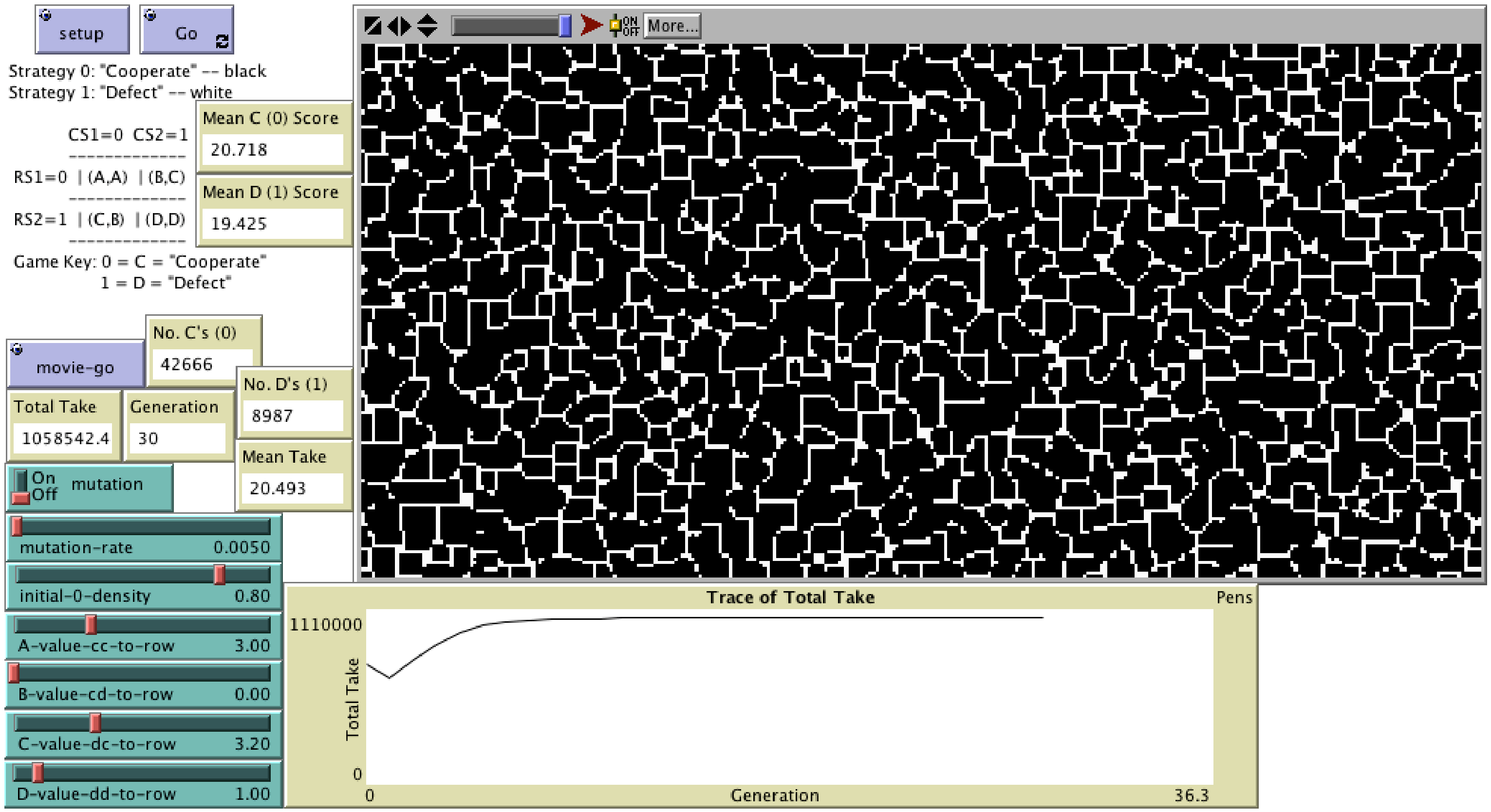

Figure 7.7: Prisoners' Dilemma Stage Game, Favorable to Cooperation, at Generation 30. Initialization was random with 80% cooperators and 20% defectors. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2. The system is in dynamic equilibrium, pictured here after generation 30.

Figure 7.8 shows the system at convergence in generation 44, after initialization with 50% cooperators. Notice the irregular checkerboard pattern that emerges, even more forcefully than in figure 7.7. Also, the total take of 1,163,622 is lower than in figure 7.6 but higher than in figure 7.7. The initial dip in total take, which we noticed in figure 7.7 is stronger here, with the number of cooperators dropping to a low of 1,818 in generation 2, then climbing steadily after that. Finally, notice in the southeast quadrant (in the neighborhood of patch 91-17) a small, changing L-shaped figure that appears in generation 2 and becomes more apparent in later generations. It is eventually eliminated. Why?

See also the QuickTime movie `PD-T3dot2-50Cinit.mov`, in the AGEbook Web site.

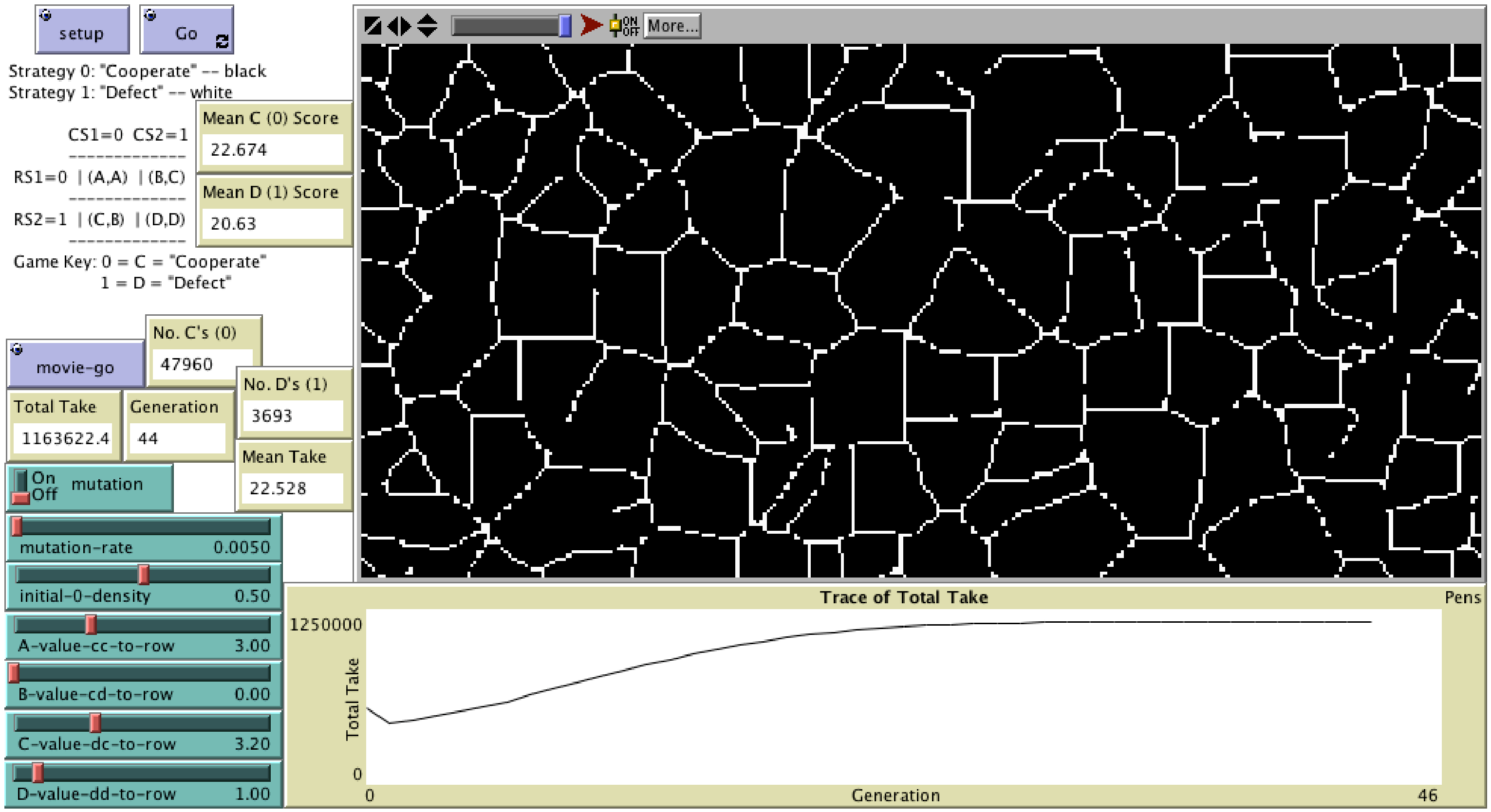

Figure 7.8: Prisoners' Dilemma Stage Game, Favorable to Cooperation, at Generation 44. Initialization was random with 50% cooperators and 50% defectors. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2. The system is in dynamic equilibrium, pictured here after generation 44.

Figure 7.9 shows the system at convergence in generation 98, after initialization with 20% cooperators. Notice the irregular checkerboard pattern that emerges, even more forcefully than in the previous figures of this series. Also, the total take of 1,217,601 is highest of all the cases examined. The initial dip in total take, which we first noticed in figure 7.7 is stronger here, with the number of cooperators dropping to a low of 118 in generation 2, then climbing steadily after that. Thus, we see that with *fewer* initial cooperators the number of cooperators declines at first to a very low level, then climbs to a level higher than in any of the other cases.

See also the QuickTime movie `PD-T3do22-20Cinit.mov`, in the AGEbook Web site.

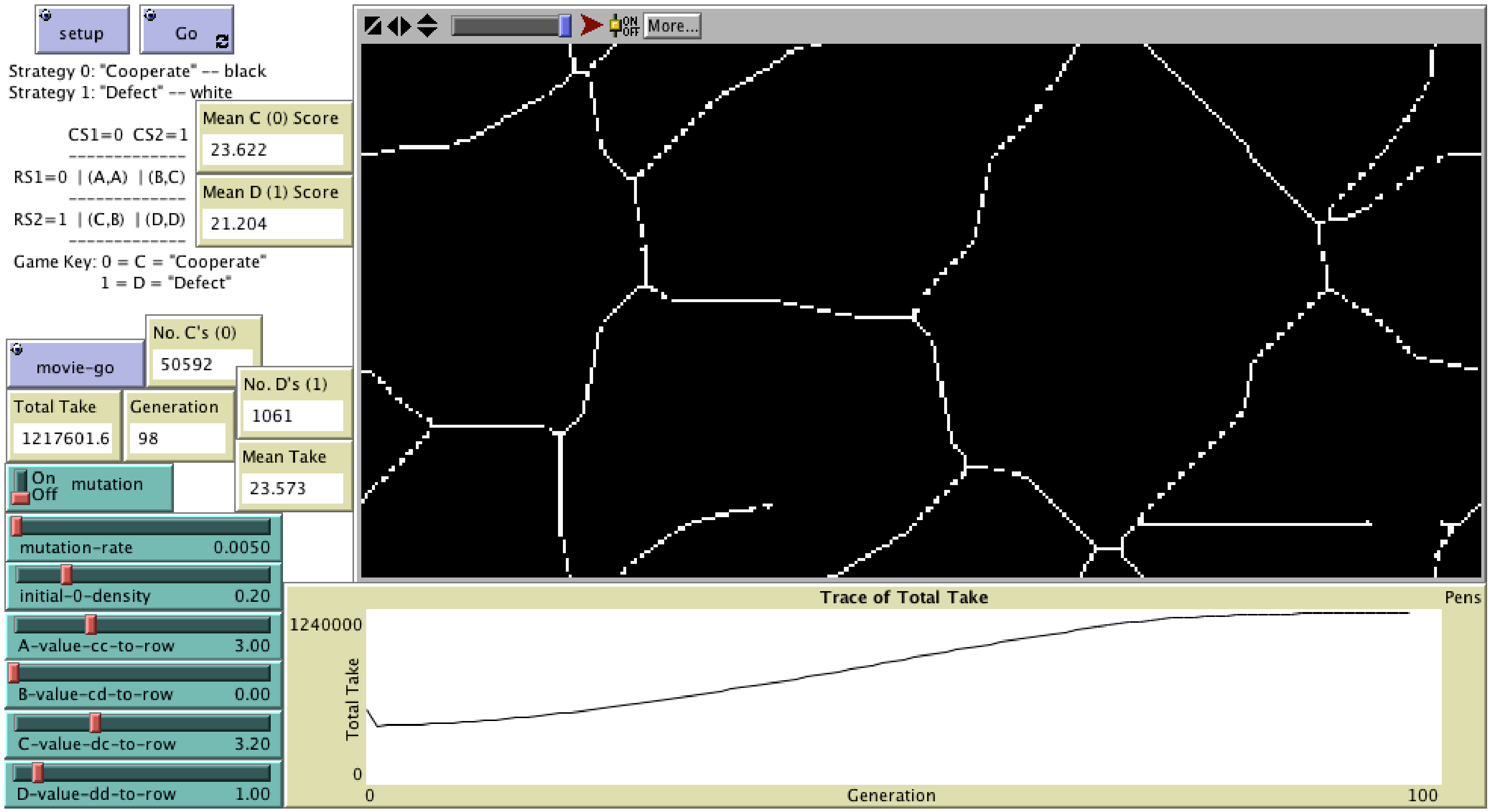

Figure 7.9: Prisoners' Dilemma Stage Game, Favorable to Cooperation, at Generation 44. Initialization was random with 20% cooperators and 50% defectors. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2. The system is in static equilibrium, pictured here after generation 98.

7.2 Introducing Mutation

Mutation, the random flipping of an agent's policy type, arguably introduces an element of greater realism in the gridscape models. Agents err, mechanisms fail, the environment randomly intervenes. How will this affect results on the gridscape?

Compare figure 7.10, mutation on, with figure 7.6. Except for mutation, the setups are identical. Figure 7.10 resembles figure 7.11, a different setup, much more than it resembles figure 7.6. Note as well figure 7.12, a longer-run version of figure 7.10. Despite differing initialization rates and run lengths, figures 7.10–7.12 are quite similar. Mutation nullifies the differences in initial configuration.

See also the QuickTime movies `PD-T3dot2_99Cinit-gen40_mut005.mov` and `PD-T3dot2_20Cinit-gen200_mut005.mov` at the AGEbook Web site.

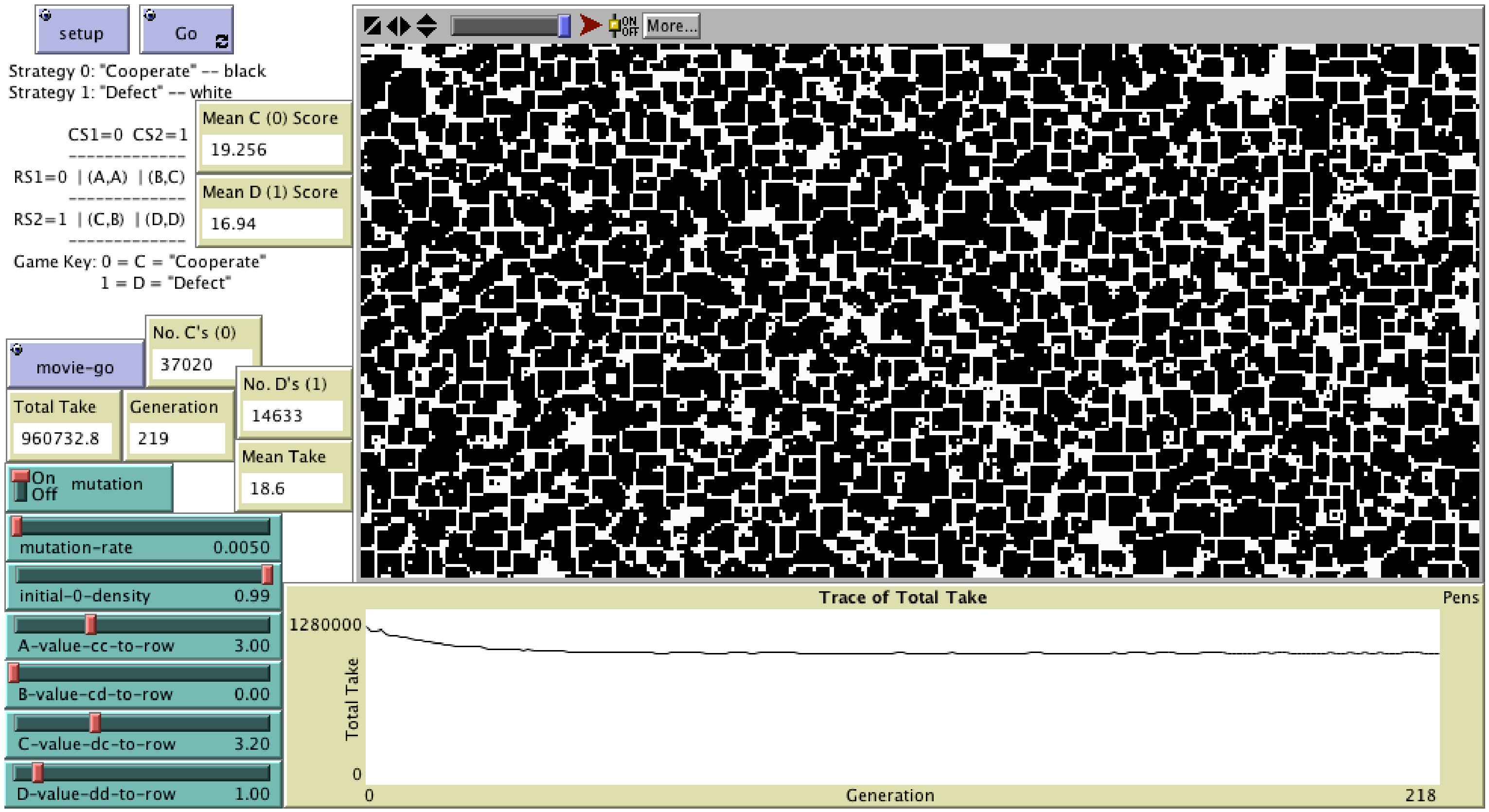

Figure 7.10: Prisoners' Dilemma Stage Game, Favorable to Cooperation, at Generation 219. Initialization was random with 99% cooperators and 1% defectors. Mutation occurs at a rate of 0.005 per agent per generation. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2.

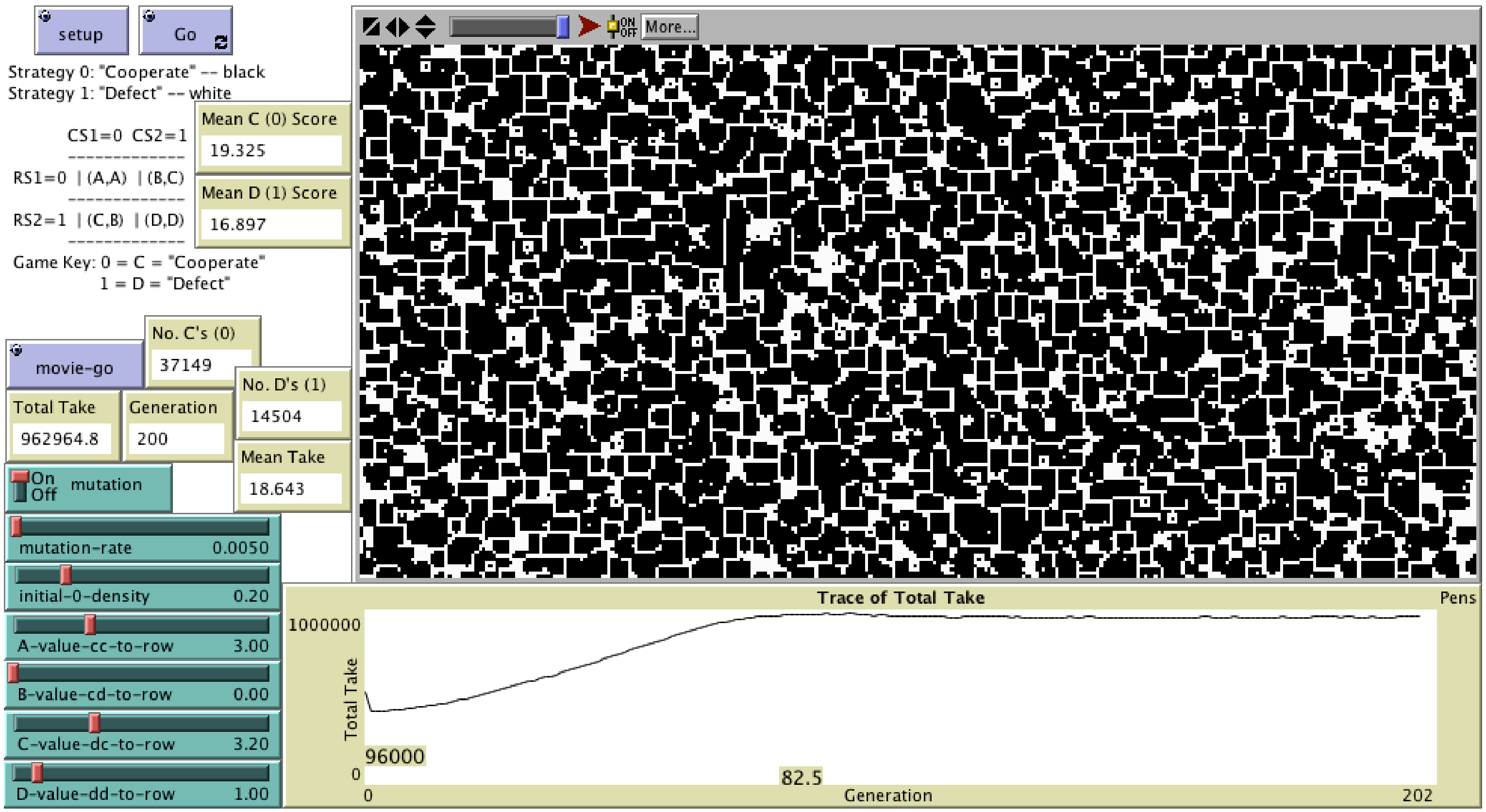

Figure 7.11: Prisoners' Dilemma Stage Game, Favorable to Cooperation, at Generation 200. Initialization was random with 20% cooperators and 80% defectors. Mutation occurs at a rate of 0.005 per agent per generation. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2.

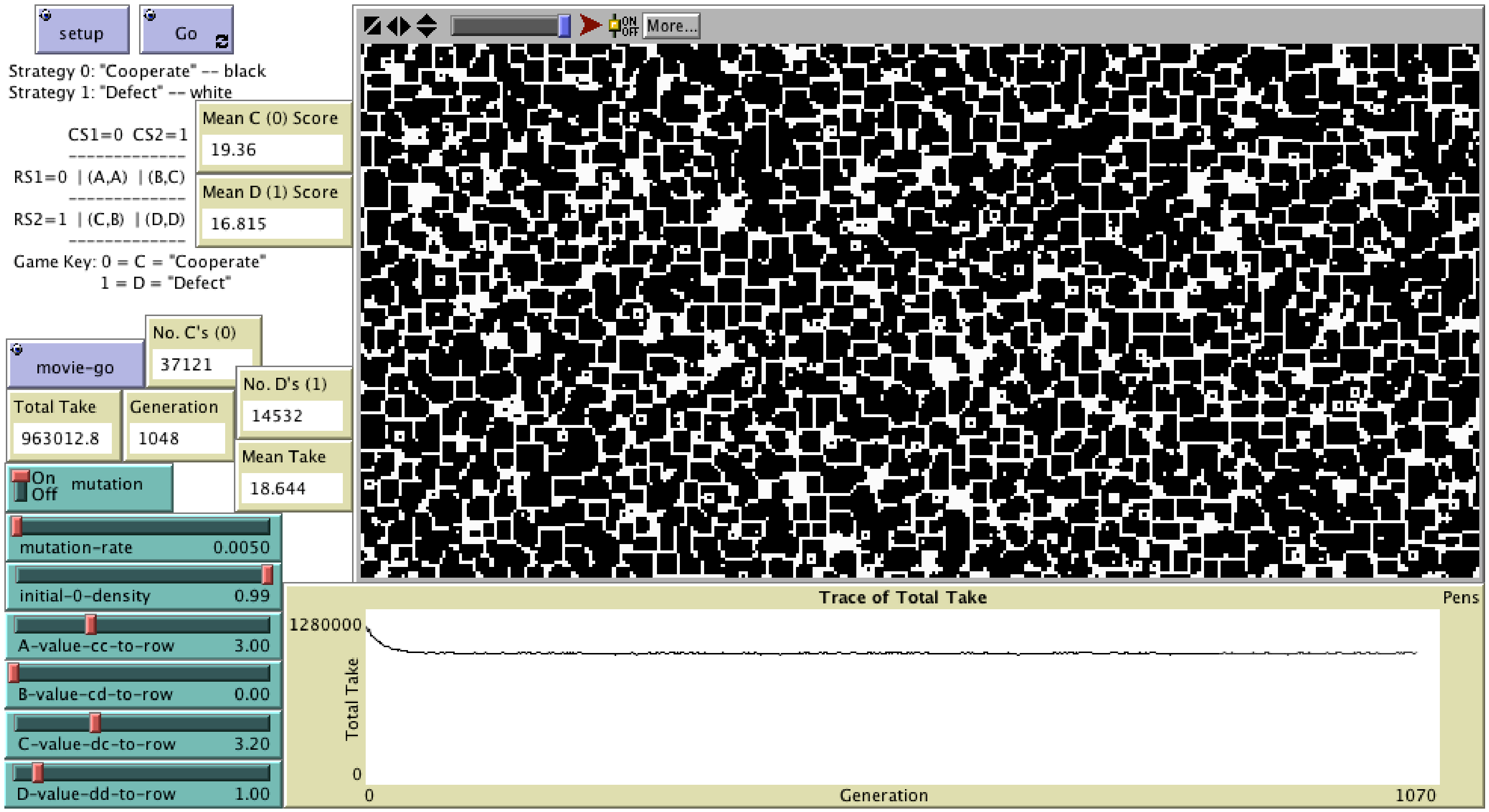

Figure 7.12: Prisoners' Dilemma Stage Game, Favorable to Cooperation, at Generation 1048. Initialization was random with 99% cooperators and 1% defectors. Mutation occurs at a rate of 0.005 per agent per generation. symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2.

7.3 Stag Hunt

In general for Stag Hunt we require that $A > C > D > B$. For Standard Stag Hunt $A = 3$, $B = 0$, $C = 2$, and $D = 1$. Standard Stag Hunt on the gridscape (under our update regime of imitating the best of the Moore neighbors) is trivial: hunting stag prevails. More carefully, since $5A + 3B = 15 > \max\{5D + 3C, 8D\} = 11$, 3-cubes of stag hunters will expand in a field of hare hunters. Moreover, isolated hare hunters are eliminated. A single hare hunter in a field of stag hunters gets $8 \cdot 2 = 16$ points, while its neighbors get $7 \cdot 3 = 21$ points. So long as at least one 3-cube of stag hunters is present, stag hunters will expand to total conquest of the gridscape.

More interesting in the present context is Extreme Stag Hunt, in which $A = C$. Hunting hare is now weakly dominant in the stage game. Letting $A = 3$, $B = 0$, $C = 3$, and $D = 1$, we see that 3-cubes of stag hunters will expand in a field of hare hunters, since $5A + 3B = 15 > \max\{5D + 3C, 8D\} = 14$. Singleton hare hunters are, however, safe from conquest in a field of stag hunters, and can expand if there are other hare hunters nearby.

Figures 7.13 and 7.14 show static equilibria for two different initializations of Extreme Stag Hunt (5% stag hunters and 90% stag hunters). Their qualitative similarities are striking and even more so is the resemblance to figure 7.7 for Prisoners' Dilemma initialized with 80% cooperators. Note that this version of Prisoners' Dilemma is numerically close to Extreme Stag Hunt, although as one-shot stage games they are classified as different games entirely in standard game theory.

Standard Stag Hunt under mutation remains uninteresting. Assuming existence of some 3-cubes and no disastrous elimination by mutation, the stag hunters will prevail (mostly) except for the newly arisen hare hunters coming from mutation. Similarly, there is only minor effect from mutation (at the 0.005 level) in Extreme Stag Hunt. Figure 7.15 shows the system state with mutation 24 generations after initialization with 5% stag hunters. Its similarity to figure 7.13 is striking. Finally, figure 7.16 shows the system state in generation 40 after turning mutation off after generation 24. Notice the modest improvement in total take.

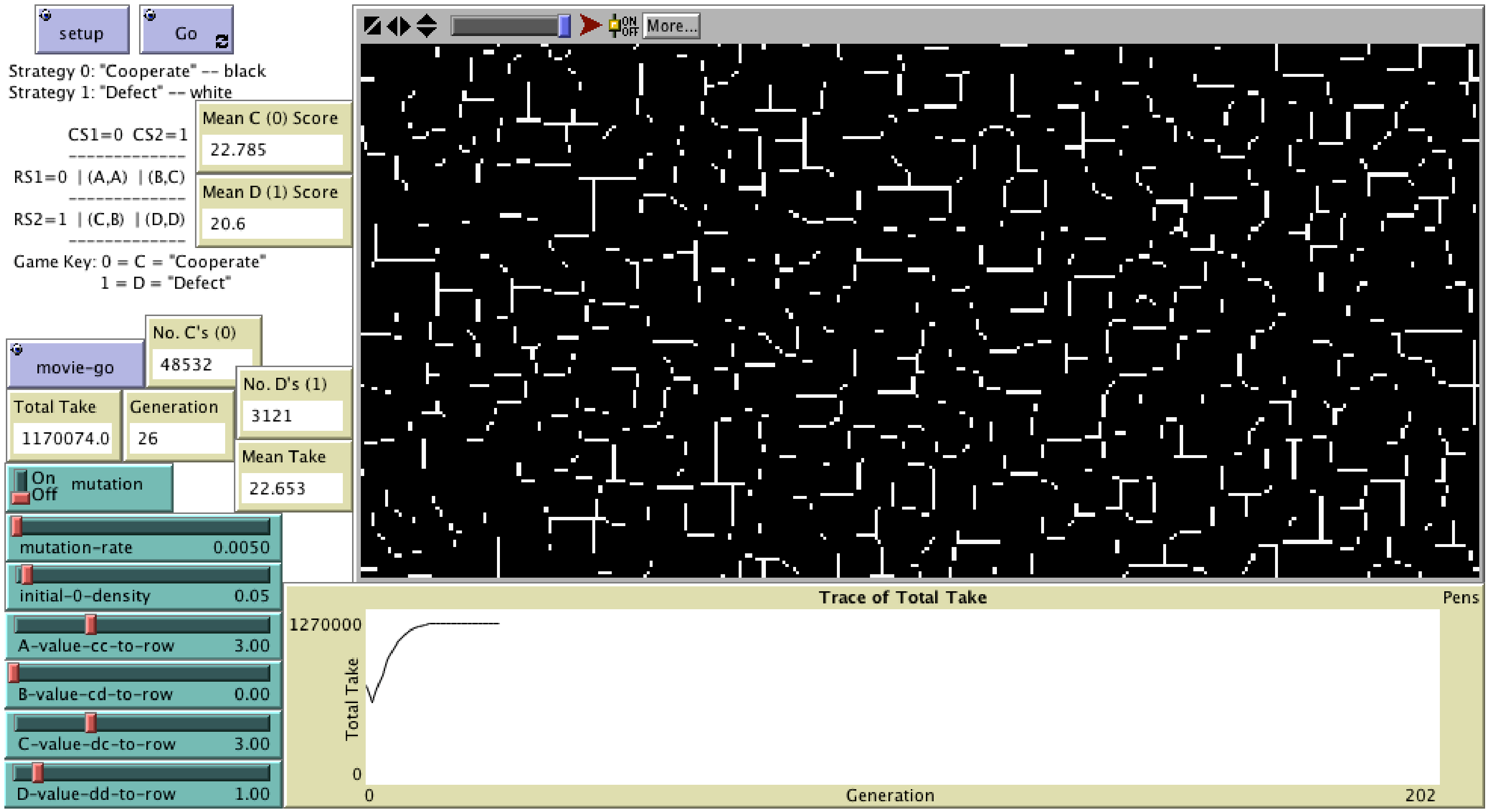

Figure 7.13: Extreme Stag Hunt Stage Game ($A = 3$, $B = 0$, $C = 3$, $D = 1$), at Generation 26 in static equilibrium. Initialization was random with 5% cooperators (stag hunters) and 95% defectors (hare hunters). symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2.

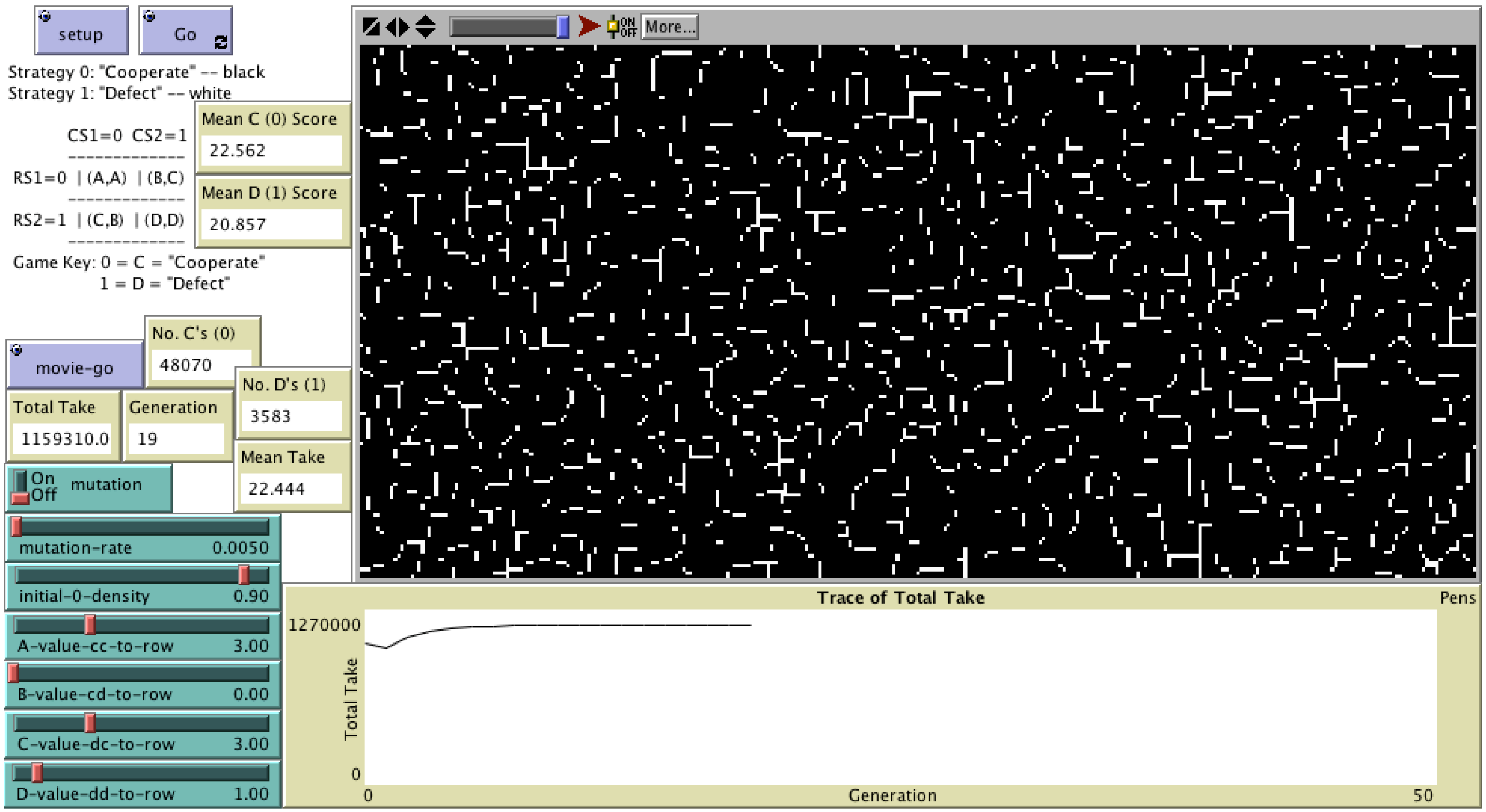

Figure 7.14: Extreme Stag Hunt Stage Game ($A = 3$, $B = 0$, $C = 3$, $D = 1$), at Generation 19 at static equilibrium. Initialization was random with 90% cooperators (stag hunters) and 10% defectors (hare hunters). symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2.

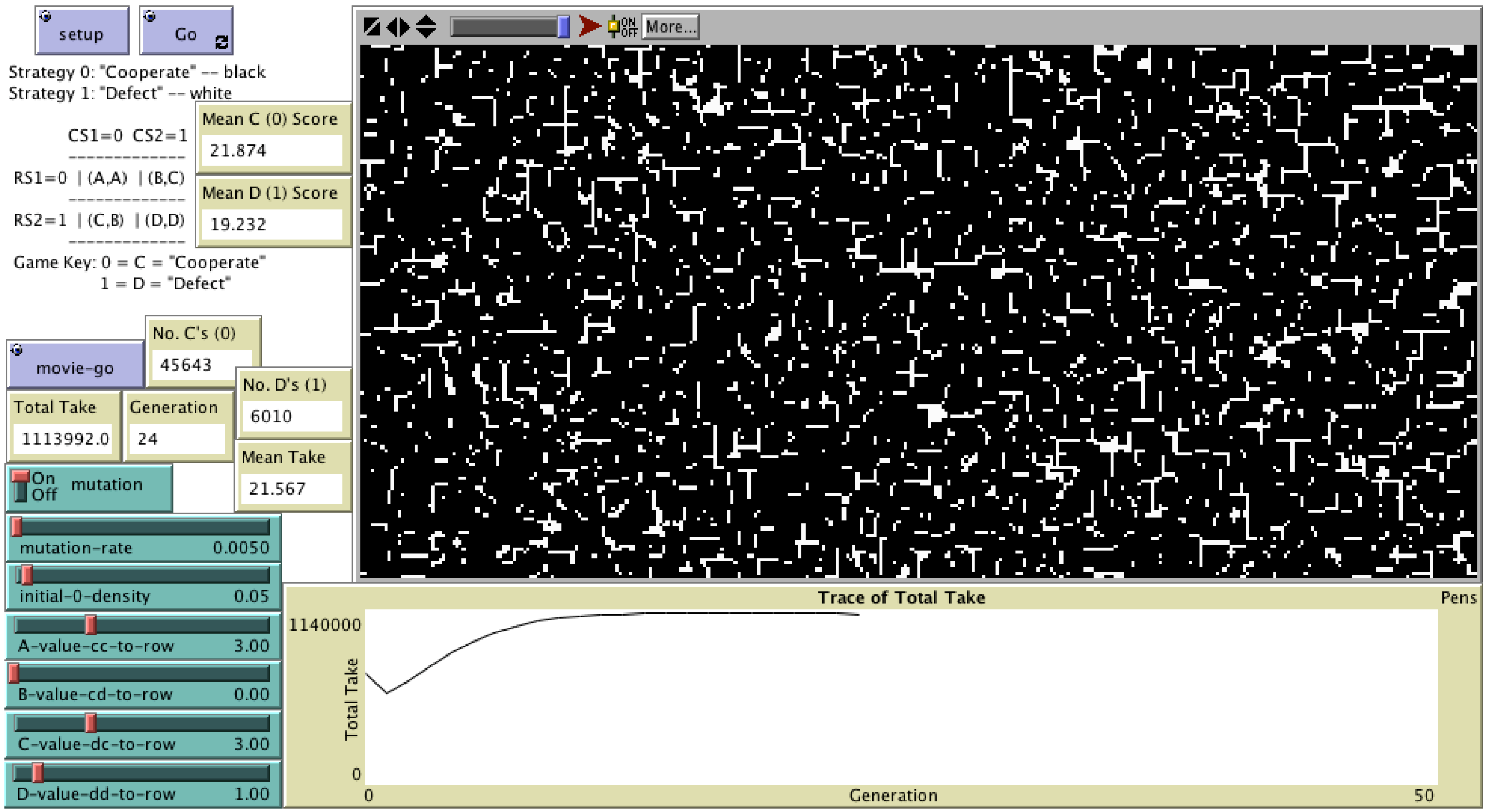

Figure 7.15: Extreme Stag Hunt Stage Game ($A = 3$, $B = 0$, $C = 3$, $D = 1$), at Generation 24 at convergence under mutation at 0.005. Initialization was random with 5% cooperators (stag hunters) and 95% defectors (hare hunters). symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2. File: `SH-extreme_05Cinit-gen24-mut005.pdf`.

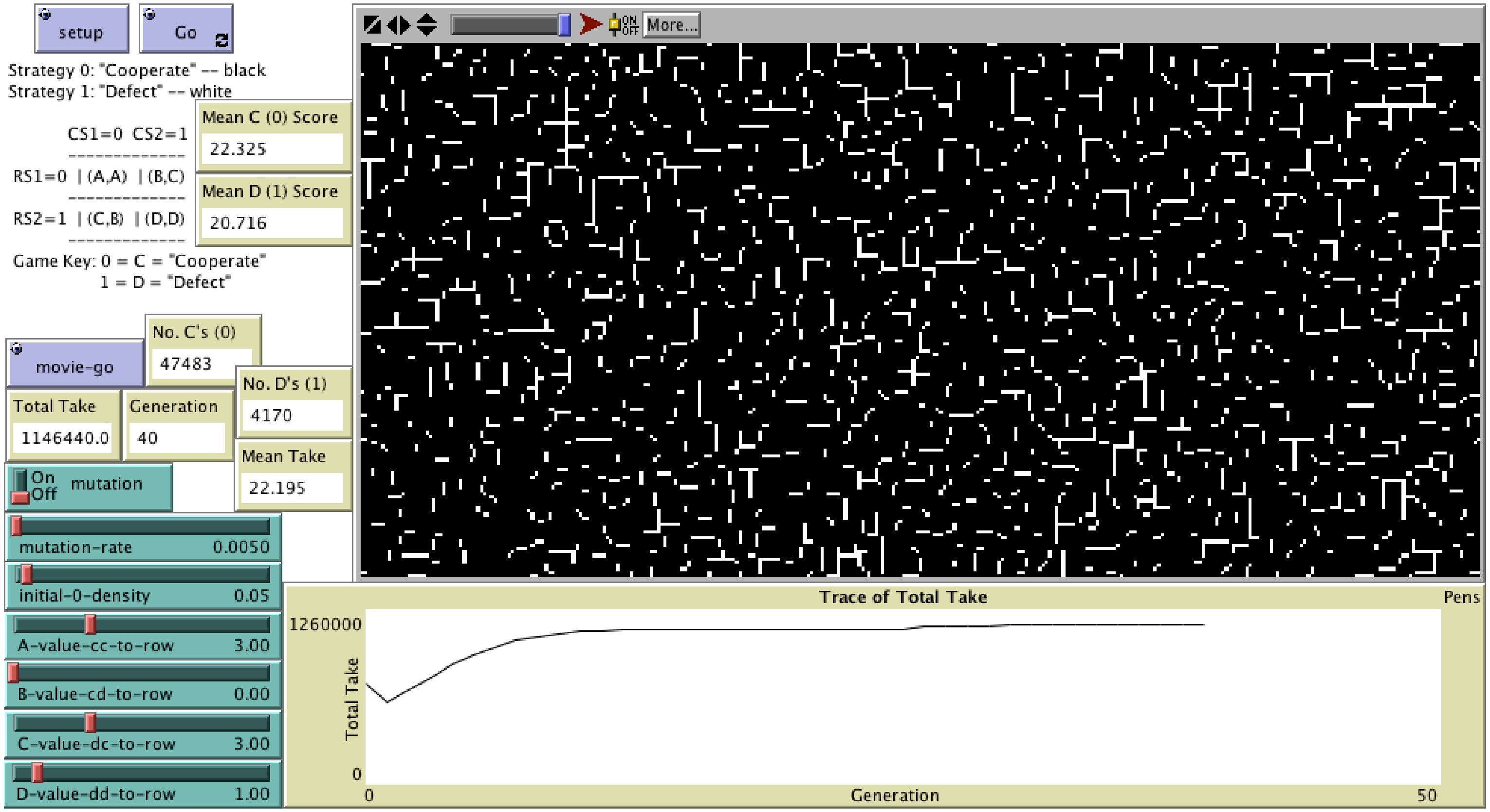

Figure 7.16: Extreme Stag Hunt Stage Game ($A = 3$, $B = 0$, $C = 3$, $D = 1$), at Generation 40 at static equilibrium under no mutation since generation 24. See figure 7.15 for mutation during generations 1-24. Initialization was random with 5% cooperators (stag hunters) and 95% defectors (hare hunters). symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2. File: `SH-extreme_05Cinit-gen40-postmut005.pdf`.

7.4 Chicken

In Standard Chicken we have $A = 2$, $B = 1$, $C = 3$, $D = 0$. Figure 7.17 shows the state of the gridscape for a typical run to stability. Cooperators—agents being chicken—outnumber defectors about 2 to 1. The mean take is 12.6. In a gridscape of all cooperators, the mean take would be $2 \cdot 8 \cdot 2 = 24$. From a social optimization perspective, these agents do not perform terribly well. Note that if both players are chicken each gets 2 points for a total of 4. If one player plays chicken and the other is reckless, the point total is still 4 = 3 + 1. In figure 7.18 we see the results of making the game a bit more favorable to mutual cooperation. Now $A = 2.1$ so the payoff sum from mutual cooperation, $4.2 = 2.1 + 2.1$, is greater than the 4 obtained by taking turns cooperating and defecting. Indeed, the ratio of cooperators to defectors goes up to about 2.5 to 1, as does the mean take, which is now 13.6.

Figure 7.19 shows the results from a typical run in which the total payoff for mutual cooperation averages *less* than that for taking turns being reckless. The ratio of cooperators to defectors is now about 2.7 to 1. The mean take is 13, higher than the 12.6 obtained in the Standard Chicken game with larger payoffs for cooperation! Perhaps Chicken is an especially difficult game for agents without memory.

See movie files at the AGEbook site: `Chicken-standard-C200_gen38.mov`, `Chicken-friendly-C210_gen38.mov`, and `Chicken-unfriendly-C190_gen38.mov`.

Figure 7.17: Standard Chicken Game ($A = 2$, $B = 1$, $C = 3$, $D = 0$), at Generation 38. Initialization was random with 7% cooperators (chickens) and 93% defectors (reckless). File: `symmetric-2x2.nlogo`, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2.

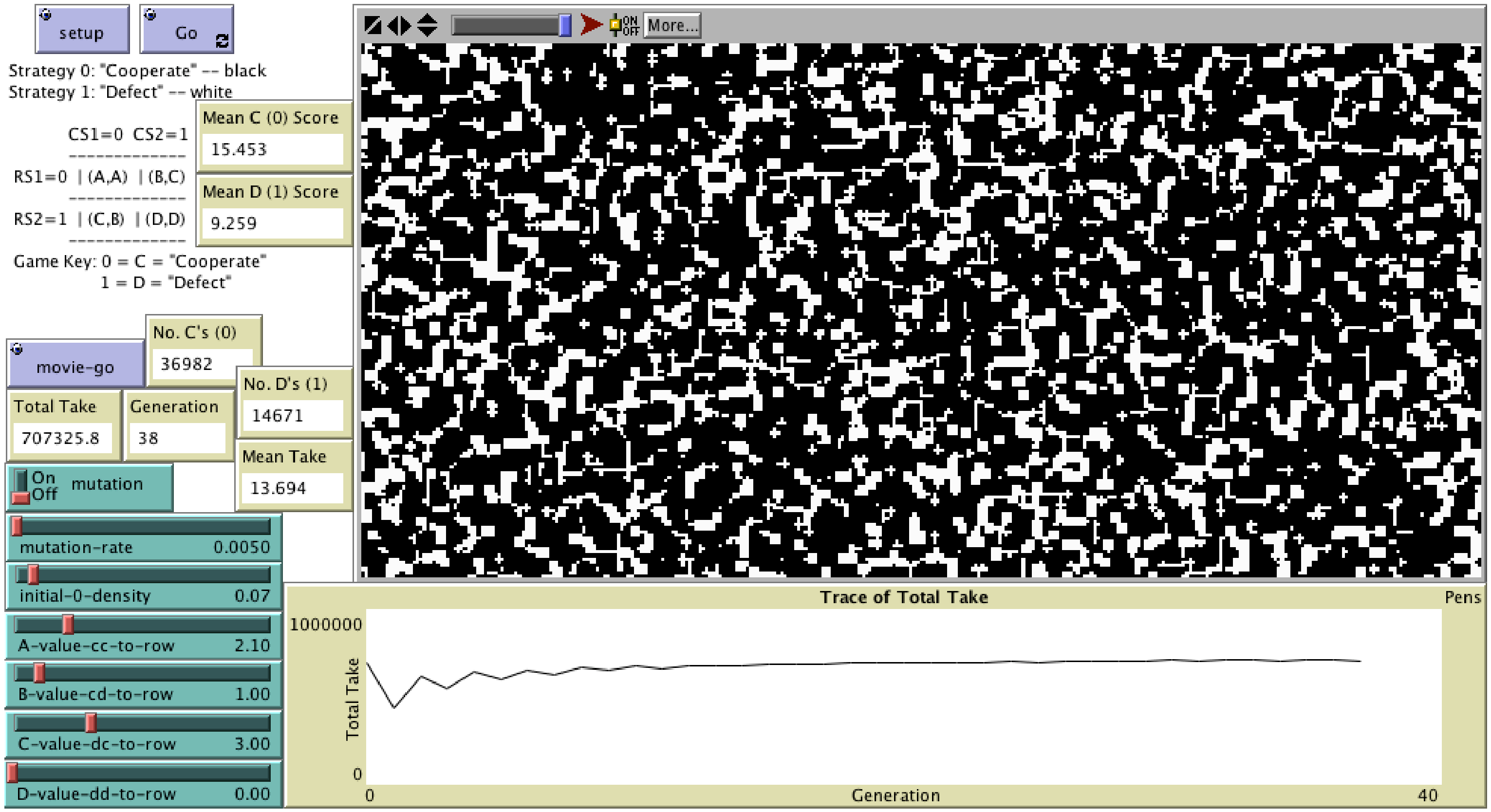

Figure 7.18: Moderately Friendly Chicken Game ($A = 2.1$, $B = 1$, $C = 3$, $D = 0$), at Generation 38. Initialization was random with 7% cooperators (chickens) and 93% defectors (reckless; Jim Kniffens). symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2.

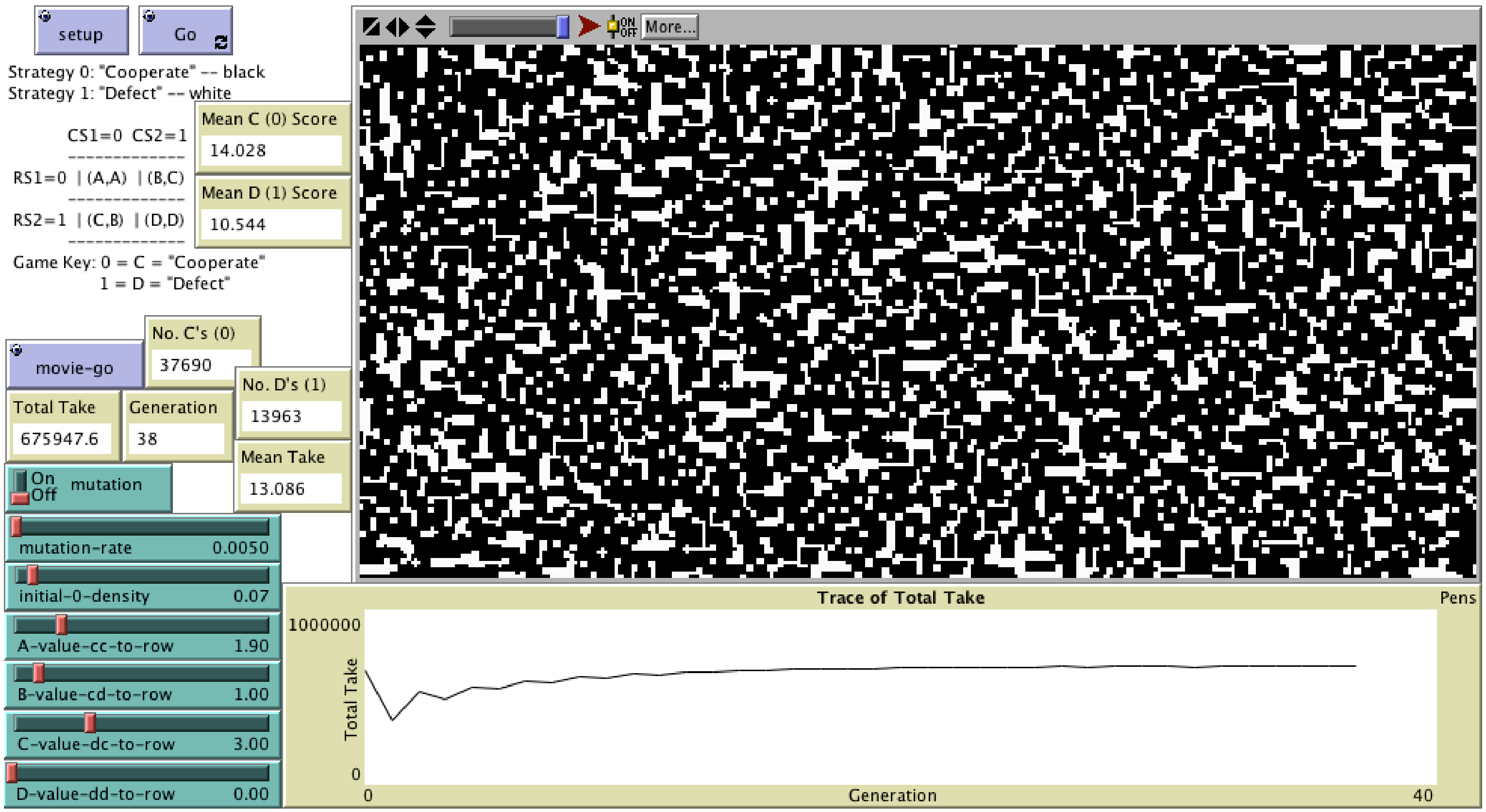

Figure 7.19: Moderately Unfriendly Chicken Game ($A = 1.9$, $B = 1$, $C = 3$, $D = 0$), at Generation 38. Initialization was random with 7% cooperators (chickens) and 93% defectors (reckless; Jim Kniffens). symmetrix-2x2.nlogo, v1.7. screen-size-x = 329 screen-size-y = 157. The total number of patches is 51653; the random seed is 2.

7.5 **Symmetric Games with M1 Policies**

/* to be completed */

7.6 **Discussion**

/* to be completed */

Glossary

constructive game theory
See page 52.
equilibrium game theory
See page 52.
fundamental rationality
See page 41.
IER: individually economically rational
See page 43.
iterated play
The literature does not speak with a unified voice regarding this terminology. By stipulation, I shall use *iterated play* to describe a game played multiple times by an agent, which play is with unknown, normally different counter-players. See: *repeated play*.
Nash equilibrium
See page 44.
outcome of a game
See page 43.
payoff of a game
See page 43.
play of a game
See page 43.
practicable accessibility
See page 51.
repeated play
The literature does not speak with a unified voice regarding this terminology. By stipulation, I shall use *repeated play* to describe a game played multiple times by an agent, which play is with the same counter-players.See: *iterated play*.

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